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DEVOTED TO THE INTERESTS OF
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The AMERICAN MATHEMATICAL MONTHLY

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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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MENASHA, WIS., AND EVANSTON, ILL.

SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Richmond, on Saturday, May 11, 1940. This meeting was held during the hundredth-year anniversary of the granting of the Charter to the University. The chairman of the Section, Dr. L. S. Dederick, presided over both sessions, morning and afternoon.

The meeting was attended by forty-five persons, including the following thirty-two members of the Association: O. S. Adams, M. W. Aylor, H. C. Ayres, C. C. Bramble, Scott Buchanan, Eleanor Calkins, Randolph Church, G. R. Clements, Nancy Cole, L. S. Dederick, Alexander Dillingham, R. E. Gaines, Michael Goldberg, Isabel Harris, G. A. Hedlund, Wilfred Kaplan, J. L. Kelley, L. M. Kells, R. H. Knox, Jr., W. D. Lambert, A. E. Landry, S. B. Littauer, E. J. McShane, Sister Thomas Marie Maloney, H. A. Perkins, O. J. Ramler, R. E. Root, T. McN. Simpson, Jr., J. M. Stetson, Carrie B. Taliaferro, G. C. Vedova, and C. H. Wheeler, III.

The Section accepted invitations from Trinity College, Washington, D. C., for the fall meeting of 1940, and from the U. S. Naval Academy for the spring meeting of 1941. Invitations from Georgetown University and from Randolph-Macon College were acknowledged. The following officers were elected for 1940–1941: Chairman, T. McN. Simpson, Jr., Randolph-Macon College; Secretary, C. H. Wheeler, III, University of Richmond; to the Executive Committee, Florence P. Lewis, Goucher College, and L. M. Kells, U. S. Naval Academy. The Section gave a rising vote of thanks to the University of Richmond for its generous hospitality.

At the invitation of the Section, Professor R. E. Gaines of the University of Richmond, and Dean Scott Buchanan of St. John's College led a symposium on "The aims of college mathematics teaching."

After an address of welcome by President Boatwright of the University of Richmond, the following six papers were read:

1. "An analysis of the indirect method of proof" by Professor G. C. Vedova, St. John's College.

2. "Convex sets" by T. A. Botts, University of Virginia, introduced by Professor McShane.

3. "On a problem in the calculus of variations" by Dr. Nancy Cole, Sweet Briar College.

4. "Topology as applied to mechanics" by Dr. Wilfred Kaplan, College of William and Mary.

5. "Certain changes in student preparation during the last fifty years" by Professor R. E. Gaines, University of Richmond.

6. "Mathematics from the point of view of the 'New Plan' " by Dean Scott Buchanan, St. John's College.

Abstracts of the papers follow in the order numbered above:

1. Professor Vedova considered a universe, U, of elements x, in which the following postulates were assumed to hold: (1) Laws L_i, and their corresponding negatives $-L_i$, can be phrased concerning the elements x; and, (2) For any x and any L_i, one and only one of L_i and $-L_i$ is determinately satisfied. Then, denoting by C_i and $-C_i$ the classes of elements x determined by L_i and $-L_i$, respectively, and making use of the symbolism of logic he laid down the definition: Given a set of laws $L_1, L_2, \cdots, L_n$ determining the classes $C_1, C_2 \cdots, C_n$, and an element x of U, the proof that x is in C_j is said to be *direct* if it is shown that x satisfies the law L_j, and *indirect* if it is shown that, (a) x satisfies the law $(L_1 \vee L_2 \vee \cdots \vee L_n)$ but, (b) x does not satisfy the law $(L_1 \vee L_2 \vee \cdots \vee L_{j-1} \vee L_{j+1} \vee \cdots \vee L_n)$. The classes C_i were then studied under various hypotheses as to exclusiveness, overlapping, exhaustiveness, *etc.*, and it was shown that the question of whether the indirect method of proof is applicable or not is always a determinate one.

2. Mr. Botts outlined two short analytic proofs of the theorem that a convex body K in a euclidean space of any dimensions has a supporting plane passing through any point x of its boundary. He first showed that the closure of the set obtained by drawing all rays from x through points of K was a convex cone C with external points, and that C then had at least one plane of support, any of which passed through x. Mr. Botts showed that if p were a point of maximum distance from K on the surface of the unit sphere about x, then x was the point of K closest to p, from which it followed that the plane through x with normal xp was a plane of support of K.

3. Dr. Cole discussed conjugate points for a problem in euclidean m-space in which the integrand is discontinuous, the basic curve g being a broken extremal with one corner at which g is cut across by a regular $(m-1)$-manifold of class C^2, not tangent to either arc of g at the corner, and at which g satisfies a set of "primary incidence relations." She defined conjugate points on g of a fixed point a of g in terms of the zeros, $t \neq a$, of the conjugate point determinant $D_g(t, a)$, whose m columns represent certain linearly independent solutions of the Jacobi equations determined by g and then showed that if a and b were any two fixed points of g, the numbers of zeros of the conjugate point determinants $D_g(t, a)$ and $D_g(t, b)$ on any finite interval of g differed by at most $m-1$.

4. Dr. Kaplan illustrated the possibilities of applying the qualitative method of topology to problems of mechanics, in particular the problems involving the solution of differential equations. The basic idea, due to Poincaré and later developed by Birkhoff and others, was shown to be the consideration of sets of solutions and the configurations they formed, as opposed to single solutions. Dr. Kaplan illustrated the method in the case of the problems of two and three bodies and several simpler problems.

5. Professor Gaines presented a brief survey of changes in the preparation of liberal arts college students of mathematics during the past fifty years. He

then classified students as to preparation, ability, and objective, and described his experience with the varied groups. He then raised the question: In the light of the widely varied degrees of preparation in mathematics with which liberal arts college freshmen are provided, and in the light of their varied objectives, what shall be the teachers' principal objectives?

6. Dean Buchanan referred to three ways of describing the teaching of mathematics in the St. John's program. From one view it was the reading of the original classic books in mathematics, from Euclid's *Elements* to Russell's *Principles of Mathematics*. Or it could be viewed as a study of the history or genesis of mathematical knowledge in the European mind. Finally (and this view includes the preceding two views), it could be seen as the transfiguration of a subject-matter into a set of habits or arts in the student; that is to say, a pattern of disciplines which traditionally has been called the quadrivium in the liberal arts, arithmetic, geometry, music, and astronomy. Dean Buchanan referred to mathematical symbols as the tools and to mathematical concepts as the formal medium for dealing with truths about nature, man, and God; to the reading of the great books in mathematics as the active use of mathematical symbols in imitation of the masters; and to the study of the historical development of mathematical knowledge with its transformation of symbols and its dialectical conflicts of concept as the force which brings the mathematical artist to an understanding of the underlying mathematical objects. Dean Buchanan then amplified the meanings of the quadrivium and showed their place in the St. John's program.

S. B. LITTAUER, *Secretary*

ON GROUPS OF SUBTRACTION AND DIVISION

E. J. FINAN, The Catholic University of America

1. Introduction. The group commonly called the Cross-Ratio Group* may be generated by successive application to the indeterminate x of the operators O_1 and O_2, where $O_1(x)=1/x$ and $O_2(x)=1-x$. The following six functions are obtained: x, $1/x$, $1/(1-x)$, $1-x$, $(x-1)/x$, and $x/(x-1)$.

If we replace O_1 and O_2 by the more general operators $R_1(x)=r/x$ and $R_2(x)=s-x$, respectively, where r and s are any complex numbers, it is conceivable other finite groups may be generated. These operators are of period two, and if their product is of finite period n, they generate the dihedral rotation group of order $2n$. We seek the conditions on r and s, so that the group generated by them may be of finite order.

This problem was studied by G. A. Miller [2] and several interesting results

* For example, see [1] M. Bôcher, Introduction to Higher Algebra, p. 106.

[2] G. A. Miller, Groups of subtraction and division, Quarterly Journal of Mathematics, 1906, pp. 80–87.

were obtained. It is the purpose of this paper to give some new theorems pertaining to the same problem. Miller proved that if R_1 and R_2 are to generate a finite group it is necessary that s^2/r satisfy a polynomial equation. Although some such equations were exhibited, no general rule was given for their formation. In Theorem 1 we shall give explicit formulas for two sets of equations whose roots are all possible values of s^2/r.

By means of these equations it is possible to obtain Miller's result [**2**, p. 84] that the only possible non-zero rational values of s^2/r are 1, 2, 3. In Theorem 2 other properties are obtained including the interesting fact that although both s and r may be any complex numbers, the ratio s^2/r must be real.

The approach is algebraic. However, the problem may be stated geometrically. For example, if r is real and positive then $R_1(\theta)$ is an inversion with respect to the circle $x^2+y^2=r^2$, followed by a reflection in the axis of reals; and $R_2(\theta)$ is a reflection in the point $s/2$.

2. Conditions on r and s. By successive application of the two operators R_1 and R_2, we obtain a rational linear function $A(\theta)=(\alpha\theta+\beta)/(\gamma\theta+\delta)$. If R_1 and R_2 are to generate a finite group, repeated application must produce θ. If $A(\theta)=\theta$ it is necessary and sufficient that $\gamma=\beta=0$ and $\alpha=\delta$. Let $A_1=\theta$; $A_2=R_1(\theta)=r/\theta$; $A_3=R_2[R_1(\theta)]=(s\theta-r)/\theta$; and in general $A_{i+2}=R_2[R_1(A_i)]$. Since we want to know what conditions we must impose on r and s to make $A_i(\theta)=\theta$ it will be necessary to calculate formulas for α_i, β_i, γ_i, and δ_i, where we have set

$$A_i(\theta) = (\alpha_i\theta + \beta_i)/(\gamma_i\theta + \delta_i).$$

Since for even values of i, A_i is of period 2 it is necessary to make the calculations for odd values of i only. This amounts to finding conditions on r and s so that the product R_2R_1 be of finite period. A characteristic property of this product is that it is transformed into its inverse by both R_1 and R_2. We shall now prove the following:

THEOREM 1. *If $A_i(\theta)=(\alpha_i\theta+\beta_i)/(\gamma_i\theta+\delta_i)$, then the following formulas give α_i, β_i, γ_i, and δ_i for all odd values of i:*

$$\alpha_{4k+3} = sr^k\left[x^k - \binom{2k}{1}x^{k-1} + \binom{2k-1}{2}x^{k-2} + \cdots + (-1)^k\binom{k+1}{k}\right],$$

$$\beta_{4k+3} = -r\gamma_{4k+3},$$

$$\gamma_{4k+3} = r^k\left[x^k - \binom{2k-1}{1}x^{k-1} + \binom{2k-2}{2}x^{k-2} + \cdots + (-1)^k\binom{k}{k}\right],$$

$$\delta_{4k+3} = -r^k s\left[x^{k-1} - \binom{2k-1}{1}x^{k-2} + \cdots + (-1)^k\binom{k}{k-1}\right],$$

$$\alpha_{4k+1} = r^k\left[x^k - \binom{2k-1}{1}x^{k-1} - \binom{2k-2}{2}x^{k-2} + \cdots + (-1)^k\binom{k}{k}\right],$$

$$\beta_{4k+1} = -r\gamma_{4k+1},$$

$$\gamma_{4k+1} = r^{k-1}s\left[x^{k-1} - \binom{2k-2}{1}x^{k-2} + \cdots + (-1)^{k-1}\binom{k}{k-1}\right],$$

$$\delta_{4k+1} = -r^{k}\left[x^{k-1} - \binom{2k-3}{1}x^{k-2} + \cdots + (-1)^{k-1}\binom{k-1}{k-1}\right],$$

where $x = s^2/r$.

Proof. We shall first prove the formula for α_{4k+3} by induction. For $k=1$ the formula is easily verified. From the definition of $A_i(\theta)$ we have

$$A_{i+2}(\theta) = R_2[R_1(A_i)] = [(s\alpha_i - r\gamma_i)\theta + (s\beta_i - r\delta_i)]/(\alpha_i\theta + \beta_i).$$

Applying this formula again with i replaced by $i+2$, we obtain

$$A_{i+4}(\theta) = \frac{[s^2\alpha_i - (sr\gamma_i + r\alpha_i)]\theta + s^2\beta_i - (sr\delta_i + r\beta_i)}{(s\alpha_i - r\gamma_i)\theta + (s\beta_i - r\delta_i)}. \tag{1}$$

If $i=4k+3$ the coefficient of θ in the numerator is α_{4k+7}. If by using the formula for $\alpha_{4(k+1)+3}$, we obtain the same result, the induction is complete. To prove this write, using $x=s^2/r$,

$$s^2\alpha_i = sr^{k+1}\left[x^{k+1} - \binom{2k}{1}x^k + \cdots + (-1)^{i+1}\binom{2k-i}{i+1}x^{k-i} + \cdots + (-1)^k\binom{k+1}{k}x\right],$$

$$-sr\gamma_i = sr^{k+1}\left[-x^k + \cdots + (-1)^{i+1}\binom{2k-i}{i}x^{k-i} + \cdots + (-1)^k\binom{k}{k}\right],$$

$$-r\alpha_i = sr^{k+1}\left[-x^k + \cdots + (-1)^{i+1}\binom{2k-i+1}{i}x^{k-i} + \cdots + (-1)^k\binom{k+1}{k}\right].$$

The sum of the left members with $i=4k+3$ is α_{4k+7}, by (1). Adding the right members and using the identity

$$\binom{2k-i}{i+1} + \binom{2k-i}{i} + \binom{2k-i+1}{i} = \binom{2k+2-i}{i+1},$$

we obtain the value of $\alpha_{4(k+1)+3}$ as given in the statement of the theorem. This completes the proof of the first equation in the theorem.

The proofs of the other seven equations are similar to the above. We may use either the first four or the last four equations in Theorem 1 and replace the other

four by the following relations. If $i=4k+1$ then $\delta_i=-r\gamma_{i-2}$, $\gamma_i=\alpha_{i-2}$, and $\beta_i=-r\gamma_i$. For both $i=4k+1$ and $i=4k+3$ we have $\alpha_i-\delta_i=s\gamma_i$.

We are now in a position to decide under what conditions $A_i(\theta)$ reduces to θ. It is evident that a necessary condition is that γ_i shall be zero. Also, this condition is sufficient since if $\gamma_i=0$ it follows at once from the above formulas for α_i, β_i, and δ_i, that $\beta_i=0$ and $\alpha_i=\delta_i+s\gamma_i=\delta_i$. Hence $A_i(\theta)=\theta$ if and only if $\gamma_i=0$. Using the above formula for γ_i, and using $rs\neq 0$, we have the following:

COROLLARY. *A necessary and sufficient condition that the operators, $R_1(x)=r/x$ and $R_2(x)=s-x$, where $rs\neq 0$,* generate a finite group is that s^2/r be a root of one of the two equations*:

$$x^k-\binom{2k-1}{1}x^{k-1}+\binom{2k-2}{2}x^{k-2}+\cdots+(-1)^k\binom{k}{k}=0, \tag{2}$$

$$x^{k-1}-\binom{2k-2}{1}x^{k-2}+\cdots+(-1)^{k-1}\binom{k}{k-1}=0. \tag{3}$$

The results are essentially the same if the operators are applied in the opposite order, beginning $A_1=\theta$, $A_2=R_2(\theta)=s-\theta$, $A_3=R_1[R_2(\theta)]$.

3. Nature of r and s. From the above discussion it is evident that r and s may be any complex numbers as long as the ratio s^2/r satisfies either equation in the corollary. We may show that the only possible rational roots of equations (2) and (3) are 1, 2, and 3. Since Miller [**2**, p. 84] obtained this result by another method, the proof will be omitted. However, we can obtain considerable additional information regarding the roots of equations (2) and (3). Let $f_i(x)=0$, of degree i, $(i=1, 2, 3 \cdots)$, stand for either set of equations (2) or (3). Then by substitution we can verify the identity

$$f_{i+1}(x)=(x-2)f_i(x)-f_{i-1}(x). \tag{4}$$

We shall now prove the following:

THEOREM 2. *Let $f_i(x)=0$, of degree i, $(i=1, 2, 3 \cdots)$, be either set of equations* (2) *or* (3). *Then, for any value of k,*

(a) *all the roots of $f_k(x)=0$ are real and distinct*;

(b) *the roots of $f_{k-1}(x)=0$ separate those of $f_k(x)=0$*;

(c) *no root of $f_k(x)=0$ equals a root of $f_{k-1}(x)=0$*;

(d) *if the roots of $f_i(x)=0$ are $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ii}$ in order of magnitude, then* $f_k(\alpha_{k-1,k-1})<0$.

Proof. The above properties hold for all of the equations of degree less than 3. We shall assume that they hold for all equations of degree less than or equal to i and prove that they hold for all those of degree less than or equal to $i+1$. To prove (d), substitute α_{ii} for x in (4). We get $f_{i+1}(\alpha_{ii})=-f_{i-1}(\alpha_{ii})$, since α_{ii} is

* If $s=0$ and $r\neq 0$ we obtain a group of order 4. See [**2**, p. 80].

a root of the equation $f_i(x)=0$. But since, by (b), α_{ii} is greater than the largest root of the equation $f_{i-1}(x)=0$, it follows that $f_{i-1}(\alpha_{ii})>0$, and (d) is proved. To prove (a), (b), and (c) we shall show that if α_{is} and $\alpha_{i,s+1}$ are any two consecutive roots of $f_i(x)=0$, then there is a root γ of the equation $f_{i+1}(x)=0$ which lies between them. By substituting in (4) we get the equations

$$f_{i+1}(\alpha_{is}) = -f_{i-1}(\alpha_{is}), \qquad f_{i+1}(\alpha_{i,s+1}) = -f_{i-1}(\alpha_{i,s+1}).$$

But by assumption, since the roots of the equation $f_{i-1}(x)=0$ separate those of the equation $f_i(x)=0$, it follows that $f_{i-1}(\alpha_{is})$ and $f_{i-1}(\alpha_{i,s+1})$ are opposite in sign and that there is a root γ of the equation $f_{i+1}(x)=0$ between any two consecutive roots of the equation $f_i(x)=0$. Since the degree of the polynomial $f_{i+1}(x)$ is one more than that of $f_i(x)$, this accounts for all the roots of the equation $f_{i+1}(x)=0$ except two. By (d) there is one root of the equation $f_{i+1}(x)=0$ greater than the largest root of the equation $f_i(x)=0$. There is one more root of the equation $f_{i+1}(x)=0$ to account for. Since the coefficients of the polynomial $f_{i+1}(x)$ are real it is evident from the above discussion that this root must be less than any root of the equation $f_i(x)=0$. This completes the proof of (a), (b), and (c).

4. Groups generated by R_1 and R_2. Miller [2, p. 86] proves that there exist values of r and s such that R_1 and R_2 will generate any given dihedral rotation group. This is equivalent to showing that in the equations (2) and (3), any equation of degree k contains at least one root not found in any equation in either set of lower degree. Since R_1 and R_2 are of period 2, no other groups arise from these operators. One definition [3] of a dihedral group is a group of order $2n$ formed by extending a cyclic group of order n by an element of period 2 which transforms each element of the cyclic group into its inverse. In the above discussion the cyclic group is the set of distinct powers of R_2R_1 (or R_1R_2), and either R_1 or R_2 is the element of period 2. For example, $R_1^{-1}R_2R_1R_1=R_1^{-1}R_2=R_1^{-1}R_2^{-1}=(R_2R_1)^{-1}$. Hence to generate a dihedral group of order $4k+2$ we assign values to r and s so that s^2/r is a root of an equation of type (2) of degree k and not a root of one of either type of lower degree. A similar remark holds for groups of order $4k$ with the exception that we use an equation of type (3) of degree $k-1$. The number of distinct values of s^2/r is not evident from the order of the group. For example, to generate the dihedral rotation group of order 22 let s^2/r be any such root of the equation

$$x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1 = 0.$$

All 5 roots satisfy the above conditions. But to generate a similar group of order 24 the number s^2/r must satisfy the equation

$$x^5 - 10x^4 + 36x^3 - 56x^2 + 35x - 6 = 0.$$

There are only 2 such roots to this equation since 1, 2, and 3 all satisfy equations of lower degree.

[3] L. C. Mathewson, Elementary Finite Groups, p. 96.

SOME HYDRODYNAMICAL PROBLEMS RELATED TO BALLISTICS*

R. H. KENT, Aberdeen Proving Ground

1. Introduction. The subject of ballistics is naturally divided into two parts, interior ballistics, concerned with the motion of a projectile inside a gun, and exterior ballistics, concerned with its motion outside the gun. That part of the theory of exterior ballistics underlying the computation of trajectories was studied during the World War, and the computations have recently been much facilitated by the use of the Bush differential analyzer. While not complete, it is in a relatively satisfactory state. We may well, therefore, turn our attention to other phases of ballistics, particularly to the application of hydrodynamics to ballistics. This deals with such problems as the motion of the powder charge, the motion of air about the shell, the motion of the gas from a detonated bomb, and the motion of the surrounding air. In this paper we shall describe briefly some problems which have been solved or partly solved and some unsolved problems of importance. The fundamental theoretical work is that of Riemann [**I, 3**] on plane waves of finite amplitude and of Rankine [**I, 4**] on plane waves of permanent type. Lamb's *Hydrodynamics* [**I, 8**] may be consulted for an introductory exposition of this work.

2. Lagrange's ballistic problem. In the time of Lagrange, quick-burning black powder was used, as opposed to the slow-burning smokeless powders of today. As a result, the powder was completely burned, developing its maximum

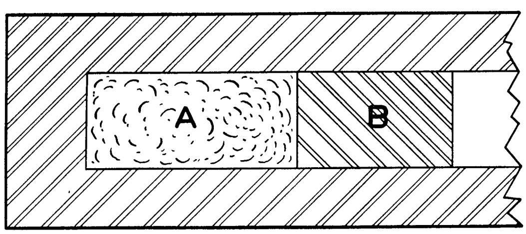


FIG. 1

pressure before the projectile moved very far inside the gun. An idealization of these conditions leads to the following problem. Figure 1 shows a section of a cylinder in which A is a chamber filled with gas at rest and at uniform pressure, and B is a projectile held at rest. At time $t=0$ the projectile is released. What will be the subsequent motion of the projectile and the gas? This problem was proposed by Lagrange, who contributed little to the solution. His work was published in a paper by Poisson [**I, 1**]. Important contributions to its solution were made by Hugoniot [**I, 5**], Gossot and Liouville [**II, 4**], and Love and Pidduck [**II, 5**]. The nature of the solution can be understood from the descrip-

* This paper is a slightly revised and augmented version of a manuscript kindly prepared by Dr. A. E. Pitcher after listening to my extemporaneous talk at the Hanover meeting of the Mathematical Association of America, September 9, 1940.

tion of affairs in Figure 2 [from Love and Pidduck, II, **5**]. Here the gas pressure is plotted as a function of the distance from the breech of the gun at various times. Along the broken lines time is constant. The position of the base of the shell is given by the curved line. As the projectile starts to move a pressure wave starts backward. At $t = .0004772$ it is roughly halfway back to the breech. At $t = .0009544$ it reaches the breech and is reflected forward. At $t = .002117$ it has reached the projectile where it is reflected backward. Alternate reflections continue until the projectile leaves the gun, which took place in this case at about the time of the second reflection from the shell.

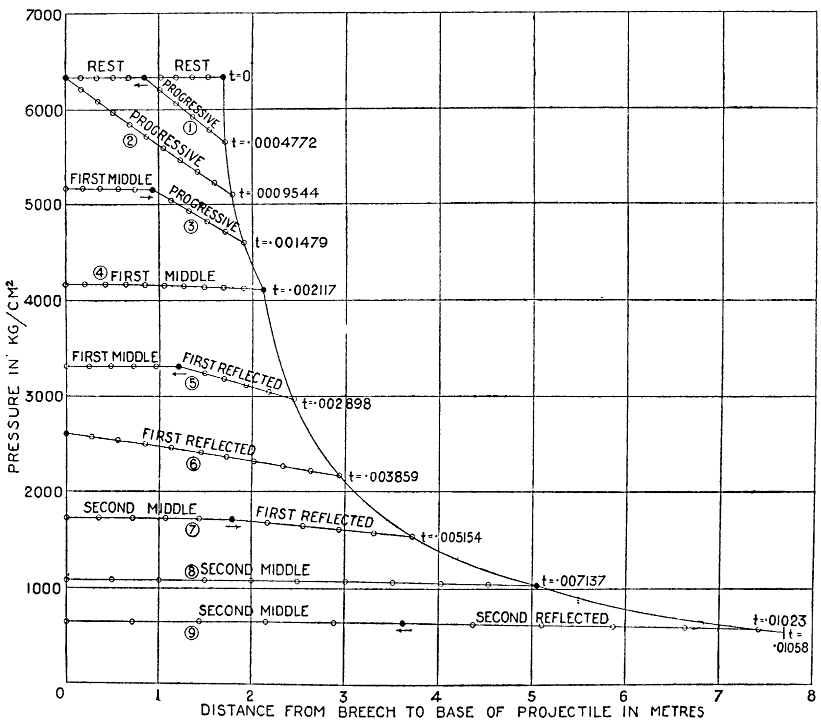


FIG. 2

In the same paper Pidduck gives a non-oscillatory special solution for the equation of the motion of the charge, which has turned out to be of considerable practical value in estimating the ratio of the pressure on the projectile to the pressure on the breech.

The actual situation differs from the idealization of the problem in that the projectile is not released suddenly as assumed. With slower burning powders and suitable distribution of the charge, the pressure is built up more gradually and a smooth pressure-time curve is obtained. With other distributions of the charge, however, pressure waves of considerable amplitude are imposed on the smooth pressure-time curve, and in some instances waves with exceedingly steep fronts, known as "shock waves," are obtained.

3. Hydrodynamical problems in exterior ballistics. The first computation of the pressure on a moving projectile was made by Lord Rayleigh [I, 6] for the case of a projectile in the shape of a right circular cylinder. He computed the pressure at the stagnation point at the center of the leading end. A solution based on Bernoulli's equation yielded too large an answer. Assuming that a "shock wave"* (which may be conveniently regarded as a discontinuity in pressure) preceded the shell, and following the variation in pressure from the wave to the shell, he was able to obtain a result in reasonable agreement with the experimental data at hand.

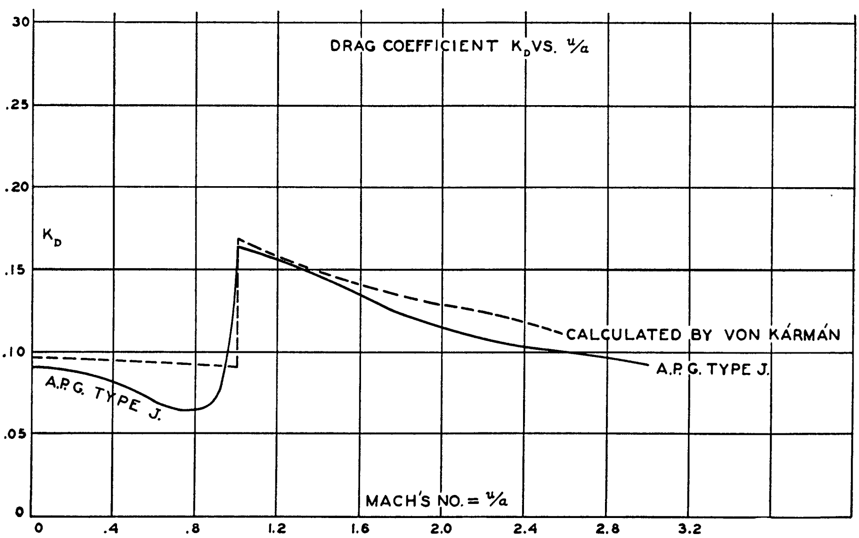


FIG. 3

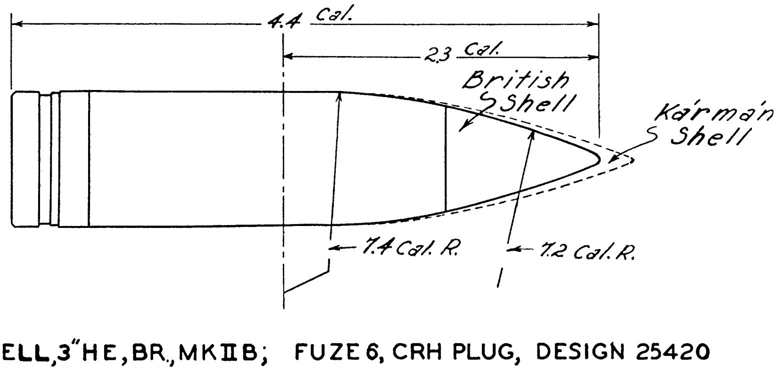
von Kármán and Moore [III, 4] attempted to compute velocity and density distributions in the air surrounding the slender head of a projectile moving at supersonic speeds. They neglected second order terms and obtained a solution to the problem without recourse to a shock wave.† Their results have been compared with experimental data obtained at the Aberdeen Proving Ground (Fig. 3) and in England (Fig. 4).‡ The agreement with experimental data is not too good. The figures show fortunate and unfortunate comparisons. In these figures, the abscissa is Mach's Number, the ratio of the velocity of the projectile, u,

* The pair of lines diverging from the nose of the shell in Figure 5 is the silhouette of the shock wave. It shows in the photograph because of the refraction of light due to the difference in density of the air on the two sides of the wave front.

† For a linear equation it is possible to have boundary conditions at infinity satisfied by a continuous solution. This may not be possible for the equation of higher degree.

‡ The experimental drag coefficient shown in Figure 4 is taken from an article by Fowler, Gallup, Lock, and Richmond, Phil. Trans. A, vol. 221, 1920, p. 295.

to the normal velocity of sound, a. The ordinate is the drag coefficient.* The figures show experimental and computed values for each of two shell shapes. The experimental results in Figure 3 are for a shell having a sharper point than that of von Kármán, while the shapes of the experimental shell and von Kármán's shell are shown in Figure 4.



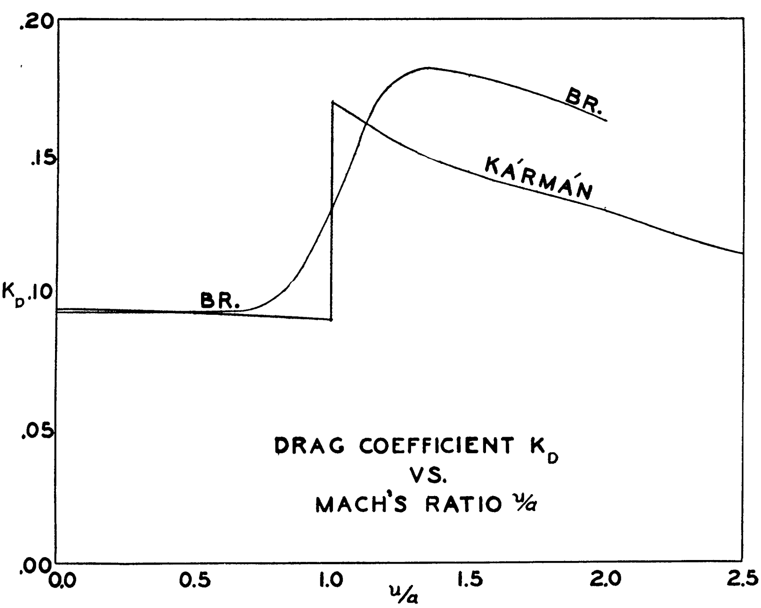


FIG. 4

Taylor and Maccoll [III, 5] carried out an exact solution of the corresponding problem for the special case of the infinite conical projectile assuming uniform distribution of density, pressure, and velocity over any coaxial cone

* The drag is the component of force parallel to the trajectory. The drag coefficient is the drag divided by ρd^2u^2, in which ρ is the air density, d is the diameter of the projectile, and u is the air speed of the projectile.

with vertex at the tip of the shell. They found a solution with a conical shock wave when the shell velocity was sufficiently high, but found such impossible when the velocity was low. In Figures 5 and 6 are reproduced photographs from Maccoll's paper [III, 7] showing that this is indeed the case. In his recent paper Maccoll [III, 7] also compares the computed and observed angles of the conical shock wave. The remarkably good agreement gives one confidence in the adequateness of the theory.

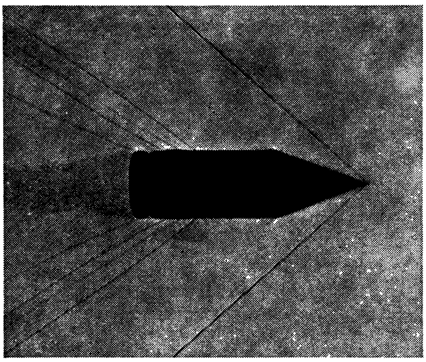


FIG. 5. Conical shock wave at $u/a=1.794$.

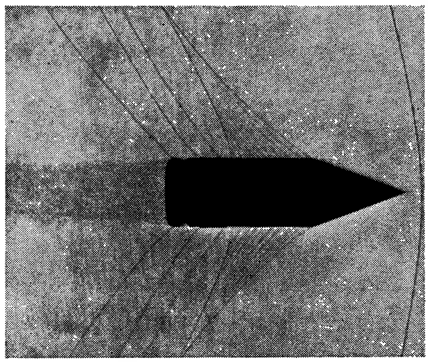


FIG. 6. Shock wave not conical at $u/a=1.090$.

4. Some unsolved problems. We have considered some solved and partly solved problems in the preceding sections. We shall now turn our attention to some problems which need to be solved.

In the problem initiated by von Kármán and described in the preceding section, further development of methods of obtaining solutions for a shell of an arbitrarily shaped nose is needed. If such solutions can be obtained, they will be of great importance in the future design of projectiles.

The theory of *plane* waves of finite amplitude developed by Riemann, Rankine and others is now in a satisfactory state. However, practically no sound work has been done on the theory of *spherical* waves of finite amplitude. Such a theory must be developed if we are to understand the phenomena accompanying the detonation of a bomb in the air or of a mine in the sea. The differential equations for the continuous spherical wave are the following with Lagrangian coördinates:

$$\rho \frac{\partial^2 r}{\partial t^2} = -\frac{\partial p}{\partial r}, \qquad \text{(equation of motion)}, \tag{1}$$

$$\frac{\partial r}{\partial r_0} = \frac{\rho_0 r_0^2}{\rho r^2}, \qquad \text{(equation of continuity)}, \tag{2}$$

$$p = f(\rho), \qquad \text{(equation of state)}. \tag{3}$$

In these equations,

r_0 = distance of element of gas from origin at $t=0$,
r = distance of element of gas from origin at time, t,
ρ_0 = density of gas at $t=0$, in general, function of r_0,
ρ = density of gas at time, t,
p = pressure at time, t.

If the gas is assumed to be perfect and the expansion adiabatic so that $p=K\rho^\gamma$, where K is a constant and γ is the ratio of the specific heats, a single equation,

$$\frac{\partial^2 r}{\partial t^2} = \frac{\gamma K \rho_0^{\gamma-1}}{\left(\dfrac{\partial r}{\partial r_0}\right)^{\gamma+1}} \cdot \frac{r_0^{2\gamma-2}}{r^{2\gamma-2}} \left[\frac{\partial^2 r}{\partial r_0^2} + \frac{2}{r}\left(\frac{\partial r}{\partial r_0}\right)^2 - \left(\frac{2}{r_0} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right)\frac{\partial r}{\partial r_0}\right] \tag{4}$$

is obtained from (1), (2), and (3).

A special solution of (4) by the method of the separation of the variables is readily obtained by taking

$$r = r_0 \phi(t).$$

It is easily shown that $\phi(t)$ and ρ_0 are given by

$$\phi''(t) = B/[\phi(t)]^{3\gamma-2}, \quad B r_0 = -\gamma K \rho_0^{\gamma-2} \frac{\partial \rho_0}{\partial r_0},$$

where B is a constant.

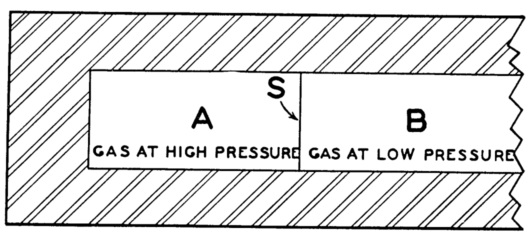


FIG. 7

The problem of obtaining a general solution of equations (1), (2), and (3) or even of (4) appears so formidable that it seems expedient that an attack should first be made on an easier problem proposed by Hugoniot [I, **5**]. Figure 7 represents an infinite cylinder closed at one end. S is a thin partition. Chambers A and B are filled with different gases at rest, the pressure in A far exceeding that in B. What is the motion of the gases after the partition is suddenly removed?* The solution of this problem should afford some insight into that of

* John von Neumann has pointed out that the solution of a similar but simpler problem has been given in Riemann-Weber's Die Partiellen Differential-Gleichungen der Math. Phys., fourth edition. This solution needs a slight modification to make it consistent with the Rankine-Hugoniot shock wave conditions.

a similar but more difficult problem concerning the detonation of a spherical shell with gas at high pressure inside and air at low pressure outside.

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A NOTE ON PARTITIONS OF THE SET OF POSITIVE INTEGERS

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In a recent note* the author studied the question of when the partition of the set of positive integers generated by the relation of congruence modulo an integer is homomorphic with respect to exponentiation. It has been pointed out† that there are partitions homomorphic with respect to addition and multiplication other than those generated by the relation of congruence. In this note we first classify the partitions homomorphic with respect to addition, and note that they are also homomorphic with respect to multiplication. We then give a necessary and sufficient condition that such a partition be homomorphic with respect to exponentiation. Proofs are given only when they differ from those in our earlier note.

THEOREM 1. *Any partition of the set of positive integers homomorphic with respect to addition has the following form. Each of the integers* $1, 2, \cdots, i-1$ *is the only member of its class. The integers greater than* $i-1$ *are divided into classes by the relation of congruence modulo an integer* m.

Proof. Suppose that $1, 2, \cdots, i-1$ are alone in their respective classes, while m is the smallest positive integer such that $i+m$ is in the same class as i. Then $i+km$ is in the same class, for any $k \geqq 0$. Also $i+j+km$ is in the same class as $i+j$, for any $j \geqq 0$ and $k \geqq 0$. Next, the class containing i consists only of the integers of the form $i+km$. For suppose that $i+km+m'$, with $0<m'<m$, is in it. By hypothesis $i+m'$ is not in the same class as i. And $i+km+m'$ is in the same class as $i+m'$. Thus we have reached a contradiction.

Define m_j to be the smallest positive integer such that $i+j+m_j$ is in the same class as $i+j$. This is defined for all $j \geqq 0$. It has been shown above, then, that $m_j \leqq m_{j-1}$. But it has also been shown that $m_m = m$. Hence

$$m = m_1 = m_2 = \cdots .$$

This completes the proof of Theorem 1.

DEFINITION. *A partition of the positive integers such as that of Theorem* 1 *will be called a partition of type* (i, m).

THEOREM 2. *A partition of type* (i, m) *is homomorphic with respect to multiplication.*

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* A theorem on partitions of the set of positive integers, this MONTHLY, vol. 47, 1940, pp. 152–154.

† The first published reference to this fact, so far as we know, is due to Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bulletin of the American Mathematical Society, vol. 40, 1934, pp. 914–920. Also, On some simple types of semirings, this MONTHLY, vol. 46, 1939, pp. 22–26. Vandiver did not show that his examples are the only type of partition homomorphic with respect to addition, but we have learned since proving Theorem 1 that this was shown by Oxtoby, who did not publish the result.

Proof. Note that

$$n(i + j + km) = n(i + j) + nkm,$$
$$n(i + j + pm) = n(i + j) + npm.$$

The cases in which we have two integers which are alone in their respective classes, or two in classes with other members, are trivial.

We consider next the question of when a partition of type (i, m) is homomorphic with respect to exponentiation. For a given i we wish to find a necessary and sufficient condition on m so that

$$(k_1 + n_1 m)^{k_2+n_2 m} \equiv k_1^{k_2} \pmod{m} \tag{1}$$

for $i \leqq k_1 < i+m$, $i \leqq k_2 < i+m$, $n_1 \geqq 0$, $n_2 \geqq 0$, and

$$k_1^{k_2+n_2 m} \equiv k_1^{k_2} \pmod{m} \tag{2}$$

for $0 < k_1 < i$, k_2 and n_2 as above. The other conditions are satisfied automatically. Expanding the left side of (1) in a binomial expansion, (1) and (2) can be combined to give the condition

$$k_1^{k_2+n_2 m} \equiv k_1^{k_2} \pmod{m}, \tag{3}$$

or

$$k_1^{k_2}(k_1^{n_2 m} - 1) \equiv 0 \pmod{m} \tag{4}$$

for $0 < k_1 < i+m$, $i \leqq k_2 < i+m$, $n_2 \geqq 0$. With these statements of the problem given we proceed to the necessary and sufficient condition.

LEMMA 2.1. *A partition of type (i, m) such that $p^e \mid m$, p a prime and $e > i$, is not homomorphic with respect to exponentiation.*

Proof. In (4) let $k_1 = p$, $k_2 = i$, $n_2 = 1$. Then

$$p^i(p^m - 1) \not\equiv 0 \pmod{m},$$

since the left member is not divisible by p^e.

LEMMA 2.2. *A partition of type (i, m) such that $p \mid m$ but $(p-1) \nmid m$, p a prime, is not homomorphic with respect to exponentiation.*

The proof is exactly like a similar proof in our earlier paper.

THEOREM 3. *A necessary and sufficient condition that a partition of type (i, m) be homomorphic with respect to exponentiation is that if $p \mid m$ then $(p-1) \mid m$, and if $p^e \mid m$ then $e \leqq i$, where p is a prime.*

Proof. The necessity is given by Lemmas 2.1 and 2.2. To prove the sufficiency assume that i and m satisfy the conditions of the hypothesis. Suppose that $p^e \mid m$ and $p \mid k_1$. Then by hypothesis $p^e \mid k_1^{k_2}$ for permissible values of k_1. Next

suppose that $p^e | m$ while $p \nmid k_1$. Now $\phi(p^e) = p^{e-1}(p-1)$. Hence by Euler's generalization of Fermat's theorem,

$$k_1^{n_2 m} - 1 \equiv 0 \pmod{p^e},$$

which completes the proof.

For partitions of type $(1, m)$, we showed in our earlier paper that there are only five values of m such that the partition is homomorphic with respect to exponentiation. We have not been able to find out how many there are for arbitrary i, or even for $i=2$. An examination of factor tables shows that in the latter case there are several hundred below ten thousand. Nevertheless we would conjecture that there are only a finite number for any value of i.

ON "ALMOST PERFECT" NUMBERS

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Although the results of this paper can be derived from a theorem of Davenport,* our point of view is different from Davenport's and the methods used here are different and considerably more elementary than those he used.

A perfect number is an integer N such that $F(N) = S(N)/N = 2$, where $S(N)$ is the sum of all of the divisors of N. A multiply perfect number of multiplicity r is an integer N such that $F(N) = r$, where r is an integer greater than 2.

For any integer N, it is true that $F(N)$ is rational, but it is not known whether there is an odd N greater than 1 for which $F(N)$ is an integer.

We define an "almost perfect" number with respect to an arbitrarily small but fixed $\epsilon > 0$ as an integer N such that $F(N)$ differs from 2 by an amount less than ϵ. In this paper we show that, for each ϵ, there are infinitely many "almost perfect" numbers. In fact we show that, for any real number $A > 1$, there are infinitely many integers N such that $F(N)$ differs from A by an amount less than ϵ.

Let

$$P = \prod_{i=1}^{\infty} \frac{p_i}{p_i - 1} = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i - 1}\right),$$

where p_i is the ith prime. It is known† that

$$\lim_{n\to\infty} \prod_{i=1}^{i=n} \frac{p_i}{p_i - 1} = +\infty.$$

Moreover, we notice that $p_i/(p_i - 1)$ decreases with increasing p_i and that $\lim_{n\to\infty} p_n/(p_n - 1) = 1$.

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* Sitzungsberichte der Preussischen Akademie, Physikalisch-Mathematische Klasse, 1933, pp. 830–837.

† Landau, Vorlesungen über Zahlentheorie, vol. 1, pp. 69–70.

We now establish the following:

THEOREM I. *For any real number $A>1$ and any $\epsilon>0$, there exist primes $q_1<q_2<q_3 \cdots <q_n$ such that $0<A-Q_n<\epsilon$, where*

$$Q_n = \prod_{i=1}^{i=n} \frac{q_i}{q_i - 1}.$$

Proof. Clearly there is in the sequence of all primes a first one, p_m, such that $p_m/(p_m-1)<A$. We denote p_m by q_1. We could equally well use any $p_i>p_m$ for our q_1 if we so desired. We have already mentioned that the product P diverges. Hence, if we let $q_1=p_m$, $q_2=p_{m+1}$, $q_3=p_{m+2}$ and continued this indefinitely, we would eventually cause Q_i to exceed A for some i. But we continue this rule for choice of the q_i's only as long as the corresponding Q_i's are less than A. Let $q_j=p_{m+j-1}$ be the last q which is chosen in this way. Then $Q_j<A$, but

$$Q_j\left(1+\frac{1}{p_{m+j}-1}\right) \geqq A.$$

We use another method for the choice of all q_i's for which $i>j$.

Let us choose b_j so that

$$Q_j\left(1+\frac{1}{b_j}\right) = A.$$

Since

$$Q_j\left(1+\frac{1}{p_{m+j}-1}\right) \geqq A,$$

it follows that $b_j \geqq p_{m+j}-1$. Notice that b_j is not necessarily integral.

Obviously, if $A>3/2$, we can use 3 for our q_1 and $p_{m+j}>3$, while, if $A\leqq 3/2$, we shall have $q_1>3$ and again $p_{m+j}>3$. Then $b_j \geqq 4$ in any case, since p_{m+j} is a prime greater than 3.

By Bertrand's postulate,* there is a prime between x and $2x-2$ for any $x>3\frac{1}{2}$. Hence there is a prime q_{j+1} such that

$$b_j+1<q_{j+1}<2b_j, \qquad b_j<q_{j+1}-1<2b_j-1<2b_j.$$

From this it is clear that

$$Q_j\left(1+\frac{1}{2b_j}\right)<Q_j\left(1+\frac{1}{q_{j+1}-1}\right)<Q_j\left(1+\frac{1}{b_j}\right), \quad Q_j+\tfrac{1}{2}Q_j\left(\frac{1}{b_j}\right)<Q_{j+1}<A.$$

We need to be sure that q_{j+1} is greater than q_j, but this is clear if we notice that $q_j=p_{m+j-1}<p_{m+j}$, while

* Serret, Cours d'Algèbre Supérieure, fifth edition, vol. II, pages 226–239. Notice that the number x need not be an integer and that the prime number is $>x$ and not merely $\geqq x$.

$$Q_j\left(1+\frac{1}{p_{m+j}-1}\right) \geqq A, \qquad Q_j\left(1+\frac{1}{q_{j+1}-1}\right) < A.$$

Since $A-Q_j=Q_j(1/b_j)$, we see that Q_{j+1} is nearer to A than it is to Q_j and so $\frac{1}{2}(A-Q_j)>A-Q_{j+1}$. If we let $Q_{j+1}(1+1/b_{j+1})=A$, we see that

$$Q_{j+1}\left(\frac{1}{b_{j+1}}\right) < \tfrac{1}{2}Q_j\left(\frac{1}{b_j}\right), \qquad \frac{1}{b_{j+1}} < \tfrac{1}{2}\left(\frac{1}{b_j}\right)\frac{Q_j}{Q_{j+1}}.$$

Then $b_{j+1}>2b_jQ_{j+1}/Q_j>2b_j$ since $Q_{j+1}>Q_j$. Thus, when we choose our q_{j+2} so that $2b_j<b_{j+1}+1<q_{j+2}<2b_{j+1}$, it will be a larger prime than the q_{j+1} last chosen.

By exactly the same process as used above, we find that, if the q_i's are chosen by the method described, then for any $i\geqq j$ we shall have $Q_i+\frac{1}{2}Q_i(1/b_i)<Q_{i+1}<A$ and $A-Q_{i+1}<\frac{1}{2}(A-Q_i)$. Thus

$$0 < A - Q_{j+1} < \tfrac{1}{2}(A - Q_j), \qquad 0 < A - Q_{j+2} < \tfrac{1}{2}(A - Q_{j+1}) < (1/2^2)(A - Q_j).$$

For every integer $t>0$, we can show by mathematical induction that $0<A-Q_{j+t}<(1/2^t)(A-Q_j)$, and hence, for sufficiently large t, we shall have $0<A-Q_{j+t}<\epsilon$. This completes the proof of Theorem I. We notice that the method of proof used here will allow us to continue choosing successive q_i's indefinitely, even after we have chosen enough of them to satisfy the inequality $0<A-Q_n<\epsilon$ for the given ϵ.

To prove the final result, we need another auxiliary theorem which can be stated as follows:

THEOREM II. *For any set of n primes $q_1<q_2<\cdots<q_n$ and any preassigned $\epsilon>0$, there exist integral exponents $m_1, m_2, \cdots, m_n$ such that $Q_n-\epsilon<F(q_1^{n_1}q_2^{n_2}\cdots q_n^{n_n})<Q_n$ whenever $n_i>m_i$.*

Proof. We define $F(q^x)$ for all positive real x by means of the equation

$$F(q^x) = \frac{q-q^{-x}}{q-1} = \frac{q^{x+1}-1}{q^x(q-1)}\cdot$$

Differentiating $F(q^x)$, we obtain

$$F'(q^x) = \frac{\log q}{(q-1)q^x}$$

Since $q\geqq 2$, it is clear that F increases with x. We notice also that F is always greater than 0 and that $\lim_{x\to\infty} F(q^x)=q/(q-1)$. Then for any $\epsilon>0$ there is an m such that $0<q/(q-1)-F(q^n)<\epsilon$ whenever $n>m$. It follows that, for any two primes q_1 and q_2 and any preassigned $\epsilon_1>0$ and $\epsilon_2>0$, there are integers m_1 and m_2 such that for all $n_1>m_1$ and $n_2>m_2$ the following inequalities hold:

$$\frac{q_i}{q_i-1}-\epsilon_i<F(q_i^{n_i})<\frac{q_i}{q_i-1}, \qquad (i=1, 2).$$

Obviously $F(N_1N_2)=F(N_1)F(N_2)$ whenever $(N_1, N_2)=1$. Hence

$$Q_2 - \{\epsilon_1 q_2/(q_2 - 1) + \epsilon_2 Q_1 - \epsilon_1\epsilon_2\} < F(q_1^{n_1} q_2^{n_2}) < Q_2.$$

If for any preassigned $\epsilon > 0$, we use $\epsilon_1 < \epsilon(q_2 - 1)/2q_2$ and $\epsilon_2 < \epsilon/2Q_1$, we find that $Q_2 - \epsilon < F(q_1^{n_1} q_2^{n_2}) < Q_2$. We extend this to our complete set of n primes by use of mathematical induction.

It is now easy to prove the final theorem.

THEOREM III. *There exist infinitely many odd integers N such that $F(N)$ differs from A by an amount less than $\epsilon > 0$, where ϵ is arbitrarily small but fixed and A is any real constant > 1.*

Proof. To prove this, we first notice that Theorem I makes it possible to choose n odd primes so that Q_n will differ from A by an amount less than $\epsilon/2$. Theorem II makes it possible to choose the exponents of these primes in infinitely many ways so that $F(q_1^{n_1} q_2^{n_2} \cdots q_n^{n_n})$ will differ from Q_n by an amount less than $\epsilon/2$.

We can readily prove a theorem similar to Theorem III for even N provided $A > 2$. We modify the proof of Theorem I in this case by letting $q_1 = 2$ and $q_2 = p_{m+1}$, where now p_{m+1} is the first odd prime such that $2(p_{m+1})/(p_{m+1} - 1) < A$. The rest of the proof remains the same as before.

The special case $A = 2$ of Theorem III proves the existence of infinitely many odd "almost perfect" numbers. Our process will not, however, yield any odd perfect numbers since, for every n, we have $A > Q_n > F(q_1^{n_1} q_2^{n_2} \cdots q_n^{n_n})$.

SETS OF POSTULATES FOR BOOLEAN RINGS

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1. Introduction. Stone has defined Boolean rings as rings in which every element is idempotent. He has shown that every Boolean ring is equivalent to a "generalized Boolean algebra" and that every Boolean ring with unit element is equivalent to an ordinary Boolean algebra.*

It is the purpose of this paper to examine various possible sets of postulates for Boolean rings, or generalized Boolean algebras, and to establish the independence of one such set (Set III) obtained as a weaker form of Stone's postulates for Boolean rings.

2. Stone's postulate sets. In view of Stone's definition, the original postulates which he used for Boolean rings may be listed in detail as follows, under Set I. The usual restrictions that the elements and their combinations belong to the system are to be understood in Postulates 2–8. We shall follow Stone's

* M. H. Stone, Theory of representations for Boolean algebras, Transactions of the American Mathematical Society, vol. 40, 1936, pp. 37–111, especially pp. 39 to 48. This paper will be referred to as [T].

For generalized Boolean algebras, see Stone, Postulates for Boolean algebras and generalized Boolean algebras, American Journal of Mathematics, vol. 57, 1935, pp. 703–732. To be referred to as [J].

notation using "$+$" for ring addition and "$\vee$" for Boolean addition.

SET I

System $(K, +, \times)$.

1. *If* a, b *are in* K, *then* (i) $a+b$, *and* (ii) ab *are in* K.
2. $(a+b)+c=a+(b+c)$
3. $a+b=b+a$
4. *For every* a, b *in* K *there exists a solution in* K *of the equation* $x+a=b$.
5. $(ab)c=a(bc)$
6. $a(b+c)=ab+ac$
7. $(a+b)c=ac+bc$
8. $aa=a$

Here Postulates 1–7 are, of course, standard postulates for any ring. Thus the addition postulates, *by themselves*, imply that the elements of K form an additive abelian group. With the aid of Postulate 8, the idempotent law, special theorems for Boolean rings are easily deduced. For example, every element is its own inverse $(a+a=0)$ in the abelian group, making this group what Bernstein has called a "Boolean group."* Also, every Boolean ring is commutative with respect to multiplication.

Stone suggests that there may be redundancies in these postulates due to the presence of the idempotent law. It is not apparent that there are any actual redundancies, but it may be surmised that Postulates 3 and 4 (the commutative law of addition and the existence law of subtraction) are considerably stronger than necessary. That this is the case will be shown when we formulate Set III.

Stone's postulates for generalized Boolean algebras [in J] were in terms of the usual Boolean addition $(\vee)$ and multiplication. These postulates will be referred to as Set I′ but will not be listed here. Stone established [in T] the equivalence of Sets I and I′ after appropriate definitions for each type of addition in terms of the other type and multiplication.

3. A set based on one of Bernstein's sets. Another possible set of postulates for Boolean rings is suggested immediately by Bernstein's set for Boolean algebras in terms of the operation of complete disjunction ("o").† This operation in a Boolean algebra is known to be equivalent to ring addition in Boolean rings with unit element. Bernstein uses multiplication and negation (′) also as undefined terms. Inasmuch as the operation of negation automatically involves the unit element in a Boolean algebra, it is reasonable to ask whether those postulates of Bernstein's set which involve multiplication and complete disjunction, without negation, are sufficient to determine a Boolean ring. These postulates include "cyclic associative laws" for o and $\times$, left-hand distributive law for $\times$ with respect to o, idempotent law, and a special law for o: $(a \text{ o } a) \text{ o } b=b$.

* B. A. Bernstein, Sets of postulates for Boolean groups, Annals of Mathematics, vol. 40, 1939, pp. 420–422. To be referred to as [B].

† B. A. Bernstein, Postulates for Boolean algebra involving the operation of complete disjunction, Annals of Mathematics, vol. 37, 1936, pp. 317–325. To be referred to as [A].

The answer is in the affirmative. The postulates just mentioned will be listed as Set II, after replacing Bernstein's "o" by "+" The equivalence of Sets I and II will then be proved. (A postulate of Bernstein's requiring at least two elements in K will be omitted, but with the understanding that K is non-empty.)

SET II

1. *If* a, b *are in* K, *then* (i) $a+b$, *and* (ii) ab *are in* K.
2. $a+(b+c)=b+(c+a)$
3. $(a+a)+b=b$
4. $a(bc)=b(ca)$
5. $a(b+c)=ab+ac$
6. $aa=a$

It is readily seen that Set II follows from Set I. The only postulates in Set II not found in Set I are Postulates 2, 3, 4. Postulates 2 and 4 are deducible from the standard associative and commutative laws which hold in Set I either by postulation or by proof. Postulate 3 of course follows from the Boolean group established by Set I.

Conversely, Set I follows from Set II. For Bernstein [in A] deduced, without any dependence on negation postulates, the fact that his system forms a Boolean group with respect to complete disjunction ("o"). (However, the term "Boolean group" was not used in the paper referred to here. As above, it signifies an abelian group with every element its own inverse.) Thus, the postulates of Set II establish a Boolean group with respect to $+$. This means that all of the additive postulates (1–4) of Set I are valid. The only remaining postulates of Set I not found in Set II are Postulates 5 and 7, the standard associative law of multiplication, and the right-hand distributive law, respectively. Both of these can readily be deduced from Set II provided the commutative law of multiplication can first be proved. Bernstein's proof of the latter is not available here, as it made use of negation postulates. The following alternate proof* makes use only of postulates of Set II (numbers referring to postulates of Set II):

$$\begin{aligned} ab &= (ab)(ab) && \text{by } 6 \\ &= a\,[b(ab)] && \text{by } 4 \\ &= a[a(bb)] && \text{by } 4 \\ &= a(ab) && \text{by } 6 \\ &= a(ba) && \text{by } 4 \\ &= b(aa) && \text{by } 4 \\ &= ba && \text{by } 6. \end{aligned}$$

Thus Sets I and II are equivalent.

It should be noted again that Set II is simply a sub-set of Bernstein's set for Boolean algebras in terms of complete disjunction. Since Bernstein proved the latter set independent it follows that the postulates of Set II are independent.

Set II has the advantage of having two less postulates than Set I, and no existence postulate (aside from the implicit postulate of non-emptiness). On the

* This proof was suggested by Professor Marie J. Weiss.

other hand, there are two very restrictive postulates now instead of one—$(a+a)+b=b$, as well as the idempotent law, $aa=a$. Also, it should be noted that in Set II the addition postulates, *by themselves*, are strong enough to imply that K is a Boolean group,* whereas in Set I the addition postulates establish an abelian group, but the idempotent law and distributive laws are needed to obtain the Boolean property of the group.

4. A weaker set. We return now to Set I and show that Postulates 3 and 4 of that set (the commutative law of addition and existence law of subtraction) can be replaced by a symmetrical pair of weaker postulates, namely left- and right-hand cancellation laws. The resulting addition postulates, *by themselves*, will not require K to be any kind of group at all. Furthermore, as in Set II, there will be no explicit existence postulate required. Thus Set III brings out clearly the strength of the idempotent law, by making the other postulates as weak as possible.

We now list Set III in detail. It is understood in advance that K is non-empty, and, as usual, that the complete hypotheses require the elements and their combinations to be in K.

SET III

1. *If a, b are in K, then* (i) $a+b$, *and* (ii) ab *are in K.*
2. $(a+b)+c=a+(b+c)$
3. *If* $a+b_1=a+b_2$, *then* $b_1=b_2$.
4. *If* $a_1+b=a_2+b$, *then* $a_1=a_2$.
5. $(ab)c=a(bc)$
6. $a(b+c)=ab+ac$
7. $(a+b)c=ac+bc$
8. $aa=a$

It is obvious that the new Postulates 3 and 4 are consequences of Set I, inasmuch as they hold for any additive group. To prove the equivalence of Sets I and III it remains to show that Postulates 3 and 4 of Set I can be deduced from Set III. The proof is very simple and can be formulated by the following theorems. The unqualified numbers in the proofs refer to the postulates of Set III. The use of Postulate 1 will be implicit, except for two cases where it is noted for emphasis.

THEOREM 1. *For every a in K, $a+a=(a+a)+(a+a)$.*

Proof.
$$\begin{aligned} a+a &= (a+a)(a+a) && \text{by } 8\\ &= a(a+a)+a(a+a) && \text{by } 7\\ &= (aa+aa)+(aa+aa) && \text{by } 6\\ &= (a+a)+(a+a) && \text{by } 8. \end{aligned}$$

DEFINITION 1. *If a is any element of K, then x_a will be written for $a+a$.*

* The addition postulates in Set II, with an extra postulate requiring at least two elements in K, are found as one of numerous sets of postulates for Boolean groups listed in Bernstein's article [B].

THEOREM 2. *If a is in K, the element x_a is an element of K such that $a=a+x_a=x_a+a$.*

$$
\begin{array}{lll}
\textit{Proof.} & x_a=a+a \text{ is in } K & \text{by 1 (i)} \\
& a+a=(a+a)+(a+a) & \text{by Theorem 1} \\
& \quad =a+[a+(a+a)] & \text{by 2} \\
\therefore & a=a+(a+a)=a+x_a & \text{by 3} \\
\text{and} & a=(a+a)+a=x_a+a & \text{by 2.}
\end{array}
$$

THEOREM 3. *If a, b are in K, then $x_a=x_b$.*

$$
\begin{array}{lll}
\textit{Proof.} & \left.\begin{array}{l} a+b=(a+x_a)+b \\ a+b=a+(x_b+b) \end{array}\right\} & \text{by Theorem 2} \\
\therefore & (a+x_a)+b=a+(x_b+b) & \\
& \quad =(a+x_b)+b & \text{by 2} \\
& a+x_a=a+x_b & \text{by 4} \\
\therefore & x_a=x_b & \text{by 3.}
\end{array}
$$

DEFINITION 2. *If k is any element of K, then $O=x_k(=k+k)$.*

THEOREM 4. *For every a in K, $a+O=O+a=a$.*

COROLLARY 1. *If $a+b=c$, then $a=c+b$.*

COROLLARY 2. *If $a+b=O$, then $a=b$.*

THEOREM 5. *For every a, b in K there exists a solution in K of the equation $x+a=b$, namely, $x=b+a$.*

$$
\begin{array}{lll}
\textit{Proof.} & x=b+a \text{ is in } K & \text{by 1(i)} \\
& x=b+a \text{ is a solution.} & \\
\text{For} & (b+a)+a=b+(a+a) & \text{by 2} \\
& \quad =b+O=b & \text{by Definition 2, Theorem 4.}
\end{array}
$$

(Incidentally, by Corollary 1 of Theorem 4 the solution is unique.)

THEOREM 6. *If a, b are in K, then $a+b=b+a$.*

Proof. By Corollary 2 of Theorem 4 it is sufficient to show that $(a+b)+(b+a)=O$.

$$
\begin{array}{ll}
\left.\begin{array}{l} (a+b)+(b+a)=[(a+b)+b]+a \\ \quad =[a+(b+b)]+a \end{array}\right\} & \text{by 2} \\
\quad =(a+O)+a=a+a & \text{by Definition 2, Theorem 4} \\
\quad =O & \text{by Definition 2.}
\end{array}
$$

The first part of Theorem 5 is identical with Postulate 4 of Set I, and Theorem 6 is identical with Postulate 3 of Set I. Thus Sets I and III are completely equivalent.

It is interesting to note that all of the postulates of Set III have been used in proving the theorems, with the exception of Postulate 5, the associative law of multiplication.

5. Other possible sets. A variation of Set III which does not seem quite as weak as that set is obtained by replacing one of the cancellation postulates, say 4, by the following:

4′. *There is not more than one element x in K such that $x+x=x$.*

The proof that this Set III′ is equivalent to Set III can easily be given.

Another set, III″, which is very simple in form, but stronger than necessary, can be formed from Set I as follows. Substitute the left-hand cancellation law for the subtraction existence law, retaining the commutative law of addition, and (for symmetry) substitute the commutative law of multiplication for the right-hand distributive law. It is obvious that this set is equivalent to Set III.

Consider now the question of duality in connection with Set III. Is it possible that a Boolean ring will satisfy completely a set of postulates dual to Set III formed by replacing multiplication by Boolean addition, and ring addition by a dual operation? (Boolean addition is to be understood as defined by $a \vee b = (a+b)+ab$.) It is clear that if our Boolean ring is a Boolean algebra, complete dualization will be possible when we replace $+$ by the standard operation dual to complete disjunction, as well as $\times$ by $\vee$. This possibility has been discussed by Bernstein [in A]. On the other hand, in the case of a Boolean ring without unit element there can be no dual for the zero element (in view of the fact that $a \vee O = a$) so that complete dualization must be impossible.

This point can be clarified from a slightly different point of view. Suppose it is possible to find an operation $\square$ in a Boolean ring determined by Set III such that on replacing $+$ by $\square$ and $\times$ by the defined $\vee$, all postulates of Set III are satisfied again. Then it follows that the system must form a Boolean group with respect to $\square$ and there must be an identity in this group, say e, with the properties

$$e = a \square a$$

$$a \vee e = a \vee (a \square a) = (a \vee a) \square (a \vee a) = a \square a = e.$$

But by the original definition of $a \vee b$ we have

$$a \vee b = (a+b)+ab$$

or

$$ab = (a \vee b)+(a+b).$$

Hence

$$ae = (a \vee e)+(a+e) = e+(a+e) = a+(e+e) = a+O = a.$$

Thus e must be a unit element for the Boolean ring. The results may be summarized as follows:

THEOREM. *A Boolean ring determined by Set* III *satisfies a set of postulates completely dual to* III *(formed by replacing $\times$ by the defined operation $\vee$ and $+$ by an operation $\square$), if and only if the Boolean ring contains a unit element, that is, if and only if the ring is a Boolean algebra.*

Of course, this theorem does not prevent us from formulating a set of postulates for general Boolean rings in terms of ring addition and Boolean addition. Such a set is given below as Set IV. As might be expected, Set IV is not so simple as Set III, but the ring addition postulates remain the same as before

and again no existence postulate is required. Postulates 6 and 7 in the new set are analogs of the left- and right-hand distributive laws used in the preceding sets. Postulates 5 and 8 are ordinary duals of the associative and idempotent laws for multiplication.

SET IV

System $(K, +, \vee)$.

1. *If* a, b *are in* K, *then* (i) $a+b$, *and* (ii) $a \vee b$ *are in* K.
2. $(a+b)+c=a+(b+c)$
3. *If* $a+b_1=a+b_2$, *then* $b_1=b_2$.
4. *If* $a_1+b=a_2+b$, *then* $a_1=a_2$.
5. $(a \vee b) \vee c=a \vee (b \vee c)$
6. $a \vee (b+c)=a+[(a \vee b)+(a \vee c)]$
7. $(a+b) \vee c=[(a \vee c)+(b \vee c)]+c$
8. $a \vee a=a$

We outline a method of proving Set IV equivalent to Set III without giving complete details. First, the new postulates, 1 (ii) and 5–8, can be easily verified as properties of any Boolean ring, after making the usual definition for Boolean addition. Thus, Set IV follows from Set III.

The proof in the reverse direction, assuming Set IV and deducing Set III, is somewhat more lengthy. It is essential first of all to establish the Boolean group properties with respect to ring addition. This may be done by the following two lemmas and theorem. (Numbers in the proofs refer to postulates of Set IV.)

LEMMA 1. *For every* a *in* K, $(a+a) \vee a=(a+a)+a=a+(a+a)$.

Proof.
$$
\begin{aligned}
(a+a) \vee a &= [(a \vee a)+(a \vee a)]+a && \text{by } 7\\
&= (a+a)+a && \text{by } 8\\
&= a+(a+a) && \text{by } 2.
\end{aligned}
$$

LEMMA 2.* *For every* a *in* K, $(a+a) \vee a=a$.

Proof.
$$
\begin{aligned}
(a+a) \vee a &= (a+a) \vee (a \vee a) && \text{by } 8\\
&= [(a+a) \vee a] \vee a && \text{by } 5\\
&= [(a+a)+a] \vee a && \text{by Lemma 1}\\
&= [(a+a) \vee a]+[(a \vee a)+a] && \text{by } 7, 2\\
&= [(a+a) \vee a]+(a+a) && \text{by } 8.
\end{aligned}
$$

But also $\quad (a+a) \vee a=a+(a+a) \quad$ by Lemma 1

$\therefore \quad [(a+a) \vee a]+(a+a)=a+(a+a)$

and $\quad (a+a) \vee a=a \quad$ by 4.

THEOREM. *For every* a *in* K, $a=(a+a)+a=a+(a+a)$.

This theorem is equivalent to Theorem 2 in the development following Set III (where x_a is written for $a+a$). The rest of that development holds equally well here and makes it certain that the system K of Set IV is an abelian group

* This proof was suggested by Professor Marie J. Weiss.

with respect to ring addition, with every element its own inverse. Thus K is a Boolean group.

Next we introduce the following:

DEFINITION. *If a, b are in K, then* $ab=(a \vee b)+(a+b)$.

This definition immediately allows us to deduce the closure postulate for multiplication and the idempotent law (Postulates 1 (ii) and 8 of Set III, respectively). The only remaining postulates to be proved are the associative law of multiplication, and left- and right-hand distributive laws (Postulates 5, 6, 7 of Set III). These proofs can be supplied by making use of the quasi-distributive postulates (6 and 7 of Set IV) and of the Boolean group properties. Thus Set III follows from Set IV.

6. Independence proofs for Set III. We shall give independence proofs for Set III alone. (Set II, of course, is already known to be independent.) One question of interest will be left open: if we add the postulate requiring existence of a unit element, thus obtaining a set of postulates for an ordinary Boolean algebra, does the enlarged set remain independent?

The following familiar system (ring of integers (mod 2)) satisfies all of the postulates of Set III and can be used as a formal consistency proof, if one is desired for reference:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

The independence of the postulates of Set III is established by the following K-systems, each of which satisfies all postulates except the one indicated by the number of the system (and referred to briefly in parentheses).

K 1 (i) (Closure for ring addition.) $K=$ the numbers 0, 1
$+=$ ordinary addition, $\times=$ ordinary multiplication.

K 1 (ii) (Closure for multiplication.)

+	0	1
0	0	1
1	1	0

×	0	1	
0	0	2	
1	0	1	(2 not in K)

K 2 (Associative law for addition.)

+	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

×	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

Here $(0+0)+1 \neq 0+(0+1)$.

K 3 (Left-hand cancellation law.)

+	0	1
0	0	0
1	1	1

×	0	1
0	0	0
1	0	1

Here $1+0=1+1$, but $0\neq 1$.

K 4 (Right-hand cancellation law.)

+	0	1
0	0	1
1	0	1

×	0	1
0	0	0
1	0	1

Here $0+0=1+0$, but $0\neq 1$.

K 5 (Associative law for multiplication.)

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Here $(1\cdot 1)2\neq 1(1\cdot 2)$.

The elements 0, 1, 2, 3 form a Boolean group with respect to addition. Furthermore, any system *K* which proves the independence of Postulate 5 will necessarily exhibit this same property; for, whenever all of the other postulates hold we know by the theorems proved from Set III, without any use of Postulate 5, that *K* must form a Boolean group.

K 6 (Left-hand distributive law.)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Here $1(2+2)\neq 1\cdot 2+1\cdot 2$.

K 7 (Right-hand distributive law.)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Here $(1+1)\ 2\neq 1\cdot 2+1\cdot 2$.

K 8 (Idempotent law.)

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	0

Here $1\cdot 1\neq 1$.

ON TRANSFORMATION OF MULTIPLE INTEGRALS*

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1. Introduction. The proof of the formula for transformation of a multiple Riemann integral under a change of variables

$$x_i = x_i(u_1, \cdots, u_n) = x_i(u), \qquad (i = 1, \cdots, n), \tag{1}$$

depends on the formula

$$V = \int_R \cdots \int |J| \, du_1 du_2 \cdots du_n, \tag{2}$$

for the volume V in (x)-space of the image, under (1), of a solid R in (u)-space, where J is the Jacobian

$$\frac{D(x_1, \cdots, x_n)}{D(u_1, \cdots, u_n)} \equiv J(u).$$

We assume that R is a Jordan measurable closed region; that R is a sub-set of a bounded region $\mathcal{K}$ in (u)-space; that the derivatives in $J(u)$ exist and are continuous over $\mathcal{K}$; that $J \neq 0$ in $\mathcal{K}$; and that (1) *transforms $\mathcal{K}$ in one-to-one fashion into a region of (x)-space.*

The standard proof of (2) includes expressing V as a surface integral and using a parametric representation of the surface. For $n \geqq 3$ it is fairly involved.

Another method considers the image in (x)-space of a rectangular solid in (u)-space. This method gives the answer quickly and naturally, but a complete proof by this method does not appear in the literature, to the writer's knowledge. *It is the purpose of the present note to give a complete demonstration of* (2) *along the lines of the second method, while keeping the detailed work within reasonable bounds.*

2. The proof. We begin with a special case of (1), namely,

$$x_i - \overset{0}{x}_i = \sum_{j=1}^{n} a_{ij}(u_j - \overset{0}{u}_j), \qquad (i = 1, \cdots, n). \tag{3}$$

LEMMA 1. *Under* (3) *any* (*measurable*) *solid is transformed so that the ratio of the volume in (x)-space to the volume in (u)-space is the absolute value of the determinant of the a_{ij}'s.*

Proof. An n-cube in (u)-space with edges of length d parallel to the axes, and with (b) as the vertex with smallest coördinates, will be carried by (3) into an n-parallelopiped such that, if

$$(c_1, c_2, \cdots, c_n) \qquad \text{corresponds to } (b_1, b_2, \cdots, b_n),$$

* Presented to the American Mathematical Society, February 24, 1940.

then

$$(c_1 + a_{11}d, \cdots, c_n + a_{n1}d) \text{ corresponds to } (b_1 + d, b_2, \cdots, b_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(c_1 + a_{1n}d, \cdots, c_n + a_{nn}d) \text{ corresponds to } (b_1, b_2, \cdots, b_n + d).$$

By a known formula,* the volume of the parallelopiped is therefore

$$\pm \begin{vmatrix} a_{11}d & \cdots & a_{n1}d \\ \cdot & \cdots & \cdot \\ a_{1n}d & \cdots & a_{nn}d \end{vmatrix} = \pm\, d^n \begin{vmatrix} a_{11} & \cdots & a_{n1} \\ \cdot & \cdots & \cdot \\ a_{1n} & \cdots & a_{nn} \end{vmatrix}.$$

Since the volume of the original cube is d^n, the ratio of volumes under (3), in the case of a cube with edges parallel to the axes, is thus seen to be the absolute value of the determinant of the a_{ij}'s. Hence the same ratio is obtained when (3) is applied to any solid, and the lemma is proved.

We now return to (1). For any fixed point (u^0) in R we consider the linear approximation to (1):

$$(4) \qquad x_i - x_i(\overset{0}{u}) = \sum_{j=1}^{n} (\partial x_i/\partial u_j)_0 (u_j - \overset{0}{u_j}), \qquad (i = 1, \cdots, n),$$

$(\partial x_i/\partial u_j)_0$ being evaluated at (u^0). The inverse of (4), with u_i replaced by u_i', is

$$(5) \qquad u_i' - \overset{0}{u_i} = J_i/\overset{0}{J}, \qquad (i = 1, \cdots, n),$$

where $J^0 = J(u^0)$, and J_i is obtained from J^0 by replacing the elements of the ith column by $x_1 - x_1(u^0), \cdots, x_n - x_n(u^0)$. Let

$$(6) \qquad u_i' = u_i'(u), \qquad (i = 1, \cdots, n),$$

be the transformation in (u)-space obtained by combining (1) and (5). Then

$$(7) \qquad \partial u_i'/\partial u_j = J_{ij}/J^0, \qquad (i, j = 1, \cdots, n),$$

where J_{ij} is obtained from J^0 by replacing the elements of the ith column by $\partial x_1/\partial u_j, \cdots, \partial x_n/\partial u_j$. Hence, at $(u) = (u^0)$,

$$(8) \qquad (\partial u_i'/\partial u_j) = \delta_{ij},$$

where δ_{ij} is 1 or 0 according as $i = j$ or $i \neq j$.

Let $h > 0$ be a constant such that if (u^0) is on R and $|u_i - u_i^0| \leqq h$, $(i = 1, \cdots, n)$, then (u) is in $\mathfrak{K}$. The set of all points $(u_1, \cdots, u_n, u_1^0, \cdots, u_n^0)$ so obtained forms a set C closed in (u, u^0)-space. Therefore the right-hand side of (7) is uniformly continuous, for each i, j, over C. Hence we infer from (8) that for a given $\epsilon > 0$, there corresponds a Δ, with $0 < \Delta < h$, such that if (u^0) is on R and

* See theorem in appendix.

$$\text{(A)} \qquad |u_i - \overset{0}{u}_i| \leqq \Delta, \qquad (i = 1, \cdots, n),$$

then

$$\text{(9)} \qquad |\partial u_i'/\partial u_j - \delta_{ij}| < \epsilon, \qquad (i, j = 1, \cdots, n).$$

With (u^0) fixed, suppose that (u) is on A, the locus defined by (A). By the law of the mean,

$$|u_i'(u) - u_i| = \left| \overset{0}{u}_i + \sum_j \left[\frac{\partial u_i'(U)}{\partial u_j} \right] \cdot (u_j - \overset{0}{u}_j) - u_i \right|, \qquad [(U) \text{ on } A],$$

$$\text{(10)} \qquad = \left| \overset{0}{u}_i + \sum_j (\delta_{ij} + \theta_{ij})(u_j - \overset{0}{u}_j) - u_i \right|, \qquad [|\theta_{ij}| < \epsilon, \text{ by (9)}],$$

$$= \left| \sum_j \theta_{ij}(u_j - \overset{0}{u}_j) \right| < n\epsilon\Delta, \qquad (i = 1, \cdots, n).$$

We assume $n\epsilon < 1$.

The boundary of A consists of those points satisfying (A) for which at least one equality holds. If a point of the boundary were carried by (6) into a point of A_1:

$$(\text{A}_1) \qquad |u_i - \overset{0}{u}_i| \leqq (1 - n\epsilon)\Delta, \qquad (i = 1, \cdots, n),$$

or into a point not an inner point of A_2:

$$(\text{A}_2) \qquad |u_i - \overset{0}{u}_i| \leqq (1 + n\epsilon)\Delta, \qquad (i = 1, \cdots, n),$$

then at least one of its coördinates would be changed by at least $n\epsilon\Delta$, contrary to (10). Hence the transform of A under (6), say A', is a sub-set of A_2; and, since it has (u^0) in common with A_1, it contains all of A_1. We infer, letting the letters denote n-dimensional volumes, that

$$[2(1 - n\epsilon)\Delta]^n = A_1 < A' < A_2 = [2(1 + n\epsilon)\Delta]^n.$$

Since $A = (2\Delta)^n$, this gives us

$$\text{(11)} \qquad (1 - n\epsilon)^n < \frac{A'}{A} < (1 + n\epsilon)^n.$$

Now (6) is the product of (1) and (5). Since (5) is the inverse of (4), we infer from Lemma 1 that the ratio of volumes in the units for the respective spaces, under (5), is $1/|J^0|$. Let B denote the volume in (x)-units of the transform of A under (1). Then (5) transforms B into A', and $A'/B = 1/|J^0|$. Thus

$$\frac{B}{A} \cdot \frac{1}{|J^0|} = \frac{B}{A} \cdot \frac{A'}{B} = \frac{A'}{A} = 1 + \gamma,$$

where $|\gamma|$ can be made arbitrarily small by taking ϵ small, as we see from (11).

Hence

$$\frac{B}{A} = |J^0| + \gamma |J^0| = |J^0| + \mu. \tag{12}$$

Since $|J^0|$ is bounded, $|\mu| = |\gamma \cdot J^0|$ can be made arbitrarily close to zero.

Formula (2) can now be derived easily from (12).

3. Appendix. It is desirable to give a simple proof of the following:

THEOREM. *If a parallelopiped in n-space has edges joining* $(x_{01}, \cdots, x_{0n})$ *to each of the points* $(x_{11}, \cdots, x_{1n}), \cdots, (x_{n1}, \cdots, x_{nn})$, *its volume is*

$$V = \pm M = \pm \begin{vmatrix} x_{11} \cdots x_{1n} & 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ x_{n1} \cdots x_{nn} & 1 \\ x_{01} \cdots x_{0n} & 1 \end{vmatrix} = \pm \begin{vmatrix} x_{11} - x_{01} \cdots x_{1n} - x_{0n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{n1} - x_{01} \cdots x_{nn} - x_{0n} \end{vmatrix}. \tag{13}$$

We begin with the following:

LEMMA 2. *Given an ordered set of n directed lines through a point in the space of the n variables* $x_1, \cdots, x_n$, *not lying in any* $(n-1)$-*plane, the directions of the lines can be changed continuously so that the first* $n-1$ *of them finally point in the directions of the* $x_1, x_2, \cdots, x_{n-1}$-*axes, respectively, and the nth in the direction of the positive or negative* x_n-*axis, and so that at no stage of the motion are the n directed lines in any* $(n-1)$-*plane.*

The proof can be given easily by induction. We return now to the theorem.

If the elements of the last row are subtracted from those of each of the other rows in the first determinant in (13), the equality of the two determinants becomes evident.*

The rows of the second determinant in (13) are sets of direction numbers of the respective given edges. If the parallelopiped is degenerate, so that the n edges are co-planar, (13) becomes $0=0$. Assume that this is not the case.

Suppose $(x_{11}, \cdots, x_{1n})$ changes its position, while the other n points remain fixed. If $x_{11}, \cdots, x_{1n}$ are considered as coördinates of a variable point, $M=0$ is an equation of the hyperplane through the remaining n points. Hence the numerator of the corresponding formula for distance from $(x_{11}, \cdots, x_{1n})$ to that plane is $\pm M$. As the volume of the parallelopiped is proportional to that distance when the base containing the other n points is fixed, it follows that the volume is proportional to M as $(x_{11}, \cdots, x_{1n})$ changes position while the other given points remain fixed and the volume does not become zero; similarly, it is proportional to M when any one of the points $(x_{i1}, \cdots, x_{in})$, $(i=1, \cdots, n)$, changes position while the others are fixed and the volume is not zero. Since any continuous change in the n points can be approximated to by a finite num-

* If $n=2$, the first form in (13) will be recognized as twice the familiar formula for the area of a triangle with given vertices. In the following proof we use the second determinant in (13).

ber of steps in each of which only one of the points changes, we infer that the volume is proportional to M as all n points change simultaneously. Now with $(x_{01}, \cdots, x_{0n})$ fixed, let the directions of the edges of the parallelopiped change to their final positions under Lemma 2, while their lengths all change to unity. In the final position M becomes

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \pm 1 \end{vmatrix} = \pm 1.$$

But the volume is 1 for this final position. Hence the constant of proportionality is either $+1$ or -1, and (13) is proved.

MODULI OF THE ROOTS OF POLYNOMIALS AND POWER SERIES*

LOUIS WEISNER, Hunter College

One of the most useful methods for determining bounds for the moduli of the roots of a polynomial in the field of complex numbers is provided by the following theorem due to Cauchy:† *The modulus of every root of the polynomial*

$$F(z) = c_0 + c_1 z + \cdots + c_n z^n, \qquad (c_n \neq 0),$$

is not greater than the sole positive root of the polynomial

$$|c_0| + |c_1| z + \cdots + |c_{n-1}| z^{n-1} - |c_n| z^n.$$

This theorem is closely related to Pellet's theorem‡ which asserts that if the polynomial

$$|c_0| + |c_1| z + \cdots + |c_{\nu-1}| z^{\nu-1} - |c_\nu| z^\nu + |c_{\nu+1}| z^{\nu+1} + \cdots + |c_n| z^n,$$

$(0<\nu<n)$, has two positive roots r_1 and $r_2 \geqq r_1$, then $F(z)$ has just ν roots whose moduli are not greater than r_1.§ In this note I express the theorems of Cauchy and Pellet in a form in which they may be applied advantageously.

Supposing all zero terms of $F(z)$ suppressed, the polynomial has the form

$$(1) \qquad f_m(z) = a_0 z^{n_0} + \cdots + a_m z^{n_m}, \qquad (0 \leqq n_0 < n_1 < \cdots < n_m),$$

* Presented to the American Mathematical Society, April 26, 1940.

† A. L. Cauchy, Oeuvres (2), vol. 9, p. 122. See also, Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis I, p. 87, problems 17 and 18.

‡ A. E. Pellet, Sur un mode de séparation des racines des équations et la formule de Lagrange, Bulletin des sciences mathématiques (2), vol. 5, 1881, p. 393. See also, J. L. Walsh, On Pellet's theorem concerning the roots of a polynomial, Annals of Mathematics, vol. 26, 1924, p. 59; and E. Egerváry, On a generalisation of a theorem of Kakeya, Acta Szeged, vol. 5, 1930, p. 78.

§ In the event that $r_2 = r_1$, $F(z)$ may have double roots on the circle $|z| = r_1$. In this case the theorem holds if each double root *on* the circle is counted only once. See Walsh, *loc. cit.*, p. 60.

where $a_s \neq 0$, $(s=0, 1, \cdots, m)$. We suppose $m \geqq 1$, and set

$$q_s = n_s - n_{s-1}, \qquad (s = 1, \cdots, m). \tag{2}$$

THEOREM 1. *For a fixed k, $0 \leqq k \leqq m$, let $p_1, \cdots, p_m$ be positive numbers, and $p_0 = 1$, satisfying the inequality*

$$p_0 + p_0 p_1 + p_0 p_1 p_2 + \cdots + p_0 p_1 \cdots p_m \leqq 2 p_0 p_1 \cdots p_k; \tag{3}$$

and let

$$u_s = \left| \frac{p_s a_{s-1}}{a_s} \right|^{1/q_s}, \qquad (s = 1, \cdots, m). \tag{4}$$

If

$$R \geqq u_s, \qquad (s = 1, \cdots, k), \tag{5}$$

$$R \leqq u_s, \qquad (s = k+1, \cdots, m), \tag{6}$$

*the circle $|z| \leqq R$ includes just n_k roots of $f_m(z)$.**

The inequality (3) may be written more conveniently,

$$(p_1 \cdots p_k)^{-1} + (p_2 \cdots p_k)^{-1} + \cdots + p_k^{-1} + p_{k+1} + p_{k+1}p_{k+2} + \cdots + p_{k+1}p_{k+2} \cdots p_m \leqq 1, \tag{7}$$

with due modifications when $k=0$ or m. Substituting

$$p_s = \left| \frac{a_s}{a_{s-1}} \right| u_s^{q_s}$$

in (7), we have

$$|a_0| u_1^{-q_1} \cdots u_k^{-q_k} + |a_1| u_2^{-q_2} \cdots u_k^{-q_k} + \cdots + |a_{k-1}| u_k^{-q_k} + |a_{k+1}| u_{k+1}^{q_{k+1}} + |a_{k+2}| u_{k+1}^{q_{k+1}} u_{k+2}^{q_{k+2}} + \cdots + |a_m| u_{k+1}^{q_{k+1}} \cdots u_m^{q_m} \leqq |a_k|.$$

Therefore, by (2), (5), and (6),

$$|a_0| R^{n_0} + |a_1| R^{n_1} + \cdots + |a_{k-1}| R^{n_{k-1}} + |a_{k+1}| R^{n_{k+1}} + \cdots + |a_m| R^{n_m} \leqq |a_k| R^{n_k}. \tag{8}$$

It follows that when $0<k<m$, the polynomial

$$|a_0| z^{n_0} + \cdots + |a_{k-1}| z^{n_{k-1}} - |a_k| z^{n_k} + |a_{k+1}| z^{n_{k+1}} + \cdots + |a_m| z^{n_m}$$

has two positive roots r_1 and $r_2 \geqq r_1$, and that $r_1 \leqq R \leqq r_2$. Therefore, by Pellet's theorem, the circle $|z| \leqq R$ includes just n_k roots of $f_m(z)$.

In the cases $k=0$ and $k=m$ we have, in place of (8),

* The factor $p_0 = 1$ is necessary in (3) only when $k=0$, in which case (5) is to be omitted. When $k=m$, (6) is to be omitted.

$$|a_1| R^{n_1} + \cdots + |a_m| R^{n_m} \leqq |a_0| R^{n_0}$$

and

$$|a_0| R^{n_0} + \cdots + |a_{m-1}| R^{n_{m-1}} \leqq |a_m| R^{n_m},$$

respectively; and the theorem follows from Cauchy's theorem.

If at least one of the relations (3), (5), and (6) is an actual inequality, no root of $f_m(z)$ is on the circle $|z| = R$; hence just n_k roots lie in the circle $|z| < R$.

We find that (3) is satisfied as an inequality by

$$p_1 = \cdots = p_k = 3, \qquad p_{k+1} = \cdots = p_m = 1/3$$

for all $k \geqq 0$ and $m \geqq k$. (The first set is to be omitted when $k=0$ and the second when $k=m$.)

THEOREM 2. *If*

$$R \geqq \left| \frac{3a_{s-1}}{a_s} \right|^{1/q_s}, \qquad (s = 1, \cdots, k),$$

$$R \leqq \left| \frac{a_{s-1}}{3a_s} \right|^{1/q_s}, \qquad (s = k+1, \cdots, m),$$

the circle $|z| < R$ includes just n_k roots of $f_m(z)$.

In particular cases, better choices of the p's are available. When $k=m$ we find that (3) is satisfied as an equality by

$$p_1 = p_2 = \cdots = p_m = \tau_m,$$

where τ_m is the sole positive root of the equation

$$1 + z + \cdots + z^{m-1} - z^m = 0.$$

THEOREM 3.* *The modulus of every root of $f_m(z)$ is not greater than the largest of the numbers*

$$\left| \frac{\tau_m a_{s-1}}{a_s} \right|^{1/q_s}, \qquad (s = 1, \cdots, m).$$

The theorem remains valid when τ_m is replaced by 2. However, a better result is obtained by observing that when $k=m$, (3) is satisfied as an equality by

$$p_1 = 1, \ p_2 = \cdots = p_m = 2.$$

THEOREM 4.† *The modulus of every root of $f_m(z)$ is not greater than the largest*

* This theorem is proved for the case in which each $q_s = 1$ by T. Anghelutza, Una estensionne di un teorema di Hurwitz, Bolletino della Unione Matematica Italiani, vol. 12, 1934, p. 284.

† Compare T. Anghelutza, Sur une extension d'un théorème de Hurwitz, Bulletin de l'academie roumaine, vol. 16, 1934, p. 119; P. Montel, Sur quelques limites pour les modules des zéros des polynomes, Commentarii math. Helvetici, vol. 7, 1935, p. 188; T. H. Chang, Verallgemeinerung des Satzes von Kakeya, Tohôku Math. Journal, vol. 43, 1937, p. 80.

of the numbers

$$\left|\frac{a_0}{a_1}\right|^{1/q_1}, \qquad \left|\frac{2a_{s-1}}{a_s}\right|^{1/q_s}, \qquad (s = 2, \cdots, m).$$

The preceding theorems may be applied to the partial sums $f_m(z)$, $(m=1, 2, \cdots)$ of a power series

$$f(z) = \sum_{s=0}^{\infty} a_s z^{n_s}, \qquad (0 \leqq n_0 < n_1 < \cdots),$$

where $a_s \neq 0$, $(s=0, 1, \cdots)$. We define

$$q_s = n_s - n_{s-1}, \qquad (s = 1, 2, \cdots),$$

consistent with (2).

For example, it follows from Theorem 4 that *if the sequence*

$$\left|\frac{a_{s-1}}{a_s}\right|^{1/q_s}, \qquad (s = 1, 2, \cdots),$$

is bounded above, the roots of all the partial sums of $f(z)$ lie in a circle of finite radius.

Again, it follows from Theorem 2 that *if the coefficients of $f(z)$ satisfy the inequalities*

$$\left|\frac{3a_{s-1}}{a_s}\right|^{1/q_s} \leqq \left|\frac{a_s}{3a_{s+1}}\right|^{1/q_{s+1}}, \qquad (s = 1, 2, \cdots),$$

the circle

$$|z| < \left|\frac{3a_{k-1}}{a_k}\right|^{1/q_k}, \qquad (k = 1, 2, \cdots),$$

includes just n_k roots of each partial sum of $f(z)$ of degree not less than n_k. This is true of the entire function*

$$\sum_{s=0}^{\infty} a^{-s^2} z^s, \qquad |a| \geqq 3,$$

which therefore has just one root in each of the rings

$$0 < |z| < |3a|, \qquad |3a^{2k-1}| < |z| < |3a^{2k+1}|, \qquad (k = 1, 2, \cdots).$$

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* Compare G. H. Hardy, On the zeros of a class of integral functions, Messenger of Math., vol. 34, 1904, p. 99; Pólya and Szegö, Aufgaben und Lehrsätze I, p. 123, problem 200.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

COMMENTS ON THE NORTH CAROLINA PROGRAM IN FRESHMAN MATHEMATICS*

H. L. DORWART, Washington and Jefferson College

With several minor changes, the article, *A program in freshman mathematics designed to care for a wide variation in student ability*, by E. A. Cameron in this MONTHLY for August–September, 1940, might have referred to Washington and Jefferson College instead of to the University of North Carolina. The Washington and Jefferson program, which also separates the freshmen into three groups after a mathematics placement test prior to registration, was likewise put into operation in the fall of 1938. However, some differences in detail, due in part to differing conditions at a large university with an entering class of about 850 and a small arts college for men with an entering class of about 200, may be of interest to readers of Cameron's article.

As a placement test at Washington and Jefferson, the American Council's Coöperative Mathematics Pre-Test for First Year Students is used. This objective type test, requiring 45 minutes, was constructed by a committee of the Mathematical Association.† The scores on this test are entered on the high school record cards which have been previously made out for each student. Two lists are then compiled, the first containing the names of students who have made high grades in four years of mathematics in preparatory school and who have *also* made high scores on the placement test. The second list contains the names of students whose high school record *or* placement test indicates that they may have difficulty with mathematics in college.

A member of the department with the file of cards and the two lists is located near the beginning of the registration line to give information and offer advice. Students whose names occur on the first (high) list are encouraged (but not required) to take a course in analytic geometry and calculus. During each of the past three years, an increasing percentage (now 25%) of the class has elected this course, and with only a few exceptions, very good records have been made. Students on neither list are assigned to any convenient section of the regular freshman course in algebra and trigonometry, while students whose names occur on the second (low) list have their mathematics scheduled at a period reserved for this group.

At the first meeting of this group, a choice of two procedures is outlined.

* Presented to the Allegheny Mountain Section of the Mathematical Association of America, November 2, 1940.

† See Mathematical Education: Report of the Committee on Tests, this MONTHLY, vol. 47, 1940, p. 290.

Either the student may take the regular freshman algebra course for college credit or he may elect a semester course in high school algebra including logarithms and numerical trigonometry, carrying no college credit but designed to remedy faulty preparation and enable the student to carry for credit the regular freshman course in trigonometry during the second semester. (The regular first-semester college algebra course then has to be made up in summer school or during a later year.) Students not ready to make an immediate choice are urged to consult members of the department, high school teachers, or anyone else who might be judged helpful. The majority of students follow the departmental recommendations, and in all but a few cases in the past three years, sound advice has apparently been given.

So far there is no compulsion in the program. To take care of the misfits (which are bound to occur under any system), an effort is made to schedule the freshmen taking the advanced course at the same hour as the two groups just described; thus during the first month, changes can be made back and forth from the advanced course to the regular or from the regular to the preparatory without disrupting the student's schedule. At the end of the first month, students failing in the advanced course are automatically shifted back to the regular course and likewise those failing in the regular course go back to the preparatory course. No further changes are then permitted.

At this point it might be remarked that the North Carolina idea of having the poor students meet for five hours a week but giving them only three hours credit is a good one if not adopted by too many other departments. The writer knows of an extreme case where it was discovered that a few students were carrying so many required, extra, non-credit hours, that, making due allowance for study periods and normal outside activities, there was very little time left for eating or sleeping.

As at North Carolina, the new program at Washington and Jefferson has been responsible for a decrease in the number of failures in mathematics. Also, the instructors agree that the greatest benefit comes from the separation of the very good and the very poor from the average students. This has "increased interest and decreased discouragement" not only "on the part of the students" but also on the part of the instructors.

An additional advantage of the plan at Washington and Jefferson seems worth noting. Although the majority of the students in the preparatory course are hopeless as far as mathematics is concerned—many of them are poor in other courses also and leave college after a year or two—, there is always a small group worth saving. These are the students for whom several years have elapsed between high school and college, those who may have taken insufficient mathematics in high school either because they did not plan to attend college or because they were poorly advised, or possibly those who have had unusually poor instruction in high school. By supplying a foundation for these students, it is then possible for them to take the regular freshman course—not a dehorned substitute course—, and thus they are eligible, if they so desire, to continue with

courses in pure mathematics or to take courses in applied mathematics, physics, chemistry, statistics, *etc.*

And finally there seems to be a somewhat intangible gain, arising probably from the lack of compulsion in the program. Not only is the freshman treated as a responsible individual as regards the decision for his course, but also he starts his college mathematics work with the feeling that he is coöperating with an instructor who is also his advisor and counselor.

DIAGNOSTIC TESTING PROGRAM IN PURDUE UNIVERSITY

2. Solution of Simple Equations

M. W. KELLER, D. R. SHREVE, AND H. H. REMMERS, Purdue University

In this paper are presented the factual findings of the second* diagnostic test on the algebraic abilities of students entering first-semester freshman mathematics courses in Purdue University. An accounting is given, in tabular form, of the types and frequencies of errors which these students made in attempting to perform solutions of simple algebraic equations.

For reference there is given in the list titled "Sample Test Problems" the odd-numbered problem of each of the 18 pairs of problems of the test. The problems are divided into six groups, according to type. The reliability of the test for this population ("odds" *vs.* "evens") was 0.924. The numbers given in parentheses following each of the problems in the "Sample Test Problems" are (1) the number of engineering students who worked both problems of the pair correctly, and (2) the number of students in the School of Science algebra course who worked both problems of the pair correctly. In all, 202 engineers and 72 School of Science algebra course students were tested.

The frequency of errors of the different types is given in the "Error Analysis Table." The first column, headed Problem pair, refers to the pair of problems of which one problem is given in the Sample Test Problems. The Error class headings of the 14 columns indicate the code numbers of the types of errors listed in the "Code Classification of Errors."

NUMBER OF CORRECT ANSWERS TO SAMPLE TEST PROBLEMS

(By 202 Engineers, 72 School of Science students)

Group A. Linear Equations; One Variable

1. $2x=6;\ x=$ (199, 72)
2. $6y=3;\ y=$ (185, 54)
3. $g-5=-8;\ g=$ (180, 53)
4. $3x+7=-2;\ x=$ (179, 54)
5. $2(x-3)=3-3x;\ x=$ (140, 40)

Group B. Two Linear Equations; Two Variables

6. $\begin{cases} 2x+7a=3, \\ 3x-5a=51;\ a= \end{cases}$ (94, 19)
7. $\begin{cases} -b+2a=5, \\ 3b-6a=2;\ b= \end{cases}$ (86, 5)

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* M. W. Keller, D. R. Shreve, and H. H. Remmers, Diagnostic testing program in Purdue University, 1. Formal algebraic manipulations, this MONTHLY, October, 1940, pp. 544–548.

Group C. Fractions; One Variable

8. $\dfrac{1}{3x}=12$; $x=$ (123, 25)

9. $\dfrac{x+1}{3x-3}=\dfrac{2}{3}$; $x=$ (101, 25)

10. $\dfrac{3x-5}{2x+7}=\dfrac{3}{2}$; $x=$ (84, 8)

Group D. Quadratic Equations; One Variable

11. $b^2=36$; $b=$ (33, 5)
12. $(y+2)^2=49$; $y=$ (40, 7)
13. $a^2+a=6$; $a=$ (82, 19)
14. $2r^2+3r-7=0$; $r=$ (23, 3)
15. $2w^2+w+1=0$; $w=$ (20, 3)

Group E. One Linear Equation, One Quadratic; Two Variables

16. $\begin{cases} y-2x+5=0, \\ x^2+y^2=10; \; x= \quad y= \end{cases}$ (28, 5)

Group F. Radicals

17. $\sqrt{b}=9$; $b=$ (166, 45)
18. $a=\sqrt{x/b}$; $b=$ (99, 21)

CODE CLASSIFICATION OF ERRORS

1. No answer was given.
2. The student obviously did not understand the correct procedure.
3. Numerical error.
4. The student miscopied.
5. Incomprehensible step.
6. The student did not add equals to equals, or did not subtract equals from equals, as the problem demanded.
7. The student did not multiply equals by equals, or did not divide equals by equals, as the problem demanded.
8. The student did not distribute multiplication over an expression.
9. The student did not square a binomial correctly.
10. The student combined unlike terms.
11. The student factored an expression incorrectly.
12. The student found only one root of a quadratic equation.
13. The student failed to recognize an inconsistency.
14. Miscellaneous errors which occured too infrequently to warrant classification.

In Error class 1, no distinction was made whether the student did some work or showed no work on the problem. In Error class 3, the approximate frequency of errors was: addition or subtraction, 28%; multiplication, 23%; division or cancellation, 14%; miscellaneous, 35%. In Error class 9, the students wrote y^2+4 for $(y+2)^2$, and so forth.

In the Error Analysis Table, the entry 4 in column 7 and row 1 (for errors of classification 7 and Problem pair 1) means a total of 4 errors on Problem 1 ($2x=6$; given in the Sample Test Problems) and the equivalent problem ($3y=12$; not given in the Sample Test Problems).

Although 254 students of the 274 tested had had more than two semesters of high school algebra, only 41% could solve a system of two linear equations in

two variables for a specified variable; only 33% could recognize that a system of inconsistent linear equations did not have a solution; only 34% could recognize any unusual difficulty in Problem pair 10; only 14% knew that a quadratic equation (in one variable) has two solutions; and fewer than 10% could solve a quadratic equation by use of the quadratic formula. More disturbing, however, is the observation that 2829 of the 4862 errors fall in classes 1, 2, 6, and 7.

ERROR ANALYSIS TABLE

Problem pair	Error class														Total
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1							4								4
2	3		4		1		10								18
3	1		9			40									50
4	5	1	16			11	9							12	54
5	13	2	5	4	2	41	28					3		2	100
6	85	66	46	7	23	38	20	9						5	299
7	171	48	26	5	72	8	8	1					105	10	454
8	34	52	12		7		102							1	208
9	71	36	46	3	27	6	24	13			1			7	234
10	180	33	5	5	11	3	6						199	6	448
11	1	2			3							455			461
12	24	16	7		14	3	5		88	5	1	284		6	453
13	27	46		6	7	3	140			6	11	58			304
14	207	129	13	4	75		4	1	1	1	34	1			470
15	213	174	15		36	1	2			7	17			2	467
16	242	101		18	46	7	6		12	3	5	10			450
17	2	80		16	2										100
18	52	68		3	41		115							9	288
Total	1331	854	204	71	367	161	483	24	101	22	69	811	304	60	4862

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

CONCERNING NEARLY-EQUAL ROOTS

E. C. KENNEDY, Texas College of Arts and Industries

When two real roots of an integral rational equation are nearly equal, it is often difficult to separate them. This difficulty is frequently due to the fact that we cannot readily approximate the roots by Newton's method, since that method is very tricky near a bend point. The scheme described below for isolating such roots is usually satisfactory.

By way of illustration, consider the quartic

$$f(X) = X^4 + 8X^3 - 70X^2 - 144X + 936 = 0, \tag{1}$$

taken from Lovitt's *Elementary Theory of Equations*, p. 138. We readily find that $f(3)=171, f(4)=8, f(5)=91$. By Descartes's rule of signs we know that the equation $f(X)=0$ has two or no real positive roots. Hence we conclude that if the equation has any positive roots they are near $X=4$. Accordingly, we shift the axes horizontally by setting $X=Y+4$, obtaining

$$F(Y) = Y^4 + 24Y^3 + 122Y^2 - 64Y + 8 = 0. \tag{2}$$

Discarding the first two terms and solving the quadratic equation $122Y^2-64Y+8=0$, we get $Y_1=.206$, $Y_2=.319$, or $X_1=4.206$, $X_2=4.319$. Hence *if* (2) has any positive roots, they lie inside the interval (Y_1, Y_2). This follows from the fact that the first two terms are positive for every $Y>0$. Hence, $F(Y)$ cannot be zero except possibly for values of Y that make the quadratic negative; that is, values between Y_1 and Y_2. We can get still better results by translating the axes by means of $Y=Z+a$, where $Y_1<a<Y_2$, in general. Usually one would take $a=(Y_1+Y_2)/2$, approximately. If we take $a=1/4$, say, we get

$$G(Z) = Z^4 + 25Z^3 + \frac{1123}{8} Z^2 + \frac{25}{16} Z + \frac{1}{256} = 0, \tag{3}$$

from which we obtain, by discarding the first two terms, $Z_1=-.00734$, $Z_2=-.00379$, or $X_3=4.24266$, $X_4=4.24621$.

Hence if (3) has any real roots near zero, they lie outside the interval (Z_1, Z_2). Thus *if* (1) has any positive roots, they lie *inside* the interval (X_1, X_2) and *outside* the interval (X_3, X_4), where $X_1<X_3<X_4<X_2$. To decide the question we find the sign of $G(p)$, where p is any value between Z_1 and Z_2. For example, we find that $G(-.005)<0$ or $f(4.245)<0$. This proves the existence of two positive real roots of equation (1). As a matter of fact, it is evident that $G(Z)=0$ has a root between 0 and Z_2 because $G(0)>0$ and $G(Z_2)<0$, since $Z^3(Z+25)<0$ for negative values of Z near zero and the quadratic vanishes at Z_2.

Since Z_1 and Z_2 are very small, it follows that X_3 and X_4 are close approximations to the two roots in question. These roots are $R_1=4.24264$, $R_2=4.24622$.

Had we taken some value for a a little different from .25, we might have found that the roots of $G(Z)=0$ rested inside the interval (Z_1, Z_2). However, the values obtained for X_3, X_4 would have been close approximations to the roots R_1 and R_2 of equation (1), and the point $X=(X_3+X_4)/2$ would very likely have separated R_1 and R_2. Incidentally, this number, $(X_3+X_4)/2$, should be a very close approximation to the abscissa of the bend point in this neighborhood.

As an exercise, it is suggested that the reader find the nearly-equal roots of

$$X^3 + 17X^2 - 46X + 29 = 0$$

to four or five decimal places. He might start off by setting $X=1+Y$. The roots are 1.21313 and 1.22952.

Note by the Editor. In applying Kennedy's process, the following observation may be of use. The roots of $Ax^2+Bx+C=0$ are given by

$$r = (- B \pm \delta)/2A = 2C/(- B \mp \delta), \qquad \delta^2 = B^2 - 4AC.$$

Hence, if δ^2 is relatively small, either of the expressions $-B/2A$ or $-2C/B$ will approximate, and lie between, the two roots. Thus in Kennedy's example, we need not solve $122Y^2-64Y+8=0$; for $\delta^2=64^2-4\cdot 8\cdot 122=8^2(64-61)$ is relatively small, and so $-2C/B=1/4$ is near and between the roots. We thus obtain the value of a without computing Y_1 and Y_2. R. J. W.

GENERATORS OF THE SYMMETRIC AND ALTERNATING GROUPS

LEONARD MILLER, Brooklyn College

The purpose of this paper is to prove a generalization of the following two well known theorems on permutation groups:

THEOREM A. *A permutation group on n distinct letters generated by the permutations*

$$(1)\qquad (a_1a_2),\ (a_1a_3),\ \cdots,\ (a_1a_n),$$

is the symmetric group.

THEOREM B. *A permutation group on n distinct letters generated by the permutations*

$$(2)\qquad (a_1a_2a_3),\ (a_1a_2a_4),\ \cdots,\ (a_1a_2a_n),$$

is the alternating group.

We now prove the following:

THEOREM. *A permutation group on n distinct letters generated by the permutations*

$$(3)\qquad (a_1a_2\cdots a_ma_{m+1}),\ (a_1a_2\cdots a_ma_{m+2}),\ \cdots,\ (a_1a_2\cdots a_ma_n),$$

where $m < n-1$, *is the symmetric group if* m *is odd and the alternating group if* m *is even.*

Proof. Throughout, we designate the permutation $(a_1a_2 \cdots a_ma_i)$ by P_i, $(i=m+1, \cdots, n)$. Since $m+1<n$, there are at least two P's.

Case I. Assume m is odd; write $m-1=2r$. Consider the products

$$(P_iP_j)^r P_i^2 P_j, \tag{4}$$

$$(P_iP_j)^{r+1} P_j, \tag{5}$$

$$(P_iP_j)^{r+1} P_j^{2r+3-t} P_i^{t-1} P_j, \tag{6}$$

where, in all cases, i and j are two distinct fixed integers in the range $m+1$, $m+2, \cdots, n$, and t is an integer in the range $3, 4, \cdots, m$. It is easy to verify that these products are, respectively,

$$(a_1a_2), \quad (a_1a_i), \quad (a_1a_t). \tag{7}$$

For example, in (6) we find that

$$P_iP_j = (a_1a_3 \cdots a_ma_ia_2a_4 \cdots a_{m-1}a_j),$$

and so

$$(P_iP_j)^{r+1} = (a_1a_ia_ja_ma_{m-1} \cdots a_2).$$

Hence a_1 is transformed into a_i by $(P_iP_j)^{r+1}$, is unchanged by P_j^{2r+3-t}, is transformed into a_{t-1} by P_i^{t-1}, and into a_t by P_j. Likewise, a_t is transformed successively into a_{t-1}, a_j, a_n, and a_1. Also, a_2 is transformed into a_1, a_{m+3-t}, a_1, and a_2; and similarly $a_3, a_4, \cdots, a_{t-1}, a_{t+1}, \cdots, a_m, a_i$, and a_j are invariant under (6). Hence $(6)=(a_1a_t)$.

It follows, then, that a permutation group containing all of the permutations of (3), m odd, will contain every one of the permutations of (1) and the theorem is proved for this case.

Case II. Now suppose m is even; write $m=2s$. Consider the products

$$(P_iP_j)^{s-1} P_i^3 P_j, \tag{8}$$

$$(P_iP_j)^s P_j^2, \tag{9}$$

$$(P_iP_j)^s P_j^{2s+3-t} P_i^{t-1} P_j, \tag{10}$$

where i, j, and t have the same meanings as before except that $t \neq 3$. Again, we can verify that these products are, respectively,

$$(a_1a_2a_3), \quad (a_1a_2a_i), \quad (a_1a_2a_t). \tag{11}$$

Hence, a permutation group which contains all the permutations of (3), m even, will contain every one of the permutations of (2) and the second part of the theorem is proved.

SOME REMARKS ON COÖRDINATE SYSTEMS

JAMES SINGER, Brooklyn College

Probably every teacher of analytical geometry has been asked by his students how it is possible to graph (in polar coördinates) an equation of the form

$$r = \theta^2, \tag{1}$$

since, they say, the left-hand side represents a distance and the right-hand side the square of an angle, whatever that may be. The answers, at least if one is to judge by many of the text-books, are frequently evasive or downright false. This brief paper is an attempt to clarify the situation by a discussion of the nature of coördinate systems and some related ideas.

The student is really not to be blamed for his confusion in analytical geometry. His trouble started some time earlier when careless text-books presented him with the "formulas"

$$d = rt, \tag{2}$$

$$A = lw, \tag{3}$$

to cite but two examples, which he was instructed to read

$$\text{distance} = \text{rate} \times \text{time},$$
$$\text{area (of a rectangle)} = \text{length} \times \text{width}.$$

He was left with the impression that he actually multiplied feet per second by seconds to get feet, in the one case, and feet by feet to get square feet in the other. He was led further astray by the "dimension" concept which seemed to prove that what he was doing was correct, for that principle asserts that the dimensions of distance, rate, and time are L, L/T, and T, respectively; hence, the two formulas are "obviously" right, aside, perhaps, from constants that are pure numbers!

To straighten him out in his thinking, let us examine the equation

$$a = bc,$$

which is of the same form as the first three equations but where the letters have been purposely chosen not to suggest any particular formula or coördinate system. This equation should mean, indeed, it can only mean, "The product of the numbers b and c is the number a." The student should be taught that the arithmetic operations of addition, subtraction, multiplication, and division operate on *numbers only*. This is the crux of the whole matter and the essential point for a clear understanding of all the difficulties.

Once the student realizes that the arithmetic operations act on numbers only, he is ready to "interpret" equations as formulas of applied mathematics. He should learn to read equation (2) somewhat as follows, "If r is the number of feet per second and t is the number of seconds a body travels, then d is the number of feet through which it will move." Or, he might say, "If distance,

rate, and time are measured in correct units, then the number of units in the distance is equal to the number of units in the rate multiplied by the number of units in the time." Similarly, the student should read equation (3) as, "If the area, length, and width of a rectangle are measured in proper units, then the number of units in the area is equal to the number of units in the length multiplied by the number of units in the width." (In this connection, note that we may measure the length in inches, say, and the width in feet if we measure the area in inch-feet, where the unit area, inch-foot, is a rectangle whose dimensions are an inch and a foot.)

In short, an equation is a relationship between numbers, but these numbers may be given "dimensions" to yield formulas of applied mathematics.

These preliminary remarks will throw some light on the nature, meaning, and significance of the graph of an equation in analytical geometry. We begin with a consideration of coördinate systems. We use the euclidean plane for the sake of convenience but our remarks will apply, with slight modifications, to general abstract spaces.

An ordered pair of real numbers, written (u, v), is called a label or coördinate set. (We again have chosen letters not suggestive of any particular coördinate system.) We determine a coördinate system for the plane by associating or attaching one or more labels to each point of the plane. *This association may be effected in any manner whatsoever;* however, in order to be useful, a coördinate system should satisfy the following three conditions:

(i). Each point of the plane shall possess at least one label.

(ii). If P and Q are distinct points, no label attached to P shall be the same as any label attached to Q.

If (u, v) is a label attached to a point P, we call u and v coördinates of P.

(iii). If a point P moves continuously in the plane, the coördinates of any one of its labels shall vary continuously.

Henceforth, when we talk about coördinate systems, it shall be understood that the three conditions are satisfied. We may now define the graph of an equation $f(u, v) = 0$ as the set of all points in the plane whose coördinates satisfy the equation. It is then obvious that the shape or appearance of the graph of an equation will depend on the particular manner in which the coördinate system for the plane was determined.

Before we proceed, let us remark again that the variables in an equation and the coördinates of a point are numbers, just numbers. Hence, were it possible to pin to each point of the plane a label in the same manner that we attach an identification tag to a suitcase, a student would not be perplexed by equation (1) or the possibility of graphing it. But since we cannot pin a label to each and every point, we must invent some device which will enable us to determine the coördinates of any point. The usual cartesian and polar coördinate systems are coördinate systems in which conventional but particular devices are used to determine the coördinates of the points of the plane. Herein is the source of the student's trouble, for he automatically and erroneously imparts to the coör-

dinates of a point the dimensions used in the device for setting up the coördinate system.

We illustrate for the sake of emphasis. Let $(2, -3)$ be the coördinates of a point P in a polar coördinate system. The coördinates of P are numbers, they are not distances or angles. However, in the device used to determine the coördinates of P, it was true that 2 was the number of (linear) units in the distance from P to the pole O and -3 was the number of (angular) units in the angle between OP and the polar axis.

Another remark to help clear up another source of confusion. In setting up a device for the determination of a polar coördinate system, we are at liberty to choose any unit we please for the measurement of distance and any unit we please for the measurement of angles. Many text-books state that in graphing equations of type (1) (in distinction to equations of type $r=\sin\theta$), it is necessary to measure angles in radians. This confusion arises partly from a misunderstanding of the nature of coördinates and partly from an erroneous but often given definition of the radian as the quotient of two distances. If we define the radian as the angle formed by two radii of a circle that intercept an arc equal in length to the radius or to $1/2\pi$ of the circumference just as the degree is the angle formed by two radii of a circle that intercept an arc equal in length to $1/360$ of the circumference, we see that the difference between degrees and radians is similar to the difference between feet and meters, say. Hence, it is not necessary to measure angles in radians in setting up a polar coördinate system or in graphing equation (1). (The graph of this equation is a parabola in rectangular cartesian coördinates and a pair of spirals in polar coördinates.)

One final remark. The variables in an equation may be given "dimensions" as previously explained, but when the equation is graphed, the variables may take on other significances. Thus, in the equation

$$r = \sin\theta,$$

the numbers r and θ may be given the dimensions distance and time, respectively. The equation can then be interpreted as an equation for simple harmonic motion. But in graph work, the usual devices determine θ as the measure of a distance in cartesian coördinates and as the measure of an angle in polar coördinates.

To sum up: the variables of an equation are pure numbers; the coördinates of a point in any coördinate system are pure numbers. But the variables in an equation may be given "dimensions," often in many ways, to yield formulas of applied mathematics. In some cases, it may be desirable to give several dimensions simultaneously to one or more of the variables in an equation. Likewise, the coördinates of a point may be given dimensions; the dimensions used in the device for setting up the coördinate system or other dimensions may be arbitrarily assigned. Finally, a pair of numbers may have one set of dimensions when considered as a solution of an equation and another set of dimensions when considered as the coördinates of a point on the graph of the equation.

NOTE ON THE NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

W. E. MILNE, Oregon State College

In an earlier paper* I proposed a method of solving differential equations which consisted essentially of a step-by-step process using the formula

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_n' - y_{n-1}' + 2y_{n-2}') + \frac{28}{90}h^5y^{(5)} \tag{1}$$

as a predictor, and Simpson's rule

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n+1}' + 4y_n' + y_{n-1}') - \frac{1}{90}h^5y^{(5)} \tag{2}$$

as a corrector. The start of the computation requires four consecutive known values of y. In the original paper it was suggested that these might be found by a few terms of Taylor's series. Later L. R. Ford† gave another method, and recently F. J. Turton‡ has published a number of formulas that give the starting values by successive approximations.

The purpose of this paper is to present a set of formulas for the starting values which have proved to be rapid and convenient in practice and which insure adequate accuracy. In these formulas, as in (1) and (2) above, h denotes the length of the interval between the equally spaced values of the independent variable, and the fifth derivative of y appearing in each error term is to be taken at some intermediate point between the extreme abscissas occuring in the particular formula.

The formulas are,

$$y_1 = y_0 + \frac{h}{24}(7y_1' + 16y_0' + y_{-1}') + \frac{h^2y_0''}{4} - \frac{1}{180}h^5y^{(5)}, \tag{3}$$

with its mate

$$y_{-1} = y_0 - \frac{h}{24}(y_1' + 16y_0' + 7y_{-1}') + \frac{h^2y_0''}{4} + \frac{1}{180}h^5y^{(5)}, \tag{4}$$

and

$$y_2 = y_0 + \frac{2h}{3}(5y_1' - y_0' - y_{-1}') - 2h^2y_0'' + \frac{7}{45}h^5y^{(5)}, \tag{5}$$

together with Simpson's rule (2).

In using these formulas we have x_0 and y_0 given. Then values of y_0' and y_0'' are found from the differential equation and from the equation obtained by dif-

* W. E. Milne, this MONTHLY, vol. 33, 1926, pp. 455–460.

† L. R. Ford, Differential Equations, McGraw-Hill, 1933, pp. 147–149.

‡ F. J. Turton, Philosophical Magazine, vol. 28, 1939, pp. 381–384.

ferentiation, respectively. Trial values for y_1' and y'_{-1} are given by $y_1'=y_0'+hy_0''$ and $y'_{-1}=y_0'-hy_0''$. From these, with (3) and (4), we obtain trial values of y_1 and y_{-1}, then compute y_1' and y'_{-1} from the differential equation, and recompute y_1 and y_{-1} using the improved values. The process is repeated until no change occurs. Note that $h^2y_0''/4$ is calculated once for all, as it does not change in the recomputations. Note also that the error terms of (3) and (4) are one-half the error term of Simpson's rule, so that we may be sure that the accuracy of the starting values is fully as good as the subsequent process of integration justifies. Next a trial value of y_2 is calculated by (5), and checked and rechecked by Simpson's rule until no change occurs. The four values y_{-1}, y_0, y_1, y_2, needed for the start of the computation are now ready.

Example. We take the equation $y'=x-y$ with initial values $x_0=0$, $y_0=2$. Then $y_0'=-2$, $y_0''=3$. Then using $h=0.1$ we have the following results:

x	y	y'	
−0.1	2.215	−2.3	
0.0	2.000	−2.0	first approximation;
0.1	1.815	−1.7	
−0.1	2.2155	−2.315	
0.0	2.0000	−2.000	second approximation;
0.1	1.8145	−1.715	
−0.1	2.21551	−2.3155	
0.0	2.00000	−2.0000	third approximation;
0.1	1.81451	−1.7145	
−0.1	2.21551	−2.31551	
0.0	2.00000	−2.00000	first approximation of y_2;
0.1	1.81451	−1.71451	
0.2	1.6562	−1.4562	
−0.1	2.21551	−2.31551	
0.0	2.00000	−2.00000	final values.
0.1	1.81451	−1.71451	
0.2	1.65619	−1.45619	

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Development of Mathematics. By E. T. Bell. New York and London, McGraw-Hill Book Company, Inc., 1940. 13+583 pages. $4.50.

Statistical Procedures and Their Mathematical Bases. By C. C. Peters and W. R. Van Voorhis. New York and London, McGraw-Hill Book Company, Inc., 1940. 13+516 pages. $4.50.

Mathematical Tables. Volume VIII. Number-Divisor Tables. By J. W. L. Glaisher. British Association for the Advancement of Science. Extended and edited by the Committee for the Calculation of Mathematical Tables. Cambridge, University Press; New York, Macmillan Company, 1940. 10+100 pages. $4.25.

College Mathematics. A First Course. By W. W. Elliott and E. R. C. Miles. New York, Prentice-Hall, Inc., 1940. 13+396 pages. $3.00.

The Weight Field of Force of the Earth. By W. H. Roever. St. Louis, Washington University Publications. New Series. Science and Technology, No. 11. 1940. 84 pages. $1.50.

Punched Card Methods in Scientific Computation. By W. J. Eckert. New York, The Thomas J. J. Watson Astronomical Computing Bureau, Columbia University, 1940. 9+136 pages. $2.00, cloth; $1.75, paper.

An Introduction to Abstract Algebra. By C. C. MacDuffee. New York, John Wiley and Sons; London, Chapman and Hall, 1940. 7+303 pages. $4.00.

A Treatise on Advanced Calculus. By P. Franklin. New York, John Wiley and Sons; London, Chapman and Hall, 1940. 14+595 pages. $6.00.

A Diagnostic Study of Students' Difficulties in General Mathematics in First Year College Work. By Elizabeth N. Boyd. (Contributions to Education, No. 798.) New York, Bureau of Publications, Teachers College, Columbia University, 1940. 152 pages. $1.85.

REVIEWS

College Algebra. By P. R. Rider. New York, The Macmillan Company, 1940. 9+372 pages. $2.00.

This text-book, designed for first-year students in colleges and technical schools, was used in multigraphed form in more than twenty sections of students in the Washington University before its final printing. The book is attractive in appearance; the typography is excellent throughout. The earlier part of the book contains a thorough review of elementary and intermediate algebra; the remaining part includes the essential features of the usual college algebra. In addition, more than the usual amount of space is devoted to theory of equations

and to probability; and to meet the needs of the many students who may engage in statistical work, an elementary chapter on finite differences is included. The problems are interesting and well chosen; answers are given for the odd-numbered exercises. This book should be a very satisfactory text.

FLORENCE M. MEARS

Handbook of Mathematical Tables and Formulas. Second edition. Compiled by R. S. Burington. Sandusky, Ohio, Handbook Publishers, Inc., 1940. 275 pages.

The new edition differs only slightly from that of 1933 (reviewed in this MONTHLY, 1933, p. 554). The table of integrals has been increased from 331 to 363, and that of definite integrals from 51 to 71. The table of square and cube roots has been extended, and that of circumferences and areas of circles is now a separate table. There has been added a table of logarithms of factorials from 1 to 200, and a tabulation of the factorials and their reciprocals from 1 to 20. According to a statement in the front of the book, several reprintings since the original 1933 publication have corrected certain errors in the latter.

R. A. JOHNSON

Mathematisch-Astronomische Blätter. Heft I. Mathematisch-Astronomische Sektion der Freien Hochschule für Geisteswissenschaften, Dornach, Switzerland, 1940. 96 pages. Fr. 3.50 (Swiss).

This is the first number of a new series of publications undertaken by the Goetheanum of Dornach, primarily for teachers in the secondary schools. It presupposes only a knowledge of school mathematics. Later numbers will appear as material warrants, at irregular intervals. This first number contains essays of mathematical or philosophic content, with predominantly pedagogic interest. The papers are well written, and attractively printed and arranged.

VIRGIL SNYDER

Calculus. By C. K. Robbins and Neil Little. New York, The Macmillan Company, 1940. 7+398 pages. $3.25.

This book is a conventional text for a first course designed to develop manipulative skill in the use of calculus. It is well suited to this purpose.

The customary division of the subject-matter into the differential and the integral calculus is made here. Two chapters are devoted to the formal solution of differential equations, including some of those of higher order. The trapezoidal and Simpson's rule are also treated. The authors introduce infinite series early in the book with a formal treatment of Taylor's series which serves as an application of the derivative. The algebraic treatment of series is given in the last chapter.

The authors avoid controversies over rigor by assuming such results as are required, but which cannot be rigorously established in an elementary text. They introduce the derivative with an intuitive notion of "limit," so that the student may become familiarized with some formal differentiation before con-

sidering the definition of derivative in general terms. Consideration of limit theory is reserved for the eighth chapter. Integration is emphasized as the inverse of differentiation. All points are amply illustrated by examples, and a great number of exercises is provided.

The whole book appears in a neat and inviting format.

S. B. LITTAUER

Non-Euclidean Geometry or Three Moons in Mathesis. Second edition. By Lillian R. Lieber; drawings by H. G. Lieber. Lancaster, Pa., The Science Press, 1940. 40 pages. $1.00.

In the new edition of this popular short account of non-euclidean geometry the author has amplified certain sections and has corrected some small inaccuracies in the original text. The drawings are all new and are more subdued than the original ones.

H. W. BRINKMANN

Table of the First Ten Powers of the Integers from 1 to 1000. Works Progress Administration of New York City, Official Project No. 365-97-3-11, 1939. 8+80 pages.

Tables of the Exponential Function e^x. Federal Works Agency, Work Projects Administration for the City of New York, Project No. 765-97-3-10, 1939. 16+535 pages.

These volumes belong to a series of mathematical tables which are being prepared by the "Project for the Computation of Mathematical Tables" conducted by the Works Progress Administration (and its successor, the Work Projects Administration) under the sponsorship of the National Bureau of Standards, Washington, D. C., of which the director is Dr. Lyman J. Briggs.

The table of powers of the integers is completely described by its title. Only a limited number of copies of this table seem to be available, all in mimeograph form, because it was learned during the work that a more extensive table of powers was being prepared by the British Association for the Advancement of Science.

The set of tables of e^x is a considerably more ambitious effort, and should prove to be an important addition to the literature of such tables. The contents are as follows:

Table I. e^x, $0.0000 \leqq x < 1.0000$, to 18 places of decimals at intervals of .0001.

Table II. e^x, $1.0000 \leqq x < 2.5000$, to 15 places of decimals at intervals of .0001.

Table III. e^x, $2.500 \leqq x < 5.000$, to 15 places of decimals at intervals of .001.

Table IV. e^x, $5.00 \leqq x < 10.00$, to 12 places of decimals at intervals of .01.

Table V. e^{-x}, $0.0000 \leqq x < 2.5000$, to 18 places of decimals at intervals of .0001.

Table VI. e^x and e^{-x}, $0.000000 \leqq x \leqq 0.000099$, to 18 places of decimals at intervals of .000001.

Table VII. e^x and e^{-x}, $1 \leqq x \leqq 100$, to 19 significant digits at intervals of 1.

Table VIII. e^x and e^{-x}, $1 \times 10^{-10} \leqq x \leqq 9 \times 10^{-7}$, to 18 places of decimals at decimal intervals.

Tables VII and VIII were taken without recomputation from C. E. Van Ostrand's *Tables of the Exponential Function, etc.*, National Academy of Sciences, vol. XIV, fifth memoir. The entire set of tables is reproduced by the photo-offset process.

The methods used in the computation and checking of these two volumes are set forth in some detail in Introductions. It is claimed that each volume is entirely free from error, and the precautions taken seem to give considerable weight to this claim.

J. H. CURTISS

Introduction to the Calculus. By Arnold Dresden. New York, Henry Holt and Company, 1940. 12+428 pages. $3.40.

This book, designed to be an introduction to the calculus for American sophomores, differs widely in emphasis and method from the customary text on this subject. The tone of the book is determined by the first two chapters which contain 24 definitions and 27 theorems pertaining to point sets, limits, and continuous functions. The definitions are carefully stated and the theorems are proved in detail. Some of the topics discussed are: point sets, finite and infinite, bounded and unbounded, denumerable and undenumerable, monotone; classes of numbers with an irrational number defined as a "cut" in the set of rational numbers; the theorem of Bolzano-Weierstrass; limits of variables and sequences; fundamental theorems on limits; functions continuous at a point and on an interval; properties of continuous functions; and the Cauchy criterion for convergence. With these two chapters as a background, the author proceeds to develop the calculus, stating carefully the definitions and giving analytical proofs of most of the theorems. Though, as is necessary in a beginning course, the conditions under which some theorems are proved are broader than necessary, the author has constructed a logical analytical calculus and has minimized the need for appeal to geometric intuition.

A partial summary of the rest of the book is as follows. In Chapter III, the derivative is introduced, using the concept of a moving particle. The question of the existence of the derivative of a function, the concept of forward and backward derivatives at a point, and the idea of derivatives of higher order are immediately taken up. Chapter IV is devoted to the technique of differentiation, differentiation of implicit functions, and partial derivatives of functions defined explicitly and implicitly. Chapters V–VI discuss maxima and minima, infinitesimals, differentials, infinites, law of the mean, and indeterminate forms. Chapters VII–VIII study the trigonometric, inverse trigonometric, exponential, logarithmic, and hyperbolic functions. The existence of the limit of $(1+1/n)^n$ as $n \rightarrow \infty$ is proved. Chapter IX is the application of the calculus to geometry.

Length of curve is defined and curvature, curve tracing, singular points, and asymptotes of plane curves are discussed. The chapter ends with tangent planes and normal lines to surfaces. Chapter X is on polar coördinates and Chapter XI is an application of the calculus to mechanics, introducing vectors.

In Chapter XII, the definite integral is defined as the limit of the sum $\sum_{k=0}^{n}(p_{k+1}-p_k)g(q_k)$, in which the points $p_0=a$, $p_1, \cdots, p_n$, $p_{n+1}=b$ represent a partition of the interval (ab) of norm δ, and $p_k \leqq q_k \leqq p_{k+1}$, as $\delta \rightarrow 0$. The area under a curve is *defined* as this same limit. The existence of this limit is proved for continuous functions. The method of proof is essentially to set up the Darboux upper and lower sums and to show that these sums approach limits that are independent of the mode of partition and that the limits approached by the two sums are the same. The proof is given in detail. The rest of the chapter is given over to the properties of the definite integral and it is shown that $\int_a^b g(x) dx = G(b)-G(a)$, where $D_xG(x)=g(x)$ and $g(x)$ is continuous on (ab), and thus the fundamental theorem is proved and indefinite integrals are introduced. Chapters XIII–XIV–XV give the technique of integration and applications of the single definite integral. The use of Duhamel's theorem is avoided by the proof of the following theorem, stated in the author's rather unusual notation. *If $g(x)$ and $h(x)$ are continuous functions on the interval (ab), then the sums $\sum_{k=0}^{n}(p_{k+1}-p_k)g(q_k)h(\bar{q}_k)$ and $\sum_{k=0}^{n}(p_{k+1}-p_k)g(q_k)h(q_k)$ tend to the same limit as the norm δ of the partition $p_0=a$, $p_1, \cdots, p_n$, $p_{n+1}=b$ tends to zero, q_k and $\bar{q}_k$ being arbitrary points in the interval (p_kp_{k+1}), and the limit is $\int_a^b g(x)h(x)dx$.*

Chapters XVI–XVII–XVIII discuss Taylor's theorem, expansion of functions, infinite series and uniform convergence, and improper integrals. Chapter XIX is a rather short chapter on double and triple integrals devoted largely to theory. The definitions and proofs involved are entirely analogous to the treatment of the single integral. The fundamental theorems are arrived at by using squares and cubes as elements. Chapters XX–XXI introduce the student to differential equations. Equations of the first order and first degree and linear equations are discussed and the method of solution by the use of series is mentioned.

The problems throughout the text seem adequate and well chosen, and the appearance of the book is very satisfactory.

The author pleads, in the preface, for a more thorough understanding of the fundamental principles of the calculus and for a knowledge of the exact conditions under which its formal processes are applicable, believing that this knowledge is equally of benefit to those who desire to apply the calculus to physics, engineering, and the social sciences and to the student of pure mathematics. He has succeeded in writing a book that should go a long way towards accomplishing his aims and he believes that it is not beyond the abilities of the American sophomore to understand and appreciate. Whether the average sophomore can appreciate it or not may be a matter of opinion, but the reviewer believes that every teacher of an introductory course in the calculus should read this book.

D. S. MORSE

Advanced Calculus. By C. A. Stewart. London, Methuen & Company, Limited, 1940. 18+523 pages. 25 s.

As in the case of some advanced calculus texts published recently in the United States, the theory of functions of a real variable is included in the subject-matter, and in fact is in no way segregated from the formal developments. Such a text may prove admirable in many cases where the student, ending his mathematical work with the course in advanced calculus, obtains an introduction to the "real" mathematics. However the reviewer believes that there is a good deal to be said for the sort of text which omits the real variable theory, as the inclusion of this material may require up to an extra semester before completing the formal aspects. Where a class in an American college may consist of a majority of students of engineering and those majoring in natural sciences, who have difficulty in completing their calculus early enough to make use of it in their other studies, this delay is a handicap. The students majoring in mathematics will in most cases repeat the work in real variables in some later course.

The text is very inclusive. The one subject omitted which is often found in advanced calculus texts is differential equations. The arrangement of topics seems good, though perhaps not what one would want to follow if giving the type of course which does not include the real variable theory. The exercises are numerous, a set of between one and two hundred appearing at the end of each chapter.

Chapter I is on definition of function, Cantor definition of real number, sequences, limits, derivatives, graphs. Chapter II introduces limit point obtained by the pinching process, bounds of sets of numbers, Lebesgue measure, properties of continuous functions, functions of two variables, differentials, partial derivatives, Taylor's expansion with the remainder. Chapter III is on implicit functions, inverse functions, algebraic functions, algebraic curves, surfaces $z=f(x, y)$. Chapter IV has functions defined by sequences, infinite series, tests for convergence, power series, operations with power series, the elementary transcendental functions, functions defined by double series. Chapter V is on the indefinite integral, the definite integral, infinite (improper) integrals, differentiation under the integral sign, integration of power series, area under a curve, approximate integration. Chapter VI is on Jacobians, implicit function theorem, functional dependence, transformations. Chapter VII begins with indeterminate forms, continues with maxima and minima of functions of one variable and of several variables, and the method of Lagrange. Chapter VIII is devoted to vectors, analytic geometry of 3-space, differential geometry of curves and of surfaces, and tensors. Chapter IX has double integrals, applications to finding volumes and areas of surfaces, Green's theorem in the plane, $\int Pdx+Qdy$, triple integrals, transformation of multiple integrals, surface integrals, applications to mass, center of gravity, moment of inertia, fluid pressure. The formulas for physical quantities as integrals are not "derived" as in American texts, but immediately written down as definitions after viewing the corresponding formulas for particles. Chapter X introduces complex numbers, functions of a com-

plex variable, convergence of series of complex quantities, analytic functions, Cauchy's theorem, Taylor's series, differentiation and integration of power series, conformal representation, Laurent series, special functions, calculation of real definite integrals by contour integration. Chapter XI on infinite series and products begins with tests for convergence, then takes up uniform convergence and properties connected with it, infinite products, various expansions of analytic functions, ending with non-convergent series summable $(C, 1)$.

The principal criticism to be made concerns the errors occurring, particularly in the early parts of the text. The reviewer has read many parts of the book in detail, and lists below most of the errors which he noted. One other criticism is that frequently a statement is made with no indication to the reader as to why it is true.

Comments and corrections. Pages 3 and 4. The properties considered should apply here only to rational numbers, as they are the only ones introduced up to this point. The same kind of treatment should be given about the top of page 5 for real numbers. Also, an essential step of the development is omitted, namely a proof that if l is the real number defined by the sequence $\{a_n\}$ of rational numbers, then a_n tends to the limit l when n tends to infinity.

Page 8, 1.32. The following corrections are obvious. Insert "an $\epsilon>0$ and" after "find" in the third line, and delete "given $\epsilon(>0)$" in the fourth line. Replace "an unbounded" by "a non-convergent" in the eighth and ninth lines.

Page 13, 1.6, Note (ii). A more general definition of derivative, in terms of h_1 and h_2 which approach zero independently, is given. The statement that this may exist when the ordinary derivative does not is false. The reverse is true. The example in the text is irrelevant, since in it h_1 and h_2 are dependent.

Page 14, 1.66. The restriction $f(x)\neq f(x_1)$ means that the formula $dy/dx=(dy/dz)(dz/dx)$ is not proved in general.

Pages 26 and 27, 2.01. The statement at the top of page 27 is false, as is shown by the following example. Let the property for a given interval be that the function shall be positive for at least two-thirds of the interval; and take a function which is positive for $0\leqq x<1$ and also for $2<x\leqq 4$, but negative for $1<x<2$. Then the point P can be the point $x=2$, yet the property is not satisfied for the interval $|x-2|<\epsilon$ if $\epsilon<1$.

Page 27, the last sentence of 2.01 is false. For example, if $f(x)<0$, $(0\leqq x<1)$, $f(x)=0$, $(1\leqq x\leqq 2)$, $f(x)>0$, $(2<x\leqq 3)$, and $f(x)$ is continuous, then $f(0)$ and $f(3)$ have opposite signs, yet there is no point ξ, $(0<\xi<3)$, neighboring which $f(x)$ has opposite signs. The error is repeated in the second sentence of page 33.

Page 27, 2.021. There is no proof of the existence of the greatest and least limit points. Lines 9 and 11 of 2.021, "every" should be "any"; otherwise the upper bound would not bound from above! Here again the existence of the numbers defined is not proved.

Page 27, 2.022. Since $a_1=a_2=\cdots$ is a possibility for a sequence of quantities, limiting point should be redefined so as to apply to sequences, and not merely to sets of distinct points. The same applies to upper limit and lower limit.

Page 30. It might well be mentioned that the measure defined here is commonly called Lebesgue measure.

Top of page 36, the argument is incomplete, because uniformity is assumed without proof at the end of the first line, page 36.

Page 36, line 6 from bottom. Here we have a second definition of continuity, with no reference to the earlier definition on page 10 (repeated on page 32), nor of any proof of equivalence of the two definitions.

Page 38, theorems on point sets stated without sufficient hypotheses. One or more of the following is necessary for each of the properties mentioned: that the point set contain an infinite number of points; that it be bounded; closed; connected.

Page 39, 2.32, second line. Instead of dz, write δz. (This is the first misprint mentioned here, though a few obvious ones occur before this point.) In 2.31 $f(x, y)$ is defined to be differentiable at a point if $\delta z = A\,\delta x + B\delta y + o(\delta\rho)$, where $\delta\rho^2 = \delta x^2 + \delta y^2$, and $o(\delta\rho)/\delta\rho$ approaches zero with $\delta\rho$. However, it is *not* proved that if f, f_x, and f_y are continuous, then $f(x, y)$ is differentiable. The unfortunate result is that the student cannot know to what functions the formulas of partial differentiation can be applied, except by taking on faith the theorem stated without proof in 2.32 (iii).

Page 40, 2.34, fallacious proof. It is assumed without proof that the dz obtained by substitution is the same as would have been obtained from u, v, du, dv using the definition. Thus an important theorem is assumed, which can be proved only *after* the last line of 2.34 is established.

Page 41, line 6 of 2.51, the statement is false. *E.g.*, if f is a function of x only whose first derivative to x exists but is not continuous, then f_{xy} and f_{yx} both exist and are continuous (equal zero), yet f_x is not continuous.

Page 82, fifth line before the examples. Change "oscillation is not zero" to "saltus is $>k$." Otherwise we could not even infer discontinuity at α.

Page 91, third line of 4.26. The "series" referred to is a double series, and double series are not taken up till page 101. Hence the paragraph must be rewritten or deferred.

Page 92, 4.31. The easily proved theorem that if a power series in x converges for $x = x_1$, then it converges (absolutely) for $|x| < |x_1|$, is not given here. So far as is shown in the text, if lim $|a_n/a_{n+1}|$ does not exist, the series might converge for some values of x with $|x| > R$, the radius of convergence!

Page 92. Misprint, last line of 4.31, R should be $1/R$.

Page 102, 4.53. Incomplete proof, similar to that on page 36.

Page 107, 4.61. Here again the easily proved principal theorem for convergence of a double power series is omitted.

Page 142, fifth line from bottom, statement that "$u_n(x)$ cannot increase so that U_n is . . . non-increasing" is false, as may be shown by a simple example. However, the material for a complete treatment is given on page 143.

Page 144, fifth line from end of 5.323, after "*i.e.*" insert "except."

Page 144, next to last line, $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$ should be a definition, not

something proved, since up to this point b has been greater than a in $\int_a^b f(x)dx$.

Page 153, lines 3 and 4 of 5.51. Incorrect reason is given. The sum-definition cannot be used when the integrand becomes infinite, for, given $\delta > 0$ arbitrarily small and M arbitrarily large, one can always make the sum $> M$, with all $x_r - x_{r-1} < \delta$.

Page 158, line 4. Since lim $(a_n)^{1/n}$ may not exist, the proof is not general.

Page 202. The statement just after the Note in 7.21 is false, as is shown by the example $h^2 - k^2 + h^3$. If $h = \epsilon$ and $k = (\epsilon^2 + \epsilon^3/2)^{1/2}$, $\epsilon > 0$, then $h^2 - k^2 < 0$ but $h^2 - k^2 + h^3 > 0$. The error is repeated on page 206, in 7.32.

Page 227, bottom. The proof requires elaboration, as the point $(x+dx, y+dy, z+dz)$ is in general not on the curve.

Page 261, lines 3 and 4. The statements regarding K_n, E_n, and I_n are false. For example, if the curve is the line $y = 5$, $2 \leqq x \leqq 8$, in the square $0 \leqq x \leqq 10$, $0 \leqq y \leqq 10$, then $K_9 < 8 \times (10/9) < 9$, but $K_{10} = 9 \times 2 = 18 > K_9$.

Page 265, lines 10 and 11. The argument is fallacious, since if we have merely proved that a sum depending on rectangles with sides δx and δy approaches a limit as $\delta x > 0$ and $\delta y > 0$ approach zero, we cannot assume that we will get the same limit by first letting δx approach zero, and then δy.

Page 273, middle of the page. The statement that "a unique normal . . . does not exist at a singular point" is contradicted at the poles of a sphere for which longitude and co-latitude are taken as parameters.

Page 282, 9.33. Treatment is incomplete. If V is single-valued, as seems to be *assumed*, then W *must* be single-valued.

Page 335, 10.14. Only the case in which lim $|a_n/a_{n+1}|$ exists is covered (for power series). The general theorem referred to in connection with the work on pages 92 and 107 is again not given.

Page 340, 10.22. The proof is incomplete because even if two points z_1 and z_2 lie inside the circle $|z - z_0| < \delta$, it does not follow that

$$\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \epsilon$$

unless $z_1 = z_0$. This error may cause considerable trouble, as the entire chapter on complex variables is based on the property taken up here.

Page 348, 10.34. The treatment is incomplete as it is assumed that the two paths do not intersect.

Page 349, last line. The *method* is incorrect for the first inequality.

Page 428, 11.18. The statement that the power series is uniformly convergent for $|z| < R$ is false. It seems that the rest of this paragraph should be deleted also.

Page 428, 11.19, II. It is assumed that if the radius of convergence of the power series is 1, then lim $|a_n/a_{n+1}| = 1$. A counter-example is furnished by $1 + z^2 + z^4 + z^6 + \cdots$.

A. B. Brown

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

Beginning with this issue, the name of this department will appear under the new heading "Clubs and Allied Activities." During the past several years the number of mathematics clubs has been increasing. The activities and work carried on by these clubs will continue to hold the major attention of this section. At the same time, we should like to report on activities in colleges and universities which do not sponsor formal organizations, but which do serve the extra-classroom interests of their students. Such activities as the undergraduate lectures, reading lists in mathematics, contests, creative writing and undergraduate research, joint meetings, exhibits, mathematical films and slides, and special events will be reviewed. Will department representatives and students in these universities and colleges send us reports on the interesting and unusual activities which are carried on by them and for them.

MATHEMATICS CONFERENCE AT ALBION COLLEGE

Seventy-five students and faculty members from colleges in Michigan attended a state-wide mathematics conference sponsored by the *Michigan Alpha* chapter of *Kappa Mu Epsilon* at *Albion College* on Saturday, May 11, 1940. Representatives were present from Adrian College, Alma College, Central State Teachers College, Hillsdale College, Hope College, Kalamazoo College, Michigan State College, Michigan State Normal College, University of Michigan, Wayne University, and Western State Teachers College. *Ohio Alpha* chapter of *Kappa Mu Epsilon* at *Bowling Green University* also sent a delegation. The morning session was devoted to a program of papers presented by students. This was followed by a luncheon and social session combined with a business meeting. It was decided to continue this undergraduate program and make it a permanent feature of the Michigan universities and colleges. This conference had its inspiration in the national conferences held bi-annually by National Kappa Mu Epsilon. The program presented by the students consisted of the following papers:

Trisection of an angle; squaring the circle, by Abner Robinson of Michigan State Normal College

Mathematics in economics, by C. R. Simms of Wayne University

The slide rule solution of quadratic equations, by Cecil Sessions of Albion College

An attempted solution to the problem: $X^2+Y=7$, $X+Y^2=11$, by Edsel Farnham of Hillsdale College

"Equations of polygons" (C. O. Oakley, this MONTHLY, 1935, pp. 476–487), by Elaine Van Aken of Michigan State College

A few mathematical recreations, by Cornelius Groenewoud of Hope College

It pays to be simple, by Robert Esling of Albion College

Editorial Note. Could similar meetings be held in other states? Student participation in programs at such meetings should be encouraged. If this interest continues to grow, it might lead to some form of junior membership for undergraduate students in the Mathematical Association of America. State and sectional meetings of the Association might well set aside sections at their annual meetings for student papers for junior and potential members! E. H. C. H.

CHICAGO MATHEMATICAL MODEL CLUB

This is an informal group with no official list of members. High school students, college students, graduate students, high school teachers, college and university instructors and others have attended the meetings.

The organization was started three years ago by Professor W. A. Spencer of Armour Institute of Technology and W. W. Gorsline of Wright Junior College. Speakers at meetings have included representatives of plastic manufacturers who have explained what could be done with their product. Some of the other programs were lecture demonstrations at schools where models or charts were on exhibition, as indicated in the following table:

Speaker	*School*	*Charts and Models*
W. A. Spencer, L. R. Ford, and others	Armour Institute	For college mathematics. Faculty conceived. Made in school shops.
F. Fisher	Wright J.C.	For descriptive geometry. Student made.
W. A. Richards and others	Morton J.C.	For junior college mathematics. Student made.
J. A. Clear	Wright J.C.	Slides for descriptive geometry. Made by WPA.
M. J. Newell	Evanston H.S.	For solid geometry. Student made.
Ruth G. Mason	Wright J.C.	For freshman mathematics. Made by WPA.
Ida Fogelson	Bowen H.S.	For plane geometry. Student made.
Messrs. Maholy and Nagy	School of Design	For architecture and allied arts. Student and faculty made.
E. P. Lane	University of Chicago	For university mathematics. Professionally made.

Officers for 1939–40 were: President, W. W. Gorsline, Wright Junior College; Vice-President, W. A. Richards, Morton Junior College; Secretary-Treasurer, Ruth G. Mason, Wright Junior College.

STUNTS

Supplementing the list of questions for determining an elusive MQ (Mathematics Quotient) suggested in the November 1939 issue of this department, we have the following prepared by Isham Pemberton, Vice-President of the *Mathematics Club* of *North Texas State Teachers College* at *Denton*:

1. Write the equation whose roots are the reciprocals of the roots of the equation $ax^2+bx+c=0$.

2. Two eggs of similar shape differ only by the fact that the smaller is one-fifth shorter than the larger. What is the ratio of volume of the two eggs?

3. The couplet, "See, I have a rhyme assisting my feeble brain, its tasks oft-times resisting" provides a scheme for remembering: a. the order of multiplication of determinants; b. double angle formula; c. π to twelve decimal places; d. e to twelve decimal places.

4. A man has lived a fourth of his life as a boy, a fifth as a youth, a third as a man, and has spent 13 years in his dotage. How old is he?

5. *Mathematics and the Imagination* was written by one of the following: a. Lancelot Hogben; b. Eric Temple Bell; c. Edward Kasner and James Newman; d. Elton James Moulton.

6. One of the following is a national mathematical fraternity: a. Iota Psi; b. Sigma Xi; c. Pi Mu Epsilon; d. Delta Gamma.

7. Given a three-inch cube, with the outside surface painted red. If the cube is cut into 27 one-inch cubes, how many of these cubes would have all six sides painted red? Five sides painted red? Four? Three? Two? One? None?

8. If a car travels 30 miles in one hour, in what length of time would it have to cover an additional 30 miles to average 60 miles an hour?

9. Two horses are 30 miles apart and start toward each other at the rate of 15 miles per hour. Also, at the same time a fly starts oscillating from the nose of one horse to the nose of the other. If the fly travels at the rate of 20 miles per hour, how far would it fly by the time the two horses met?

10. The möbius band is: a. a well known swing orchestra; b. an emblem of peace and prosperity; c. a one-sided surface; d. a famous group of desperadoes.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 451. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

There are three containers, having capacities of a, b, c quarts, where $a>b>c$ (positive integers). With the largest container full and the others empty, it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible? [*Cf.* 1940, 374.]

E 452. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find a multiple of 7 whose square has eight digits of the form *ababbbcc*.

E 453. *Proposed by N. A. Court, University of Oklahoma.*

Given three skew lines a, b, c, for what positions of a point M will the harmonic inverses of M with respect to the pairs b and c, c and a, a and b be coplanar with M? (The harmonic inverse of M with respect to two skew lines is its harmonic conjugate with respect to the points in which the lines meet their transversal from M.)

E 454. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A maze consists of stations A and B, n other stations, and just one direct passage between every pair of stations. Let N be the number of distinguishable routes from A to B, going through no station more than once; and let N' be the number of such routes which go through all the other stations on the way. What is the limiting value of the ratio N/N' as n increases without limit?

E 455. *Proposed by V. W. Graham, Dublin, Ireland.*

Given a fixed straight line l and a fixed point P outside it, consider two variable points Q and R on l, such that $\angle QPR$ is constant. Let S be the point in which l meets the bisector of this angle, and let C be the center of the circle PQR. Prove that CS passes through a fixed point.

E 456. *Proposed by Leopold Infeld, University of Toronto.*

What is the smallest popular vote by which a President can be elected in the U. S. A. under the present electoral system? Assumptions: N is the total popular vote; the popular vote in each state is proportional to the electoral vote (which you will have to look up); there are just two candidates.

SOLUTIONS

E 416 [1940, 240]. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that

$$3\cdot 2^{p-1} - 2 \equiv \pm \binom{p-1}{[p/4]} (\text{mod } p^2),$$

where p is any odd prime.

Solution by J. S. Frame, Brown University.

We simplify the notation by setting $(p-1)/2 = n$, $[p/4] = m$, and defining the two residues s and t (mod p) as follows:

$$s \equiv \sum_{k=1}^{n} \frac{1}{k} (\text{mod } p), \qquad t \equiv \sum_{k=1}^{m} \frac{1}{k} (\text{mod } p).$$

Then we have

$$s - t \equiv \sum_{m+1}^{n} \frac{1}{k} \equiv 2\sum_{m+1}^{n} \frac{1}{2k-p} \equiv -2\sum_{h=1}^{n-m} \frac{1}{2h-1} \equiv -2\left(s - \frac{t}{2}\right) (\text{mod } p).$$

Hence

$$3s \equiv 2t \ (\text{mod } p).$$

Now the given binomial coefficient satisfies the congruence

$$(-1)^m \binom{p-1}{m} = \left(1 - \frac{p}{1}\right)\left(1 - \frac{p}{2}\right) \cdots \left(1 - \frac{p}{m}\right) \equiv 1 - tp \ \ (\text{mod } p^2),$$

and the Fermat quotient q satisfies the relations

$$\begin{aligned} q &= (2^{p-1} - 1)/p = \{(1+1)^p + (1-1)^p - 2\}/2p \\ &= \sum_{k=1}^{n} \frac{1}{p}\binom{p}{2k} = \sum_{1}^{n} \frac{1}{2k}\binom{p-1}{2k-1} \equiv -\sum_{1}^{n} \frac{1}{2k} (\text{mod } p), \\ q &\equiv -s/2 \equiv -t/3 \ (\text{mod } p). \end{aligned}$$

Hence

$$3\cdot 2^{p-1} - 2 = 1 + 3qp \equiv 1 - tp \equiv (-1)^m \binom{p-1}{m} (\text{mod } p^2).$$

Also solved by E. P. Starke.

E 417 [1940, 240]. *Proposed by J. F. Kenney, Northwestern University.*

Let E be any point outside a circle, ABE the diameter through E, and CDE any chord through E. In the triangle BCE, show that the side CE is divided by D into segments (CD and DE) whose ratio is less than the ratio of the angles E and C.

Solution by E. P. Starke, Rutgers University.

Since angle EBD is obtuse, both the bisector of angle EBC and the perpendicular from B upon EC meet EC outside the circle, say at P and Q, respectively. Hence

$$(1) \qquad CD/DE = (CP - DP)/(PE + DP) < CP/PE = BC/BE = \sin E/\sin C,$$

and similarly

$$(2) \qquad CD/DE < CQ/QE = (CQ/BQ)/(QE/BQ) = \tan E/ \tan C.$$

We need also the fact that, as x varies from 0 to $\pi/2$, the functions $\tan x/x$ and $\sin x/x$ are monotonic increasing and decreasing, respectively. (This is so since their derivatives, $(2x-\sin 2x)/2x^2 \cos^2 x$ and $(x-\tan x)/x^2 \cos x$, are respectively positive and negative throughout the interval.) Suppose first that angle E is greater than angle C. Then $\sin E/E<\sin C/C$ or $\sin E/\sin C<E/C$ which, with (1), proves $CD/DE<E/C$. If, on the other hand, angle E is not greater than angle C, then $\tan E/E\leqq\tan C/C$ or $\tan E/\tan C\leqq E/C$ which, with (2), proves $CD/DE<E/C$.

Also solved by the proposer, who derived the problem from a statement of Maurolycus in the *Photismi de Lumine*, Naples, 1611, p. 42, Theorem XVIII. A translation of this scholarly little book with the playful title has recently been made by Professor Henry Crew and published by the Macmillan Company.

E 418 [1940, 240]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Find triangles with rational sides and angle-bisectors.

Solution by E. P. Starke, Rutgers University.

This problem has already been solved as part of E 331 [1939, 172]. Following the notation and analysis given there, the internal angle-bisectors have lengths $2bc \cos \frac{1}{2}A/(b+c)$, $2ca \cos \frac{1}{2}B/(c+a)$, $2ab \cos \frac{1}{2}C/(a+b)$. The sides being rational, it is thus necessary and sufficient that the three half-angles have rational cosines. It is then easy to establish that they must have rational sines. Thus the analysis and results of the earlier problem apply here, and we have the following theorem:

If the sides and internal angle-bisectors are rational, so also are the external angle-bisectors, the altitudes, the area, and the five radii.

The simplest way to present the general result of E 331 seems to be as follows. Let (x_1, y_1) and (x_2, y_2) be rational solutions of $x^2+y^2=1$, with $x_1x_2>y_1y_2$. Then any numbers a, b, c, proportional to

$$x_1y_1, \qquad x_2y_2, \qquad (x_1x_2 - y_1y_2)(x_1y_2 + y_1x_2),$$

are the sides of a triangle whose angle-bisectors are

$$2bcx_1/(b + c), \qquad 2cax_2/(c + a), \qquad 2ab(x_1y_2 + y_1x_2)/(a + b).$$

Thus the solution (4/5, 3/5) used twice gives an isosceles triangle with sides

25, 25, 14 and angle-bisectors 700/39, 700/39, 24; and the two solutions (4/5, 3/5) and (12/13, 5/13) give a scalene triangle with sides 169, 125, 154 and angle-bisectors 30800/279, 48048/323, 2600/21. (Because a line of manuscript was overlooked, the last sentence of the solution to E 331 is incorrect. The numbers given there are in fact the *external* bisectors.)

Editorial Note. The solutions (3/5, 4/5) and (12/13, 5/13) give a simpler scalene triangle, with sides 169, 125, 84 and angle-bisectors 12600/209, 26208/253, 975/7.

E 419 [1940, 240]. *Proposed by V. Thébault, Le Mans, France.*

In what direction must a billiard ball be hit in order to return to its starting-point after a given even number of rebounds? Neglect spin of the ball.

Solution by E. P. Starke, Rutgers University.

Besides neglecting spin, let us assume the cushions to be perfectly elastic, and exclude rebounds from a corner. Choose for axes of reference those edges of the table for which both coördinates increase (or remain stationary) along the ball's path. Let the dimensions of the table be u and v, and let the initial position of the ball be (a, b). Then, for "almost all" values of a and b, the desired direction is that whose slope is sv/ru, where r and s are any non-negative integers whose sum is half the given number of rebounds. To see this, imagine the first quadrant filled with billiard tables of the same size, placed edge to edge (with cushions removed) and having the points of each pair of adjacent tables so oriented as to be symmetric with respect to the common edge. The actual path of the ball on the original table may now be represented as a straight line crossing this lattice. The different representations of the original position (a, b) are $(2ru \pm a, 2sv \pm b)$, $(r, s=0, 1, 2, \cdots)$. Every time the path crosses a line $x=ru$ or $y=sv$, the ball on the original table rebounds from an edge. Thus in passing from (a, b) to $(2ru+a, 2sv+b)$ there are $2r+2s$ rebounds, and the slope of the path is sv/ru. The points $(2ru-a, 2sv+b)$ and $(2ru+a, 2sv-b)$ would each give an odd number of rebounds (namely $2r+2s-1$). The point $(2ru-a, 2sv-b)$ is ruled out by one of our simplifying restrictions, since in that case the path would pass through the corner-point (ru, sv).

Also solved by the proposer, who draws attention to the fact that the possible distances traversed by the ball are independent of (a, b).

E 420 [1940, 240]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Let M be the point of intersection of the diagonals of a quadrangle inscribed in a circle with center O. Let parallels through M to the four sides meet the respective opposite sides at P, Q, R, S. Prove that these four points are collinear, that their line is perpendicular to OM, and that analogous results hold for a cyclic hexagon whose three main diagonals concur at a point M.

Solution by E. P. Starke, Rutgers University.

Let $ABCD$ be the quadrangle, and let the parallels to AB, BC, CD, DA

through M meet the respective opposite sides in P, Q, R, S. Let also AD and BC meet in Y, and AB and CD in Z. Then YZ, being the polar of M with respect to the circle, is perpendicular to OM.

From the triangle ACZ, in which RM is parallel to CZ, we have $AR/RZ = AM/MC$. From the triangle ACY, we have $AQ/QY=AM/MC$. Thus $AQ/QY=AR/RZ$, and QR is parallel to YZ. Similarly we may show that PQ and PS are each parallel to YZ. Hence P, Q, R, S are on a line which, being parallel to YZ, is perpendicular to OM.

The properties and proof for the hexagon whose main diagonals are concurrent is strictly analogous. The line YZ is here the polar of M and also the Pascal line of the hexagon.

Also solved by W. B. Clarke, who remarks that the statement can immediately be generalized to any cyclic $2n$-gon with concurrent diagonals.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3979. *Proposed by W. V. Parker, Louisiana State University.*

If A_i, $(i=1, 2, 3, 4)$, are four points on a circle in the order of the subscripts with the rectangular coördinates x_i, y_i, prove that the equation of the circle is

$$a_{23}a_{14}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + a_{12}a_{34}\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\begin{vmatrix} x & y & 1 \\ x_4 & y_4 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0,$$

where a_{ij} is the length of the side A_iA_j.

3980. *Proposed by Esther Szekeres, Budapest, Hungary.*

The symmetric polynomials $y_1, y_2, \cdots, y_n$ in the variables $x_1, x_2, \cdots, x_n$ are of the degrees indicated by the subscripts, and are algebraically independent. If $f(x_1, x_2, \cdots, x_n)$ is any given polynomial symmetric in the x's, show that it can be expressed as a polynomial in the y's.

3981. *Proposed by V. Thébault, Le Mans, France.*

Let S_i, $(i=1, 2, 3, 4, 5, 6)$, be the spheres of similitude of the spheres with centers at the vertices of the tetrahedron $A_1A_2A_3A_4$ such that the square of the radius of any one of the latter is equal to one-half of the sum of the squares of the edges of the opposite face. (1) Examine the relative positions of the spheres

S_i. (2) Show that these six spheres are orthogonal to the circumspheres of $A_1A_2A_3A_4$ and of its anticomplementary tetrahedron. (3) The powers of the extremities of an edge of $A_1A_2A_3A_4$ with respect to the sphere S_i whose center is on the opposite edge are independent of the length of that last edge. (4) The spheres S_i' symmetric to the spheres S_i with respect to the midpoint of the edges upon which they are centered intersect in two points collinear with the circumcenter of $A_1A_2A_3A_4$.

3982. *Proposed by V. Thébault, Le Mans, France.*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to one-third of the sum of the squares of the edges through the considered vertex. Show that the sphere orthogonal to the four spheres is concentric with the twelve point sphere of $ABCD$.

Note. See N. A. Court, *Modern Pure Solid Geometry*, p. 250, for the twelve point sphere.

3983. *Proposed by V. Thébault, Le Mans, France.*

The vertices of the tetrahedron $ABCD$ are centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges of the face opposite to the vertex considered, and they are also centers of spheres the squares of whose radii are equal respectively to k times the sum of the squares of the edges through the considered vertex. Let ω_1 and ω_2 be the centers of the spheres, radii R_1 and R_2, orthogonal respectively to the two sets of four spheres, G the centroid, and O the circumcenter of $ABCD$. (1) Show that the points O, G, ω_1, ω_2 are collinear, and determine their relative positions. (2) Show that $(R_1^2-R_2^2)/k$ remains constant when k varies.

SOLUTIONS

3852 [1938, 52]. *Proposed by V. Thébault, Le Mans, France.*

Let I, I_a, I_b, I_c, I_d be the center of the inscribed sphere of radius r, and the centers of the spheres (I_a), (I_b), (I_c), (I_d) escribed in the truncated trihedrals with the vertices A, B, C, D of a tetrahedron $ABCD$; let O, O_a, O_b, O_c, O_d be the circumcenters of the tetrahedrons $ABCD$, $IBCD$, $ICDA$, $IDAB$, $IABC$. Produce the segments OO_a, OO_b, OO_c, OO_d by the lengths $O_aA'=O_bB'=O_cC'=O_dD'=r/2$. (1) Prove that the circumspheres of $A'BCD$, $B'CDA$, $C'DAB$, $D'ABC$ are respectively tangent to the pairs of spheres (I), (I_a); (I), (I_b); (I), (I_c); (I), (I_d). (2) Examine the cases where (I) is replaced by each of the escribed spheres corresponding to pairs of opposite edges, when they exist.

Dedicated to N. A. Court who has studied the tetrahedron so extensively.

Solution by the Proposer.

This problem was suggested by the following theorems due to M. R. Bricard, professor at the Conservatoire des Arts et Métiers, Paris, who has furnished the proofs given below.

THEOREM I. *Given an ellipse E, with the foci F and F' and the minor axis BB'; let θ be the angle $F'BF$, and Γ and Γ' the two circles through F and F' and tangent to any given tangent of E. The two circles intersect at the angle θ.*

Proof. Let M and M' be the points where the chosen tangent to E cuts the tangents at B and B'. Then, from a theorem of Poncelet, we have $\angle FMP = \angle BMF' = \angle PF'M$, where P is the trace of MM' on FF'. From this it follows at once that the circle through F, F', M is tangent at M to MM', and this is the circle Γ. Similarly, Γ' is tangent at M' to MM'. Let the tangents at F to Γ and Γ' cut MM' in T and T'. Then we have $\angle PFT = \angle F'MF = \angle PFM - \angle FF'M$, and $\angle T'FP = \angle FM'F' = \angle M'FP - \angle M'F'F$. Addition of these two equalities gives (1) $\angle T'FT = \angle M'FM - \angle M'F'M$.

Now let Q be the point where MM' touches E. From another theorem of Poncelet, we have $\angle QF'M = \frac{1}{2} \angle QF'B$, $\angle M'F'Q = \frac{1}{2} \angle B'F'Q$. By addition we get $\angle M'F'M = \frac{1}{2} \angle B'F'B = \angle FF'B$. Similarly, we find that $\angle M'FM = \pi - \angle BFF'$. Hence we have from (1), $\angle T'FT = \pi - \angle BFF' - \angle FF'B = \angle F'BF = \theta$, which is the desired result.

THEOREM II. *Let G be a circle such that there exist triangles inscribed in G and circumscribing E. Then the two circles Γ and Γ' through F and F' tangent to G intersect at an angle equal to θ.*

Proof. Let M be one of the intersections of FF' with G. Then there exists a triangle MNP inscribed in G and circumscribing E. The line $MF'F$, which cuts G again in I, is the internal bisector of angle NMP. If a conic varies while remaining inscribed in MNP and one focus describes MI, its other focus also describes MI and the two foci are in involution with F and F' as homologous points. The incenter ω of MNP is a double point. Finally, the point I on G is the focus of a parabola inscribed in MNP so that the homologous point of I is at infinity. From this it results that $IF \cdot IF' = (I\omega)^2$. But we know that $I\omega = IN = IP$ so that (2) $IF \cdot IF' = (IN)^2 = (IP)^2$. Consider now the inversion with center I and power $IF \cdot IF'$ which leaves the points ω, N, P invariant by (2), so that G inverts into the straight line NP. The circles Γ and Γ' are also invariant; and, since they are tangent to G, they are also tangent to NP. Since NP is tangent to E, we know from Theorem I that Γ and Γ' intersect at the angle θ. This proves Theorem II.

THEOREM III. *Let E be an ellipse in the plane π, and (S) and (S') two spheres, with the respective radii R and R', tangent to π at F and F' and lying on opposite sides of π. If $R \cdot R' = b^2$, where $2b$ is the length of the minor axis of E, every sphere Σ tangent internally to (S) and (S') cuts π in a circle G such that there exist triangles inscribed in G and circumscribing E.*

Proof. Consider the cone of revolution circumscribing (S) and (S') so that the spheres lie in the same nappe. The plane π cuts this cone in an ellipse with foci F and F' (Dandelin), and we know that $R \cdot R'$ is the square of its semi-minor axis. Hence this ellipse must be E. Let I be one of the intersections of G with the plane

which bisects FF' perpendicularly. In an inversion with the pole I and power $(IF)^2=(IF')^2$, the plane π is invariant as are also (S) and (S'). The sphere Σ inverts into a plane tangent to (S) and (S'). The circle G transforms into a straight line tangent to E. The two circles through F and F' tangent to G transform into two circles through the same two points which are tangent to this straight line. Hence these last two circles intersect at the angle θ, and therefore triangles can be inscribed in G which circumscribe E.

Note. For the sake of brevity, we consider only triangles which circumscribe E in the proper, ordinary sense, and not such that E is escribed in the triangle.

THEOREM IV. *Let $ABCD$ be a tetrahedron, (I) its inscribed sphere, (I_a) its escribed sphere for the angle A. There exists a sphere (S_a) passing through B, C, D which is tangent to (I) and (I_a).*

Proof. This theorem follows immediately from the previous theorem, for the spheres (I) and (I_a) are tangent to the plane of face BCD at the foci of a conic inscribed in the triangle BCD for which the square of the minor axis is equal to $4rr_a$, r and r_a being the radii of the spheres (I) and (I_a).

We know that there are in general eight spheres each of which is tangent to the four planes of the faces of $ABCD$. They are separated into two groups: the first is composed of the four spheres (I_a), (I_b), (I_c), (I_d), of radii r_a, r_b, r_c, r_d, inscribed in the truncated trihedral angles corresponding to the vertices A, B, C, D; the second group contains the inscribed sphere (I), radius r, and the three spheres (I_1), (I_2), (I_3), of radii r_1, r_2, r_3, corresponding to pairs of opposite edges; the number of these last may in special cases reduce to two, one, or zero. Theorem IV may be completed thus:

Two spheres belonging to two different groups are tangent to the same sphere (S) passing through three vertices of the tetrahedron $ABCD$.

These theorems of M. R. Bricard enable us to solve the present problem.

THEOREM A. *If we consider the center I_i of one of the spheres tangent to the four planes of the faces of $ABCD$ and the centers O_a, O_b, O_c, O_d of the circumspheres of the tetrahedrons I_iBCD, I_iCDA, I_iDAB, I_iABC; the sphere circumscribing the tetrahedron $O_aO_bO_cO_d$ is concentric with the circumsphere of the fundamental tetrahedron* (V. Thébault, *Annales de la Société scientifique de Bruxelles*, 1924, p. 179).

Proof. For, if $I_i(O)$ and $I_i(O_a)$ denote the powers of the center I_i of the considered sphere (I_i), radius r_i, with respect to the circumsphere (O) of $ABCD$, radius R, and, for example, (O_a), the sphere circumscribing I_iBCD, we have, in magnitude and sign, $I_i(O)-I_i(O_a)=2O_aO\cdot r_i$, or (3) $O_aO=\pm(d_i^2-R^2)/2r_i$, which depends only upon the radius of (I_i) and the distance d_i of its center from that of (O). Thus (4) $O_aO=O_bO=O_cO=O_dO=\rho_i$.

THEOREM B. *The spheres tangent internally, or externally, to one of the spheres (I_i) and passing respectively through the vertices (B, C, D), (C, D, A), (D, A, B),*

(A, B, C) *are tangent to a sphere concentric with the circumsphere of the fundamental tetrahedron* (*loc. cit.*, 1938, p. 153).

Proof. Consider, for example, the spheres (S_a'), (S_b'), (S_c'), (S_d'), radii R_a', R_b', R_c', R_d', tangent internally to the insphere (I) and passing respectively through the vertices (B, C, D), (C, D, A), (D, A, B), (A, B, C); then we have, in magnitude and sign,

$$\begin{aligned} 2OS_a' \cdot r &= (IS_a')^2 - (R_a')^2 - \left[(IO)^2 - R^2\right] \\ &= (R_a' - r)^2 - (R_a')^2 - d^2 + R^2, \end{aligned}$$

so that $OS_a' + R_a' = (R^2 + r^2 - d^2)/2r$, and, finally, with the aid of (3) and (4),

$$R' = \rho + r/2 = OS_a' + R_a' = OS_b' + R_b' = OS_c' + R_c' = OS_d' + R_d',$$

where ρ is the radius of the circumsphere of $O_aO_bO_cO_d$ whose vertices are the centers of the spheres $(IBCD)$, $(ICDA)$, $(IDAB)$, $(IABC)$, and $d = OI$. The spheres (S_a'), (S_b'), (S_c'), (S_d'), whose centers S_a', S_b', S_c', S_d' are on the perpendiculars to the planes of faces BCD, CDA, DAB, ABC drawn from the center O of the circumsphere of $ABCD$, are therefore tangent internally to a sphere with center O and radius $R' = \rho + r/2$ which contains all four spheres.

The spheres (S_a'), (S_b'), (S_c'), (S_d') are therefore circumspheres respectively of $A'BCD$, $B'CDA$, $C'DAB$, $D'ABC$; and, from Theorem IV (Bricard), they coincide with the spheres (S_a), (S_b), (S_c), (S_d) passing through the points (B, C, D), (C, D, A), (D, A, B), (A, B, C) and tangent to (I) and (I_a), (I) and (I_b), (I) and (I_c), (I) and (I_d).

We obtain similar results by replacing (I) by each of the spheres (I_1), (I_2), (I_3) of the pairs of opposite edges, if they exist.

The spheres with centers A', B', C', D', *passing through the vertices* (B, C, D), (C, D, A), (D, A, B), (A, B, C), *are orthogonal to the sphere* (I) *inscribed in* $ABCD$.

For, the distance of the center A' of the sphere passing through (B, C, D) and orthogonal to (I), from the circumcenter O of $ABCD$, is $OA' = (R^2 + r^2 - d^2)/2r$. Similar theorems apply to the spheres (I_a), (I_b), (I_c), (I_d) and those corresponding to opposite edges.

Editorial Note. The proposer stated that the Bricard theorem appears to be one of the most original theorems added in recent years to the geometry of the tetrahedron, and that the theorem of his problem gives a construction of the very remarkable spheres indicated by Bricard.

In the above solution, the proof of Theorem III depends upon Theorem I and the converse of Theorem II, while a proof of Theorem IV may be obtained from Theorem II and the converse of Theorem I. It seems then that the Theorems I and II are intended to include their converses. We give below proofs of these converses. The converse of Theorem I may be set in the form:

The two circles (Γ), (Γ'), *with centers* Γ *and* Γ', *intersect in two fixed points*

F, F' at the angle θ, which is taken as the angle containing no interior point of either circle. A common tangent has the points of contact M, M' whose projections on the line of centers $\Gamma\Gamma'$ are B, B'. As the pair of circles vary so that θ remains constant, each of the two points B, B' remains fixed; and MM' envelops an ellipse E with foci F, F', with minor axis BB', and angle $F'BF$ is equal to θ.

Since FF' is the radical axis of the two circles, it bisects MM' and also BB'. The reflections of B', Γ' in FF' are B, Γ''; and, from the similar right triangles $BM\Gamma, B'M'\Gamma'$, we have $B'\Gamma'/B\Gamma = M'\Gamma'/M\Gamma = F\Gamma'/F\Gamma$, or $\Gamma''B/B\Gamma = F\Gamma''/F\Gamma$. Hence FB is the internal bisector of angle $\Gamma F\Gamma''$; and it then follows that $\angle BFB' = \angle \Gamma F\Gamma' = \pi - \theta$, and $\angle F'BF = \theta$, and B is fixed and also B'. From the circle (Γ) we see that $\angle FMM' = \angle BMF'$, and similarly for the two angles at M'. Hence F, F' are isogonal conjugate points for the triangle with vertices M, M' and the point at infinity on MB produced. Thus an ellipse E with foci F, F' is tangent to MM' and to MB, $M'B'$ at B, B'. There is a one-to-one correspondence between the circles (Γ) and the points M on the half-line BM produced; hence, as the pair of circles vary for a constant θ, we get all of the tangents to one-half of E, so that the original theorem also follows.

For the converse of Theorem II we have two circles (Γ), (Γ') intersecting in the points F, F' at the angle θ and tangent to a circle (G) which cuts the line of FF' in the points U, I. In the inversion with the pole I and power $IF\cdot IF'$, each of the circles (Γ), (Γ') is invariant, and (G) inverts into a straight line tangent to the same pair of circles at the points M, M'. Hence the inverse of U is P, the intersection of MM' with the line of FF'. The circle (I) of inversion and (G) have the common chord VW on which lie the points M, M'; the points V, W are also invariant; and (I) cuts FF' in ω, ω'. The pairs of points (U, P), (F, F') determine an involution with the double points ω, ω'. Let K be the conic inscribed in the triangle UVW with one focus at F and the other at F_1. Then K is tangent to the four straight lines J_1F, J_1F_1, J_2F, J_2F_1, where J_1 and J_2 are the circular points at infinity. The pairs of points (F, F_1), (J_1, J_2) are degenerate conics touching the same four lines; and the pairs of tangents from any point, say V, to the three conics form an involution of rays from V. Thus (VU, VP), (VF, VF_1), (VJ_1, VJ_2) are pairs in an involution. Since (I) passes through ω, ω', V, W, J_1, J_2, the pencil $V(\omega, \omega', J_1, J_2)$ is harmonic; and thus $V\omega$, $V\omega'$ are double rays for the involution (VU, VP), (VF, VF'), (VJ_1, VJ_2). Hence V, F', F_1 are collinear; and similarly W, F', F_1 are collinear. If V and W are distinct points, it follows from the fact that V, W, F' are not collinear, that $F' \equiv F_1$, and that $K \equiv E$, the ellipse with foci F, F' tangent to MM' and with the angle θ. Hence the triangle UVW is inscribed in (G) and it circumscribes E. The same kind of reasoning may be used to prove the original theorem. If $V \equiv W$, the circles (G), (I) are tangent and MM' is the common tangent at the real point V; the two tangents from U coincide; and hence U is an end of the major axis of E. This is a limit case of the above, and it may happen in two ways. The desired triangle is then the double segment UV. In other cases U, V may be a pair of conjugate imaginary points; but in the most important case where (G) contains

in its interior both (Γ) and (Γ'), the points V, W are real and distinct, and also the perpendicular bisector of FF' cuts (G) in real points. In the other cases it may happen that these last mentioned points, one of which is used later as a center of inversion, are imaginary.

In the last part of the solution it is stated that the sphere with center O and radius R' contains in its interior the four spheres such as (S_a'). In order to prove this it must be shown that no one of the four formulas, such as the one for OS_a', gives a negative value for OS_a'.

A synthetic proof of the two Poncelet theorems is given in the solution of 2971 [1923, 341].

3904 [1939, 111]. *Proposed (July 1937) by the late R. P. Baker, University of Iowa.*

ABC is a given triangle; find the condition that a point P may be constructed in the plane of ABC such that

$PA:PB:PC = p:q:r$, (p, q, r real positive constants).

I. *Solution by the Proposer.*

Let $BC=a$, $CA=b$, $AB=c$, $PA=p/\lambda$, $PB=q/\lambda$, $PC=r/\lambda$, where a, b, c, λ are real positive constants. Then the relation which must be satisfied by the distances between the four points P, A, B, C is given by

$$\begin{vmatrix} 0 & c^2 & b^2 & p^2 & 1 \\ c^2 & 0 & a^2 & q^2 & 1 \\ b^2 & a^2 & 0 & r^2 & 1 \\ p^2 & q^2 & r^2 & 0 & \lambda^2 \\ 1 & 1 & 1 & \lambda^2 & 0 \end{vmatrix} = 0. \tag{1}$$

(See Cayley, Paper 1, *On a theorem in the geometry of position*, Collected Mathematical Papers, vol. 1, Cambridge, 1889.) Expanding this determinant, and dividing by 2, we have

$$\begin{aligned} &-\lambda^4a^2b^2c^2+\lambda^2[p^2a^2(b^2+c^2-a^2)+q^2b^2(c^2+a^2-b^2)+r^2c^2(a^2+b^2-c^2)] \\ &\quad + a^2(p^2-q^2)(r^2-p^2)+b^2(q^2-r^2)(p^2-q^2)+c^2(r^2-p^2)(q^2-r^2)=0. \end{aligned} \tag{2}$$

The discriminant D of this quadratic in λ^2 is

$$\begin{aligned} &[p^2a^2(b^2+c^2-a^2)+q^2b^2(c^2+a^2-b^2)+r^2c^2(a^2+b^2-c^2)]^2 \\ &\quad + 4a^2b^2c^2[a^2(p^2-q^2)(r^2-p^2)+b^2(q^2-r^2)(p^2-q^2)+c^2(q^2-r^2)(r^2-p^2)], \end{aligned}$$

or

$$\begin{aligned} D = (a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2)(&p^4a^4+q^4b^4+r^4c^4 \\ &-2p^2q^2a^2b^2-2q^2r^2b^2c^2-2r^2p^2c^2a^2). \end{aligned} \tag{3}$$

Thus $D=(-H_1)(-H_2)$, where $H_1=(4\cdot\text{area }\Delta ABC)^2$, using the Heron formula for area, and $H_2=(4\cdot\text{area }\Delta PQR)^2$, with $PQ=rc$, $QR=pa$, $RP=qb$.

Since H_1 is positive for the given triangle ABC, the sign of D is the same as the sign of H_2. Now D cannot be negative, since λ^2 is real. Also, $D=0$ if and only if $H_2=0$. That is, a solution of (2) leading to two equal positive values of λ, and hence to a single position of P, is possible if and only if ΔPQR vanishes, *i.e.*, has area $=0$. Again, $D>0$ if and only if $H_2>0$. That is, a solution of (2) leading to two unequal positive values of λ, and hence to two distinct positions of P, is possible if and only if ΔPQR exists, *i.e.*, has a positive area.

Consequently, under the hypotheses stated, a necessary and sufficient condition for the construction of a point P in the plane of ΔABC such that $PA:PB:PC=p:q:r$, is that the area of the triangle with sides pa, qb, rc be either positive or zero.

Example 1. If $a=b=c=1$, $p:q:r=1:2:4$, ΔABC is real, but ΔPQR is not, since $pa:qb:rc=1:2:4$. Point P cannot be constructed.

Example 2. If $a=2$, $b=3$, $c=4$, $p:q:r=5:2:1$, ΔABC is real and area $\Delta PQR=0$, since $pa:qb:rc=10:6:4$. A single point P can be constructed.

Example 3. If $a=3$, $b=4$, $c=5$, $p:q:r=3:2:1$, ΔABC is real and area ΔPQR is positive, since $pa:qb:rc=9:8:5$. Two distinct points P can be constructed.

II. *Solution by Rufus Crane, Ohio Wesleyan University.*

First, if p, q, r are not equal, let it be assumed for the sake of definiteness that $p<q<r$.

The locus of P such that $PA:PB=p:q$ is a circle S_3, the extremities of whose diameter form with A and B a harmonic range. Likewise, the locus of a point P such that $PB:PC=q:r$ is a circle S_1, the extremities of whose diameter form with B and C a harmonic range. Hence, these circles are orthogonal to the circumcircle of ABC.

If these circles have a common point P, then for this point, by combination of the above ratios, $PA:PC=p:r$. Hence, this point lies on a third circle S_2 orthogonal to the circle ABC.

If there is just one common point P, the three circles are tangent to each other, their line of centers is tangent to the circle ABC, and P is the point of tangency; so that the convex cyclic quadrilateral $APBC$ has the diagonals CP and AB. For such a quadrilateral we must have

$$p\cdot a + q\cdot b = r\cdot c,$$

where, as usual, a, b, c denote the sides of the triangle.

The quantities p, q, r, a, b fix the position of the center of S_2 on AC, and of S_1 on BC, and also the lengths of their radii. If c be smaller than required by the above equality, the circles will intersect. Hence, for two points P we have

$$p\cdot a + q\cdot b > r\cdot c.$$

If c be larger, the circles will fail to meet. Hence, for no point P we have

$$p\cdot a + q\cdot b < r\cdot c.$$

Finally, if $p=q=r$, there is only one finite point P, the circumcenter of ABC, since each S circle becomes a diameter of the circumcircle ABC. If $p=q\neq r$, or $p=r\neq q$, or $q=r\neq p$, one of the S circles becomes a circumdiameter, but the reasoning is the same as above.

Editorial Note. The equation (1) in solution I results from the formula for the volume V of the parallelopiped having PA, PB, PC as three of its edges, $V^2=|b_i\cdot b_j|$, taking P as origin of vectors b_i to A, B, C. We then set $2b_i\cdot b_j=l_{0i}^2+l_{0j}^2-l_{ij}^2$, where the l's are the lengths of the edges of the tetrahedron $PABC$; see the solution of 3863 [1940, 186]. For the special case $p:q:r=a^{-1}:b^{-1}:c^{-1}$, the triangle ΔPQR (in the notation of I) is equilateral, the two distinct points P in this case are often named the isodynamic points for ABC, and the three corresponding circles in II are called the Apollonian circles for ABC; see the solution of 3891 [1940, 576].

Miss Frances E. Baker gave as illustration the geometric construction of the points P when they exist, and a discussion of the case where there is only one point P, corresponding to area $\Delta PQR=0$.

3906 [1939, 175]. *Proposed by W. B. Campbell, Drexel Institute.*

A model for a spool-type hyperboloid consists of two disks of radius R, free to rotate and slide on a rod of radius r, and a set of connecting thin strings of length $2h$. With the parts initially set up as a right circular cylinder, the strings serving as elements, the disks are rotated through equal angles θ, in opposite senses. Discuss the resulting positions of the strings, especially after they come in contact with the rod, if the latter is (a) smooth, (b) rough.

Solution by the Proposer.

Let the axis of the rod be the Z-axis, with the disks initially in the planes $z=\pm h$. The string initially connecting the points $(R, 0, \pm h)$ will, for θ not too large, connect the points $R\cos\theta$, $\pm R\sin\theta$, $\pm(h^2-R^2\sin^2\theta)^{1/2}$, having its center at $(R\cos\theta, 0, 0)$. The strings then represent the rulings on the portion of a hyperboloid of revolution lying between two bases of radius R, with neck radius $a=R\cos\theta$, the distance between the bases being $2k$, where $k=(h^2-R^2\sin^2\theta)^{1/2}$. (If extended indefinitely beyond the bases, they represent the entire surface.) In the conventional equation $(x/a)^2+(y/a)^2-(z/c)^2=1$, $c=ak/(R^2-a^2)^{1/2}$, the asymptotic cone being $(x/a)^2+(y/a)^2=(z/c)^2$.

If $h^2<R^2-r^2=b^2$, rotation will cease when $\theta=\sin^{-1}(h/R)$, with both disks in the plane $z=0$, and the strings as chords of the circles, not touching the rod. If $h^2>R^2-r^2$, the strings will begin to wind upon the rod when $\theta=\cos^{-1}(r/R)=\alpha$. If the rod is smooth, the portion upon it will always lie in a true helix, making everywhere the same acute angle γ with the intersecting elements as do the free ends extending toward the disks. The total projection t, upon the XY-plane, of the curved part of the half-string is $r(\theta-\alpha)$, that of the straight part is b; and $\gamma=\sin^{-1}(t+b)/h$, with initial value $\sin^{-1}(b/h)$. The helical parts of the strings then lie between $z=\pm r(\theta-\alpha)\cot\gamma$, and the straight parts on two hyper-

boloidal surfaces with their necks, of radius r, in these two planes, and their bases, of radius R, in the planes $z=\pm(t+b)\cot\gamma$. As θ increases, the disks slide in, compressing the helices between them, until $\theta=\alpha+(h-b)/r$, when the straight parts form a set of rays tangent to the rod, and extending to the perimeters of the coincident disks in the plane $z=0$.

If the rod is rough, each particle of string retains the position and direction it had when laid on the rod, and each string winds around the rod in a curve of decreasing pitch, until it is perpendicular to an element, and motion stops with the straight parts lying in the two final planes of the disks. If we measure u along the rod element intersecting the string at the point of first contact, and v perpendicular to it on the rod surface, the string locus is such that, at any point P on it, $dv/du=\tan\phi_s$, where $\sin\phi_s=b/(h-s)$, and s is arc length OP measured from the origin. We have

$$u_p(s) = \int_0^s \cos\phi_s ds = b(\phi_0+\cot\phi_0-\phi_s-\cot\phi_s),$$

$$v_p(s) = \int_0^s \sin\phi_s ds = b\log_e[h/(h-s)].$$

At the origin, $dv/du=\tan\phi_0=b/(h^2-b^2)^{1/2}$; the curve terminates and becomes parallel to the V-axis when $s=h-b$, at the point

$$u = b(\phi_0+\cot\phi_0-\tfrac{1}{2}\pi), \qquad v = b\log_e(h/b).$$

Let Q be the free end of a line segment of length $h-s$, tangent at P; the locus of Q in UV-plane is an involute of that of P; and we have

$$u_q(s) = u_p(s)+(h-s)\cos\phi_s = b(\cot\phi_0+\phi_0-\phi_s).$$

Moreover, $v_p(s)=t=r(\theta-\alpha)$, whence

$$h-s = he^{-(\cot\alpha)(\theta-\alpha)},$$

and $u_p(s)$ and $u_q(s)$, calculated for a particular θ, give the momentary Z-coördinates of the point where the string leaves the rod and that where it meets the disk. Motion stops when $u_p(s)=u_q(s)$, at $\theta=\alpha+(b/r)\log_e(h/b)$, and the disks and the straight parts of the strings then lie in the planes $z=\pm u_p(h-b)$ $=\pm[(h^2-b^2)^{1/2}-b\cos^{-1}(b/h)]$.

3907 [1939, 175]. *Proposed by V. W. Graham, The High School, Dublin, Ireland.*

Find all the roots of the equation

$$\frac{(x^2-x+1)^3}{x^2(x-1)^2} = \frac{(a^2-a+1)^3}{a^2(a-1)^2}.$$

I. *Solution by J. Barinaga, Madrid University, Spain.*

Setting the left member equal to $f(x)$, the given equation becomes $f(x)=f(a)$. It is easily seen that the rational function $f(x)$ admits the group of substitutions

$$x = \xi, \quad x = 1/\xi, \quad x = 1 - \xi, \quad x = 1/(1 - \xi), \quad x = (\xi - 1)/\xi, \quad x = \xi/(\xi - 1);$$

and, since evidently $x=a$ is a root, the six roots are

$$a, \quad 1/a, \quad 1 - a, \quad 1/(1 - a), \quad (a - 1)/a, \quad a/(a - 1), \quad (a \neq 0, 1).$$

For $a=1$, $f(a)$ fails to have a meaning. As $a \to 1$, the roots approach 1, 0, ∞ regarded as double roots; and similarly for $x=0$.

II. *Solution by G. A. Baker, University of California, Davis, California.*

If x is a root so is $1/x$. Hence a and $1/a$ are obvious solutions. The function on the left-hand side is symmetrical about the line $x=1/2$. Hence, $1-a$ and $(a-1)/a$ are roots. In view of the first statement, $1/(1-a)$ and $a/(a-1)$ are roots. The original equation is a sextic and hence has only the six roots listed above.

Solved also by F. A. Alfieri, J. W. Cell, H. H. Downing, William Forman, D. W. Hall, B. A. Hausmann, J. S. Leech, Veretta Mills, J. S. Morrell, E. R. Ott, W. V. Parker, E. A. Rasor, F. C. Smith, T. H. Southard, F. Underwood, C. W. Williams, and the proposer.

Editorial Note. Williams remarked that, if a is the cross-ratio of four collinear points in any given order, the six possible cross-ratios of the same four points, taken in all possible orders, are the six roots of the given equation. Cell, Hall, Morrell, Ott, Parker, Southard, and Williams observed that $f(x)$ is invariant under the group generated by the substitutions $x'=1/x$, $x'=1-x$. This group is the cross-ratio group in the above solution. Forman, Leech, Ott, Smith, Underwood, and the proposer observed that the substitution $x^2-x=y$ gives at once the roots a, $1-a$.

3908 [1939, 175]. *Proposed by Otto Dunkel, Washington University.*

Given a simplex S in n dimensions with the vertices A_i, $(i=1, 2, \cdots, n+1)$, let $p_1, p_2, \cdots, p_{n+1}$ denote the normal homogeneous coördinates of a point P with respect to the basis figure S, where the p_i's are equal to or proportional to the respective $n+1$ distances of P from the faces of S. Prove that the equation of the sphere circumscribing S is

$$\sum_{i \neq j} \frac{p_i p_j}{h_i h_j} e_{ij}^2 = 0, \qquad (n+1)n/2 \text{ terms},$$

where $e_{ij}=\overline{A_i A_j}$ and h_i is the length of the altitude of S from A_i.

Solution by H. S. M. Coxeter, University of Toronto.

The problem is clearly equivalent to that of obtaining the equation of the

circumsphere in the form

$$\sum_{i\neq j} e_{ij}^2 x_i x_j = 0,$$

where $x_1, x_2, \cdots, x_{n+1}$ are barycentric coördinates. Barycentric coördinates have the special virtue (as compared with other homogeneous coördinates) that, when one of them is put equal to zero, the rest are coördinates of the same kind referred to a face of the original simplex.

By making all but one of the coördinates vanish, we see that the most general quadric circumscribing S is of the form

$$\sum_{i\neq j} a_{ij} x_i x_j = 0$$

(free from squared terms). We make this quadric a sphere by insisting that its plane sections shall be circles. Putting all but three of the coördinates equal to zero, we obtain the section by the plane $A_1A_2A_3$ in the form

$$a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2 = 0.$$

But we know (see, for instance, W. P. Milne, *Homogeneous Coördinates*, London, 1924, pp. 104–106) that the circumcircle of a triangle ABC in areal coördinates is

$$a^2yz + b^2zx + c^2xy = 0.$$

Hence we may write $a_{23} = e_{23}^2$, $a_{31} = e_{31}^2$, $a_{12} = e_{12}^2$. By taking a sufficient number of such sections $A_iA_jA_k$ we thus obtain $a_{ij} = e_{ij}^2$, as required. Finally, we derive the normal coördinates from the barycentric coördinates by writing p_i/h_i for x_i.

Editorial Note. The more complicated derivation which follows does not require the result for $n=2$, and it may be of interest since it develops transformations which are useful for obtaining other equations in barycentric coördinates. Let the rectangular coördinates of A_i be $a_{i1}, a_{i2}, \cdots, a_{in}$ for any chosen origin, let Δ be the determinant of order $n+1$ whose ith row consists of the coördinates of A_i followed by unity and where $\Delta \neq 0$, and let $\Delta_i(x)$ be the determinant obtained from Δ by replacing the coördinates of A_i by $x_1, x_2, \cdots, x_n$. The equation of the face π_i opposite A_i is then $\Delta_i(x) = 0$. If P is any point, its distance p_i from π_i is given by $p_i = \Delta_i(x)/\sigma_i$, where the x's are coördinates of P and σ_i^2 is the sum of the squares of the coefficients of the x's. The similar distance for A_i is $h_i = \Delta/\sigma_i$ and we choose the sign of σ_i so that h_i is positive. Then $q_i = p_i/h_i = \Delta_i(x)/\Delta$, and we have

$$(1) \qquad \sum_{i=1}^{n+1} a_{ij}q_i = \sum_{i=1}^{n+1} a_{ij}\Delta_i(x)/\Delta = x_j, \qquad \sum_{i=1}^{n+1} q_i = 1,$$

for any point P whose barycentric coördinates with respect to S are $q_1, q_2, \cdots, q_{n+1}$ and whose rectangular coördinates are $x_1, x_2, \cdots, x_n$.

Now let the origin O be the circumcenter of S and R the radius of (O) whose

equation is $\sum x_t^2 = R^2$. Then

$$\sum_{t=1}^{n}\left(\sum_{i=1}^{n+1} a_{it}q_i\right)^2 = R^2 = \sum_{t=1}^{n}\left[\sum_{i=1}^{n+1} a_{it}^2 q_i^2 + 2\sum\sum_{i\neq j} a_{it}a_{jt}q_iq_j\right],$$

$$R^2 = R^2\sum_{i=1}^{n+1} q_i^2 + 2\sum_{t=1}^{n}\sum\sum_{i\neq j} a_{it}a_{jt}q_iq_j. \tag{2}$$

We have also

$$e_{ij}^2 = \sum_{t=1}^{n}(a_{it}-a_{jt})^2 = 2R^2 - 2\sum_{t=1}^{n} a_{it}a_{jt},$$

$$\sum\sum_{i\neq j} e_{ij}^2 q_iq_j = 2R^2\sum\sum_{i\neq j} q_iq_j - 2\sum_{t=1}^{n}\sum\sum_{i\neq j} a_{it}a_{jt}q_iq_j, \tag{3}$$

$$= R^2\left[2\sum\sum_{i\neq j} q_iq_j - 1 + \sum_{i=1}^{n+1} q_i^2\right] = R^2[(\sum q_i)^2 - 1] = 0,$$

where we have used (1) and (2). Thus the barycentric equation of (O) is

$$\sum\sum_{i\neq j} e_{ij}^2 q_iq_j = 0.$$

3909 [1939, 238]. *Proposed by Béla de Sz. Nagy, Szeged, Hungary.*

Let $P_1, P_2, \cdots, P_{n+2}$ be $n+2$ points in n-dimensional space, $n \geqq 2$, no three of the points on the same straight line. Let the symbol $[P_{i_1}P_{i_2}\cdots P_{i_s}]$ denote the least convex polyhedron containing in its interior the points indicated. Show that if $n=2, 3$, then there exist always subscripts i, k, such that $[P_iP_k]$ is not an edge of $[P_1P_2\cdots P_{n+2}]$. Show that if $n>3$, then this statement does not remain true.

Solution by H. S. M. Coxeter, University of Toronto.

When $n=2$, we have four coplanar points, P_1, P_2, P_3, P_4, of which any three form a triangle. By considering the various possible positions of P_4 relative to $P_1P_2P_3$, we see that $[P_1P_2P_3P_4]$ is either a triangle or a quadrangle. In fact, it is a triangle if the areal coördinates of P_4 are $(+\ +\ +)$ or $(-\ -\ +)$, and a quadrangle if they are $(-\ +\ +)$. Therefore, at most four of the six joins are edges.

When $n=3$, we have five points, P_1, P_2, P_3, P_4, P_5. If any of these lie in a plane, we have just seen that some joins of them would not be edges. Hence we may suppose that any four of the points form a tetrahedron. By considering the various possible positions of P_5 relative to $P_1P_2P_3P_4$, we see that $[P_1P_2P_3P_4P_5]$ is either a tetrahedron or a triangular bipyramid. In fact, it is a tetrahedron if the barycentric coördinates of P_5 are $(+\ +\ +\ +)$ or $(-\ -\ -\ +)$, and a bipyramid if they are $(-\ +\ +\ +)$ (apices P_1, P_5) or $(-\ -\ +\ +)$ (apices P_3, P_4). Therefore, at most nine of the ten joins are edges.

When $n=4$, we can construct a polytope composed of 6 vertices, $6+9$ edges, 18 triangles, and 9 (irregular) tetrahedra. We merely have to take triangles

$P_1P_2P_3$, $P_4P_5P_6$ in two planes whose only common point is interior to both triangles; for instance, we may take two equilateral triangles in absolutely perpendicular planes. In this case *all* the 15 joins of the 6 points are edges.

When $n>4$, we take six of the $n+2$ points as the vertices of such a four-dimensional polytope, and the remaining $n-4$ points in general position, so as to make a kind of generalized pyramid, having $(n+2)(n+1)/2$ edges.

Solved also by the proposer.

Editorial Note. After the discussion of the cases $n=2$, 3, the proposer gave the following for $n>3$. Let the points P_i, $(i=1, 2, \cdots, n)$, have the coördinates $x_1^i, x_2^i, \cdots, x_n^i$, where $x_i^i=1$, and the other $n-1$ coördinates are zero; P_{n+1} is the origin; and P_{n+2} has the coördinates $1, 1, -1, -1, \cdots, -1$. It will be shown that $[P_iP_j]$ is an edge of $[P_1P_2 \cdots P_{n+2}]$ whatever i and j may be, $i<j$. The hyperplane $L(x)\equiv x_1+x_2+\cdots+x_n-1=0$ contains the vertices of the $(n-1)$-dimensional simplex $[P_1P_2 \cdots P_n]$, and $L(P_{n+1})=-1$, $L(P_{n+2})=3-n<0$: hence $[P_1P_2 \cdots P_n]$ is on the boundary of $[P_1P_2 \cdots P_{n+2}]$. Thus $[P_iP_j]$ is an edge of $[P_1P_2 \cdots P_{n+2}]$ for $1\leqq i<j\leqq n$.

Consider next the hyperplanes $L_{rs}(x)\equiv x_r+x_s=0$ with $r=1, 2; s=3, 4, \cdots, n$. We see that $L_{rs}(P_i)=0$ for $i\neq r, s$, and that $L_{rs}(P_r)=L_{rs}(P_s)=1$. This means that the $(n-1)$-dimensional simplex determined by all the P's except P_r and P_s is on the boundary of $[P_1P_2 \cdots P_{n+2}]$; its edges $[P_iP_k]$, $(i<k, i\neq r, k\neq s)$, are therefore also edges of $[P_1P_2 \cdots P_{n+2}]$. Hence it has been shown that every $[P_iP_k]$, $(i<k)$, is an edge of $[P_1P_2 \cdots P_{n+2}]$.

3911 [1939, 238]. *Proposed by J. R. Musselman, Western Reserve University.*

If N be the center of the nine-point circle of triangle ABC, and L', M', N' be the symmetrics of A, B, C respectively as to N, show that the circles $AM'N'$, $BN'L'$, and $CL'M'$ meet on the circumcircle of ABC at Φ, the point of Feuerbach for the tangential triangle of ABC, that is, the triangle formed by the tangents to the circumcircle of ABC at its vertices.

Solution by Frank Ayres, Jr., Dickinson College.

Let the coördinates of the points A, B, $C\equiv A_i$, $(i=1, 2, 3)$ be denoted by the turns t_i, satisfying $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$. Then, N has the coördinate $\sigma_1/2$, and L', M', $N'\equiv B_i$ have the coördinates σ_1-t_i. The circle $A_iB_jB_k$ is centered at the point $\sigma_2/(\sigma_1-t_i)$, and its equation is

$$t_i(\sigma_1 - t_i)^2x\bar{x} - \sigma_1(\sigma_1 - t_i)x - t_i\sigma_2(\sigma_1 - t_i)\bar{x} = \sigma_3 - \sigma_1\sigma_2.$$

This circle passes through the point of Feuerbach for the tangential triangle of coördinate σ_2/σ_1.

Solved also by O. J. Ramler and R. Goormaghtigh.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The National Council of Teachers of Mathematics will have its twenty-first annual meeting in Atlantic City on February 21 and 22, 1941. The theme of the meeting will be "Mathematics in a Defense Program." Details of the program and places of meeting will appear in the February issue of *The Mathematics Teacher.*

Stanford University is celebrating its Fiftieth Anniversary during the year 1941. In this connection, a larger than usual Summer Quarter program is planned. Professor G. T. Whyburn from the University of Virginia is announced for a course on Combinatorial Topology, Professor Einar Hille from Yale University for a course on Selected Topics in Theory of Functions. During the summer, presumably in the first part of August, a two-day Symposium is planned with conferences on Topology, Calculus of Probability and Statistics, Theory of Functions, and general subjects. The following speakers have been secured so far: E. T. Bell, H. F. Blichfeldt, Einar Hille, A. D. Michal, J. Neyman, G. Pólya, G. T. Whyburn, A. Zygmund. The program will be announced later.

Dr. K. C. Schraut has been appointed to an instructorship at the University of Dayton.

The Editor wishes to express his appreciation to the following persons who refereed papers or otherwise assisted in the work of editing the MONTHLY for the year 1940:

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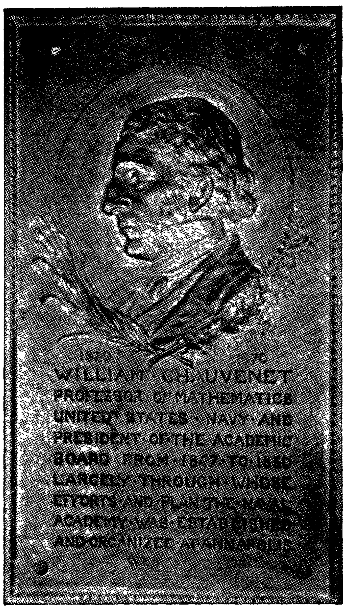
NAVAL ACADEMY GIVES CHAUVENET MEMORIAL

The Mathematical Association of America established in 1925 the Chauvenet Prize "for the best expository paper published in English" during a certain period of years. This may render especially interesting to readers of the MONTHLY the following news item from the *Alumni Bulletin* of Washington University:



THE CHAUVENET PLAQUE

William Chauvenet, second chancellor of Washington University, is now commemorated by a plaque on the wall of Ridgley Arcade, presented by the United States Naval Academy and unveiled on Navy Day, October 27.

The plaque is a replica of one in Mahan Hall at the Academy, where Chauvenet is venerated as one of the founders of that institution. He left it to become professor of mathematics and astronomy here in 1852; was made chancellor in 1862 and served until his death in 1869.

Before an audience of some twelve hundred people in the Quadrangle Capt. Mark C. Bowman, USN, presented the plaque on behalf of the Academy. It was unveiled by Louis Chauvenet, grandson of the former chancellor, and accepted by Chancellor George R. Throop. Capt. Bowman holds the chair of Seamanship and Navigation once occupied by Professor Chauvenet at the Academy.

The ceremonies constituted the local observance of Navy Day, and were attended by the St. Louis alumni of the Naval Academy, who were instrumental in arranging for the gift. Governor Lloyd C. Stark delivered the principal address, and the British Consul, Mr. H. B. McClelland, read a greeting from the Admiralty.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 2, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May.
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3 or 4.
OKLAHOMA
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, April.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, March 28–29.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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CONTENTS

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EDITORIAL

In the present number of the MONTHLY we have the announcement of a new series of Papers named in honor of the late Professor Herbert Ellsworth Slaught. We believe that the Association is launching a project of great importance to mathematics in America, one which would have received the most enthusiastic support of the man whose life it will commemorate. Under the guidance of a capable and enthusiastic committee, headed by Professor Langer, the new series should bring out its first number this year, as a supplement to the MONTHLY, to be sent free to all subscribers.

At the time of his death in 1937 Professor Slaught had served as an editor of the MONTHLY for thirty years. It was his vigorous interest in collegiate mathematics which was largely responsible for the organization of the Mathematical Association of America in 1915. In 1933 his exceptional contributions to the Association were signally recognized when he was elected Honorary President of the Association for life. The 1938 volume of the MONTHLY was dedicated to him "in appreciation of his exceptional services" to the Association and to the MONTHLY, and the leading articles were devoted to memorial papers by his close associates Professors Cairns and Bliss; the volume also contained a series of expository articles contributed by his former students and friends as an especial token of high regard for Professor Slaught, and in appreciation of his interest in expository writing of mathematical papers.

I wish to express my personal satisfaction in having this opportunity to assist in further honoring Professor Slaught, under whose inspiring guidance I first became acquainted with the calculus, differential equations, infinite series, and elliptic integrals. Among a distinguished group of teachers under whom it was my good fortune to study, including Professors Bôcher, Bolza, Bliss, Dickson, Lunn, Maschke, Max Mason, E. H. Moore, F. R. Moulton, Osgood, Skinner, E. B. Van Vleck, and Wilczynski, I rank Professor Slaught very high as an expositor of collegiate mathematics, and without a peer in his long-continued helpful personal interest in his students. He liked students, he liked to teach mathematics, and by his devoted service to mathematics and mathematicians he made a conspicuously important contribution to the development and dissemination of our science.

E. J. M.

THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

At its recent meeting, and at the instance of the committee report published below, the Mathematical Association took action toward the initiation of a series of publications which it envisages as being primarily of an expository nature. In length these publications are to be intermediate between the relatively few pages of an ordinary journal paper and the relatively many of a book. In form they are to be issued as pamphlets ranging from forty to eighty pages in length. It is hoped that in time the gamut of subjects covered may be extensive. The treatments are thought of as likely to vary with the subject, and to range from informative syntheses of topics of contemporary research at the one extreme, to

quite elementary concise expositions of established mathematical theories and applications on the other. Critical and philosophical essays, and historical or biographical accounts and the like may well find a place in the series.

In undertaking the sponsorship of such a series as this the Association believes itself to be responding to a genuine desideratum of its members. In return it bespeaks that support which can best be given through an adequate recognition of the high achievement and service which authorship of any scholarly readable mathematical presentation embodies. Expository writing is exacting in its demands upon time, talent, and clarity of thought. It rewards mainly by tendering the means of teaching the many rather than the few.

The editorial committee through the undersigned will welcome constructive suggestions from all who have them to offer.

R. E. LANGER

REPORT OF THE NEWSOM COMMITTEE ON THE SLAUGHT MEMORIAL PAPERS

In recent years American mathematicians have become insistent that the number and variety of expository treatises in the field of mathematics be increased. Frequently it has been advocated that the Mathematical Association of America should accept considerable responsibility for the stimulation of such expository writing and for the establishment of additional media through which expository tracts can be made available to mathematical scholars. The Carus Monographs were established to further the opportunity for the publication of expository works. Moreover, the Association's publication, THE AMERICAN MATHEMATICAL MONTHLY, has published a goodly number of excellent expository articles. The question still remains, however, whether the present program of the Association is adequate.

During the summer meeting of 1938 the trustees of the Association authorized the appointment of a committee to review the activities of the Association. The resulting committee of eight under the chairmanship of Professor R. E. Langer considered many aspects of the Association's activities. In the report of the committee accepted in December, 1939, appears the following recommendation:

"The encouragement and sponsorship of expository and critical writing is one of the objectives of the Association which enjoys the unanimous support of the members. There is a ready welcome and a general demand for more readable scholarly papers on all kinds of mathematical subjects from the classical to the modern, from the elementary to the advanced, on theory, on application, on history, or on philosophy. In the past there have been, of course, the Carus Monographs, and from time to time excellent papers in the MONTHLY. There seems, however, to be at the present time little or no means for the ready publication of writings which in length are intermediate between the relatively few pages of a journal paper, and the relatively many pages of a complete monograph. Such papers, say in length between twenty and a hundred pages, could be profitably written on subjects in many categories, including among others, elementary

introductory expositions of theories and their applications, more advanced expositions and interpretations of modern viewpoints and theories, philosophical essays and criticisms, broad historical accounts of important schools, or biographical accounts of individuals. The Association could perhaps well undertake to publish such papers in the form of a series of pamphlets to be made available to the members at cost. And we believe that these costs could well be made to include small token honoraria to the authors. It is not our intention in this connection to suggest that papers be bought and paid for, but rather that the Association recognize that this branch of authorship calls for talent and involves much work, while yielding but little in return. We believe that a material token will do much toward enlisting the ablest authors, and that this same end will be furthered if the series is properly dignified in form and name. We recommend, therefore,

"*That the Association create a standing 'Committee on Expository Writing,' to be charged with receiving, refereeing, and at its discretion inviting papers of intermediate length. That such papers be published in the form of a series of pamphlets, which could appropriately be designated the Herbert Ellsworth Slaught Memorial Publications, and that the Association undertake to award a token honorarium to each author (from $25 to $75 is suggested) in appreciation of his authorship and of his donation of the paper to the Association.*"

Subsequent to the acceptance of the report of the "Langer Committee," the trustees of the Association authorized the appointment of a committee "to study the desirability of the establishment of the H. E. Slaught Memorial Publications, and to formulate definite plans, including the possible alternative method of serial publications in the MONTHLY and available reprints, the committee to report to the Trustees." The undersigned committee was appointed by Professor W. B. Carver, President of the Association, in May, 1940, for the specified purpose.

In carrying out the commission assigned to us, it seemed imperative that we first investigate the need for additional expositions upon mathematical topics as well as the programs of the various agencies now sponsoring expository publications. Many individuals known to be familiar with such matters were consulted, and, in addition, a cross-section of the Association's membership was interrogated.

The principal conclusion reached by this investigation is that there is a widespread interest in additional expository writing of the type discussed in the report of the "Langer Committee," and that those who are now sponsoring series of expository monographs would welcome the creation of additional opportunities for the publication of studies pertaining to mathematical subjects. In truth, the members of the committee have been impressed with the enthusiasm which has been displayed by those who have given opinions relative to the possibility of a new publication program sponsored by the Association. Syntheses of modern investigations in many fields of mathematics seem to be wanted by college men who do not have an opportunity to follow developments in the mathematical literature. Instructors in our junior colleges and secondary schools who may have

a limited preparation in mathematics are seeking easily accessible accounts of some of the older theories. Some correspondents have expressed the belief that there is an amazing dearth of readable mathematical material for college students who have studied little beyond the calculus. And finally, some have emphasized that the interest of the American public in mathematical attainments and methods needs to be cultivated; this interest is attested to by the recent wide sale of a few popular books upon mathematics.

It is the belief of the committee, moreover, that the Mathematical Association of America, in view of its purpose and objectives, is the appropriate organization to create additional publication facilities dedicated to more and better expository writing. In many respects, the strong demand for more expository material represents a challenge to the Association's leadership.

Pursuant to these conclusions, therefore, the following specific recommendations are made for consideration by the Board of Governors of the Association:

(1) That the Mathematical Association of America sponsor the publication of a series of pamphlets from forty to eighty pages in length containing one or more expository articles upon mathematics and related subjects.

(2) That the articles published in such a series be designated the "Herbert Ellsworth Slaught Memorial Papers."

(3) That the Association award an honorarium to each author which shall be computed upon the basis of one dollar per page of printed manuscript.

(4) That each pamphlet in the series be published as a supplement to a regular issue of the AMERICAN MATHEMATICAL MONTHLY.

(5) That a new associate editor be appointed in the customary manner to the staff of the MONTHLY as editor of the H. E. Slaught Memorial Papers under the general supervision of the Editor-in-Chief. It will be his duty, if necessary, to invite particular men to write papers on specified subjects, and in any case to take the responsibility for receiving, refereeing, and preparing papers for the printer, and reading proof as the papers are prepared for publication.

By way of discussion of the above proposals, certain remarks are pertinent. First of all, the committee is conscious of the fact that the success of the proposed series of expository works depends to a great extent upon the person appointed to the associate editorship. The individual appointed must have the complete confidence of other mathematicians, and he himself should have an established reputation in expository writing.

In regard to the fourth proposal above, it is the thought of the committee that the form of the proposed supplements should resemble that now employed in the publication of the register of officers and members, with the single exception that the sub-title, "Register of Officers and Members," which appears upon the cover, should be replaced by the designation, "Herbert Ellsworth Slaught Memorial Papers." It appears that such a provision satisfies postal regulations in regard to the mailing of second class material, but, of course, official confirmation should be obtained.

The committee desires to emphasize that the honorarium to be awarded to each author should not be regarded as payment for his manuscript. Rather it is

to be hoped that the award will partially defray the expenses incidental to the preparation of the manuscript as well as signify the appreciation of the Association for services rendered. The token payment may furnish some incentive to prospective authors.

The committee recommends that the program proposed above be inaugurated upon a rather modest basis. It is doubtful if the number of supplements appearing in any one year should exceed three. If the series be initiated next year with one noteworthy supplement, we would regard the start as auspicious.

It is apparent from the discussion immediately above that the committee does not anticipate a tremendous strain upon the financial resources of the Association. In fact, it is our belief that for the present, at least, it should not be necessary to increase the annual dues charged to members of the Association in order to finance the project which we propose. Under ordinary circumstances the revenues of the Association exceed expenditures, and a sizeable financial surplus has accrued. Moreover, the subventions to the *Annals* and the *Duke Journal* are being withdrawn, thereby releasing additional funds for the conduct of Association enterprises. Also, a recent action of the Governors would indicate that the subvention to the *Mathematical Reviews* is not to be regarded as a permanent encumbrance upon Association funds. Thus the financial outlook for the Association is definitely favorable.

Nevertheless, it might be necessary to effect certain economies in order to finance the proposed program within the present income of the Association. For instance, the suggestion has been made that the directory of members could be printed less often, and the money saved be diverted to the enterprise which we propose. Also, the supplements might be printed by the off-set method, thereby reducing their cost.

The sale of the supplements to non-members of the Association at fifty cents per copy should be profitable. The register of members of the National Council of Teachers of Mathematics contains the names of many potential subscribers.

For the present, however, the chief concern of the committee is that the publication of the Slaught Memorial Papers be accepted as one of the activities of the Association. If at some future date additional funds are needed to properly maintain the Association's program, we believe that the members will gladly increase their annual contribution—especially if the program has their sincere approval. The committee has a definite indication from several members of the Association that they would be willing to have the annual dues increased to five dollars per year if additional expository material could be made available to them. Of course, an increase in dues could be justified to the membership as a whole only after one or more excellent supplements had already been provided. Therefore, we recommend that the publication of the H. E. Slaught Memorial Papers be instituted during the year 1941, with at least one supplement distributed free to each member of the Association.

Respectfully submitted,

GORDON FULLER
E. J. MOULTON
C. O. OAKLEY
G. B. PRICE
C. V. NEWSOM, *Chairman*

THE FALL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The fifteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Grove City College, Grove City, Pennsylvania, on Saturday, November 2, 1940. Professor J. S. Taylor, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was thirty-five, including the following seventeen members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, W. B. Brown, P. N. Carpenter, L. L. Dines, H. C. Hicks, B. P. Hoover, M. L. Manning, David Moskovitz, C. T. Oergel, E. G. Olds, Elizabeth Renwick, S. R. Smith, R. G. Sturm, J. S. Taylor, E. D. Wells.

The following officers were elected for the coming year: Chairman, J. S. Taylor, University of Pittsburgh; Secretary, David Moskovitz, Carnegie Institute of Technology; member of Executive Committee, C. H. Vehse, West Virginia University. Professor H. L. Dorwart, Washington and Jefferson College, continues in office for the second year of his term as the additional member of the Executive Committee.

At the close of the afternoon session, tea was served for the guests. A vote of thanks was extended to the staff of the college for their generous hospitality.

After the opening address by President Weir C. Ketler of Grove City College, the following six papers were read:

1. "Stress and deflection in reciprocating parts of internal combustion engines" by T. O. Kuivinen, Engineer, Cooper-Bessemer Corporation, introduced by Professor Carpenter.

2. "Fitting the engineering student into industry" by W. R. Crooks, Resident Engineer, Cooper-Bessemer Corporation, introduced by Professor Carpenter.

3. "Plane collineations" by Professor K. H. Stahl, State Teachers College, California, introduced by the Secretary.

4. "Lagrange's expansion and its use in solving certain equations" by H. G. Landau, Carnegie Institute of Technology, introduced by the Secretary.

5. "Groups of reflections in n dimensions" by E. L. Kaplan, Carnegie Institute of Technology, introduced by the Secretary.

6. "Comments on the North Carolina program in freshman mathematics" by Professor H. L. Dorwart, Washington and Jefferson College.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Kuivinen presented a joint paper with Mr. R. L. Boyer, Chief Engineer of the Cooper-Bessemer Corporation, in which they considered stresses and deflections in reciprocating parts of internal combustion engines from the viewpoint of the heavy-duty engine builder. The authors gave suggested rules of design for pistons, piston pins, connecting rods and piston rods, bolts and connecting rod caps. All recommendations given were based on what has proved to be successful practice in this class of machinery.

2. Mr. Crooks presented his personal experiences and observations during the

past seventeen years with large corporations in the heavy goods industry, and concerned himself with the fitting of the engineering graduate into industry.

3. Professor Stahl rewrote a common form for plane collineations, and the eight parameters of this new form were taken as the coördinates of four points. These points, the "pattern points" of the collineation, determined the collineation, which in turn may be used to find the transforms of the points themselves; thus revealing that two of the points were the transforms of the other two. To distinguish them, the primed pair were taken as the transforms of the unprimed pair. Lines were drawn from a general point (x, y) to the unprimed pattern points, and the points themselves were joined to complete a triangle. The slopes of the sides of this triangle determined a pair of lines through the primed pattern points and which intersected at (x', y'), the transform of (x, y). Further investigation showed that the pattern points remained unchanged for a translation of axes, but moved along straight lines, in pairs perpendicular to each other, when the axes were rotated. These lines have been used as "intrinsic axes" which form a basis for a method of locating the transforms of points.

4. Mr. Landau gave an exposition of Lagrange's expansion theorem, with examples of its use in solving some algebraic and transcendental equations.

5. Mr. Kaplan presented a study of groups of transformations of the form $x_i=\epsilon_i x_i'$, $(i=1, 2, \cdots, n)$, where each ϵ_i represented a definite one of the numbers 1 or -1, all signs being independent. The principal properties of abstract self-inversive groups, of which the groups of reflections are examples, were first established. The number of groups of reflections of given order and given number of dimensions was found. The notion of class was introduced and investigated; a class of groups includes all groups of reflections obtainable from a given group by a permutation of the variables x_i. Each class may be thought of as representing a particular type of symmetry. The resultant of two or more classes was used to express all classes in terms of a few elemental classes. All the 277 classes (32,501 groups) of dimensions 1 to 7 inclusive were constructed and tabulated. The tabulation of classes was used to disprove several plausible statements concerning classes in general.

6. This paper, which was read at the meeting by Professor Atchison, appeared in the January, 1941, issue of this MONTHLY.

DAVID MOSKOVITZ, *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The fifteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 30, 1940, Professor J. A. Shohat presiding.

The attendance was forty-five, including the following twenty-seven members of the Association: E. F. Allen, C. B. Allendoerfer, Robert Atkinson, J. A.

Benner, H. W. Brinkmann, W. B. Campbell, P. A. Caris, Mary L. Constable, E. H. Cutler, D. R. Davis, J. E. Davis, Arnold Dresden, Janet C. Durand, W. H. Fagerstrom, Helen G. Fudge, J. R. Kline, V. V. Latshaw, Marguerite Lehr, H. H. Mitchell, G. E. Raynor, C. J. Rees, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, P. I. Speicher, J. W. Tukey.

At the business meeting the following officers were elected for next year: Chairman, Arnold Dresden, Swarthmore College; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, C. O. Oakley, W. M. Smith, and J. A. Shohat. It was voted to hold the 1941 meeting at Swarthmore College, Swarthmore, Pennsylvania. The time was left to the discretion of Professor Dresden and the Secretary. They have chosen November 29, 1941.

The following four papers were presented:

1. "Transitive flows" by Professor J. C. Oxtoby, Bryn Mawr College, introduced by Professor Dresden.

2. "Modern methods in differential geometry" by Dr. J. L. Vanderslice, Lehigh University, introduced by Professor Dresden.

3. "On Dedekind sums" by Professor H. A. Rademacher, University of Pennsylvania, introduced by Professor Dresden.

4. "Statistics involved in College Entrance Examinations" by Professor S. S. Wilks, Princeton University, introduced by Professor Dresden.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Oxtoby stated that a continuous flow is defined as a one-parameter continuous group of automorphisms of a space. It is called topologically transitive if there exists an everywhere dense streamline. A construction was described whereby a transitive flow can be defined in the cube. Extensions of the result and connections with the theory of differential equations were mentioned briefly.

2. Dr. Vanderslice presented an essentially expository paper embracing the following subject-matter: (i) the generalizations of classical differential geometry; (ii) unification through the notion of the "geometric object"; (iii) methods of developing the theory of a geometric object. In (iii) the author tried to show the indispensability of tensors and tensor differentiation, and discussed general methods of applying the tensor technique to geometries based on a geometric object.

3. Professor Rademacher spoke about "Dedekind sums." These are certain arithmetical expressions which appear in the theory of modular functions. They have been introduced by R. Dedekind in his commentary on fragmentary notes of Riemann, published in Riemann's *Gesammelte Werke*. The Dedekind sums have interesting arithmetical properties, the most important one of which can be stated as a "theorem of reciprocity." A detailed account of the theory is given in a paper by A. Whiteman and the speaker, which will be published shortly.

4. Professor Wilks discussed statistical problems which arise in the construction and reading of examinations, with particular reference to examinations

constructed and administered by the College Entrance Examination Board. As far as an examination is concerned, a field of subject-matter is defined as an aggregate of questions which would be declared appropriate by experts in the field. The difficulty of any given question in a given field must be defined with respect to a given aggregate or population of individuals for whom the question would be regarded as appropriate. The determination of the difficulty of an item for any given group of individuals depends on an actual try-out of the question. The problem of making up an examination is one of sampling the aggregate of questions which define the field. In general, the longer the examination, the greater will be the degree of discrimination between each pair of individuals taking the test, in the sense that similar examinations of the same length will tend to produce approximately the same ranking of individuals. The mathematics underlying this statement was briefly discussed. To secure a maximum amount of reliability in an examination with a given number of questions, care must be taken to select items of suitable difficulty satisfying the subject-matter requirements such that the correct and incorrect responses on each pair of questions are highly positively correlated. Practical methods involved were discussed.

Statistical procedures as used by the Board for isolating the extent of unreliability of scores due to inconsistent marking of papers were described. Still further work is done in post-mortem statistical analysis of results and in checking on the relationship between examination scores and subsequent performance in college courses.

P. A. CARIS, *Secretary*

SERIES WITH DELETED TERMS*

I. E. PERLIN, Illinois Institute of Technology

1. Introduction. In 1914 A. J. Kempner† examined the harmonic series and showed that the series resulting after deleting the terms containing the digit 9 at least once is convergent. This result was generalized by E. J. Moulton‡ and Frank Irwin.§ In the present paper the author considers a general series and establishes sufficient conditions that the series resulting after deleting certain terms is convergent. The results due to Kempner, Moulton, and Irwin follow as special cases of the theorems developed here.

An application is made to series with positive and negative terms. Sufficient conditions that such series diverge are given in terms of the frequency of the terms of one sign.

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* Presented to the Illinois Section of the Mathematical Association of America, May 10, 1940.

† A. J. Kempner, this MONTHLY, vol. 21, 1914, pp. 48–50.

‡ E. J. Moulton, this MONTHLY, vol. 23, 1916, pp. 302–303.

§ Frank Irwin, this MONTHLY, vol. 23, 1916, pp. 149–152.

2. Series with deleted terms. Let $\sum_{n=1}^{\infty} a_n$ denote a series of real or complex numbers. We shall partition this series into groups of terms, the ith group containing N_i terms. Let $\{M_i\}$ denote a sequence of positive numbers such that for every term $a_{n,i}$ in the ith group,

$$|a_{n,i}| \leqq M_i.$$

Let $\{a_{n_j}\}$ be a sequence of terms of the series to be deleted. Let k_i be the number of terms in the ith group to be deleted. Let $p_i = 1 - k_i/N_i$. We shall denote the series resulting after deleting the sequence of terms $\{a_{n_j}\}$ by $\sum_{n=1}^{\prime\infty} a_n$. We now prove the following:

THEOREM 1. *If $\sum_{i=1}^{\infty} p_i$ and $\sum_{i=1}^{\infty} |M_{i+1}N_{i+1} - M_iN_i|$ converge, then $\sum_{n=1}^{\prime\infty} a_n$ converges.*

Let $s_n = \sum_{i=1}^{n} p_i$ and $s = \sum_{i=1}^{\infty} p_i$. Then

$$\sum_{v=n+1}^{n+t} p_v M_v N_v = \sum_{v=n+1}^{n+t} s_v(M_v N_v - M_{v+1}N_{v+1}) + s_n(M_{n+t+1}N_{n+t+1} - M_{n+1}N_{n+1}) + (s_{n+t} - s_n)M_{n+t+1}N_{n+t+1}, \tag{1}$$

and

$$\left| \sum_{v=n+1}^{n+t} p_v M_v N_v \right| \leqq 2s \sum_{v=n+1}^{n+t} |M_v N_v - M_{v+1}N_{v+1}| + |s_{n+t} - s_n| M_{n+t+1}N_{n+t+1}.$$

Hence,

$$\lim_{n=\infty} \left| \sum_{v=n+1}^{n+t} p_v M_v N_v \right| = 0,$$

uniformly in t.

THEOREM 2. *If $\sum_{i=1}^{\infty} p_i$ converges and $\{M_iN_i\}$ is monotonically decreasing, then $\sum_{n=1}^{\prime\infty} a_n$ converges.*

From (1) it follows that

$$\left| \sum_{v=n+1}^{n+t} p_v M_v N_v \right| \leqq s_{n+t}(M_{n+1}N_{n+1} - M_{n+t+1}N_{n+t+1}) - s_n M_{n+1}N_{n+1} + s_{n+t}M_{n+t+1}N_{n+t+1},$$

$$\left| \sum_{v=n+1}^{n+t} p_v M_v N_v \right| \leqq (s_{n+t} - s_n)M_{n+1}N_{n+1}.$$

$$\lim_{n=\infty} \left| \sum_{v=n+1}^{n+t} p_v M_v N_v \right| = 0,$$

uniformly in t.

These theorems can be deduced directly from known results.*

Let us consider the harmonic series $\sum_{n=1}^{\infty} 1/n$. Let us partition this series into groups. The ith group shall consist of those terms $1/n$ for which n contains exactly i digits. We see that the number of terms in the ith group, N_i, equals $9 \cdot 10^{i-1}$. We can choose $M_i = 10^{1-i}$, since

$$1/n \leqq M_i$$

if n has exactly i digits. If we delete all terms containing the digit 9 at least once, we can easily calculate $p_i = \frac{8}{9} \cdot (\frac{9}{10})^{i-1}$. We see then, that Theorem 1 or Theorem 2 will yield the result established by A. J. Kempner.

For the extension of Kempner's work as given by E. J. Moulton and Frank Irwin we choose N_i and M_i as above, and proceed to calculate p_i. It is easily seen that the number of terms remaining in the ith group after deleting those terms containing the digit 0 at least a times is

$$9\left\{9^{i-1} + \binom{i-1}{1} \cdot 9^{i-2} + \cdots + \binom{i-1}{a-1} \cdot 9^{i-a}\right\}.$$

The number of terms remaining in the ith group after deleting all terms containing the digit 9 at least a times is

$$9\left\{9^{i-1} + \binom{i-1}{1} \cdot 9^{i-2} + \cdots + \binom{i-1}{a-2} \cdot 9^{i-a+1} + \frac{8}{9}\binom{i-1}{a-1} \cdot 9^{i-a}\right\}.$$

The results for the other digits are the same as that for the digit 9. We now calculate the number of terms remaining in the ith group after deleting all terms containing the digit 9 at least a times, the digit 8 at least b times, $\cdots$, and the digit 0 at least j times.

Every term remaining in the ith group must satisfy at least one of the following conditions:

1. It contains the digit 9 at most $a-1$ times.
2. It contains the digit 8 at most $b-1$ times.

. . . .

10. It contains the digit 0 at most $j-1$ times.

Hence, the number of terms in the ith group to be calculated is not more than

$$10 \cdot 9\left\{9^{i-1} + \binom{i-1}{1} \cdot 9^{i-2} + \cdots + \binom{i-1}{q-1} \cdot 9^{i-q}\right\},$$

where $q = a+b+ \cdots +j$. If $i > 2q-1$, then

$$10 \cdot 9\left\{9^{i-1} + \binom{i-1}{1} \cdot 9^{i-2} + \cdots + \binom{i-1}{q-1} \cdot 9^{i-q}\right\} \leqq 10 \cdot 9^i q(i-1)^{q-1}.$$

* See Hadamard, Acta Mathematica, vol. 27, 1903, pp. 177–184.

Hence,

$$p_i \leqq 10 \cdot q(i-1)^{q-1}\left(\frac{9}{10}\right)^{i-1}, \qquad (i > 2q-1).$$

Applying Theorem 1 or Theorem 2 to the present case we obtain the result established by both E. J. Moulton and Frank Irwin, that the series resulting after deleting all those terms of the harmonic series containing the digit 9 at least a times, the digit 8 at least b times, $\cdots$, and the digit 0 at least j times is convergent.

3. Special theorems on the harmonic series. The above results can be generalized further. As before, let us choose $N_i = 9 \cdot 10^{i-1}$ and $M_i = 10^{1-i}$. Now let $a_i, b_i, \cdots, j_i$ be positive integers or zero for every value of i. We can establish the following:

THEOREM 3. *Let us partition the harmonic series into groups. The ith group shall consist of those terms $1/n$ where n contains exactly i digits. In the ith group we shall delete those terms containing the digit 9 at least a_i times, the digit 8 at least b_i times, $\cdots$, and the digit 0 at least j_i times. If the functions $a_i, b_i, \cdots, j_i$ are bounded, then the resulting series is convergent.*

The proof of this theorem follows that given for the application of Theorem 1 or Theorem 2 to obtain the result of Moulton and Irwin.

THEOREM 4. *The series resulting after deleting all those terms of the harmonic series for which two consecutive digits are alike is convergent.*

As before, choose $N_i = 9 \cdot 10^{i-1}$ and $M_i = 10^{1-i}$. The number of terms remaining in the ith group after deleting the terms for which two consecutive digits are alike is seen to be 9^i. Hence, $p_i = (\frac{9}{10})^{i-1}$ and the result follows directly from Theorem 1 or Theorem 2.

THEOREM 5. *The series resulting after deleting all those terms of the harmonic series for which the first digit is repeated at least a times is convergent.*

As before, let $N_i = 9 \cdot 10^{i-1}$ and $M_i = 10^{1-i}$. Then

$$k_i = 9\left\{\binom{i-1}{a-1} \cdot 9^{i-a} + \binom{i-1}{a} \cdot 9^{i-a-1} + \cdots + \binom{i-1}{i-1}\right\},$$

and

$$p_i = \frac{9 \cdot 10^{i-1} - k_i}{N_i},$$

$$p_i = \frac{9^{i-1} + \binom{i-1}{1} \cdot 9^{i-2} + \cdots + \binom{i-1}{a-2}}{10^{i-1}}.$$

If $i > 2a - 2$, then

$$p_i \leqq \frac{(a-1) \cdot 9^{i-1}(i-1)^{a-2}}{10^{i-1}}.$$

Theorem 1 or Theorem 2 can be applied now, and the result follows.

4. Real series with positive and negative terms. Let us consider an arbitrary series of real terms $\sum_{n=1}^{\infty} a_n$. Let us partition this series into groups. Let N_i be the number of terms in the ith group, and M_i be such that for every term $a_{n,i}$ in the ith group,

$$|a_{n,i}| \leqq M_i.$$

Let k_i denote the number of negative terms in the ith group. Let $p_i = 1 - k_i/N_i$. We prove the following:

THEOREM 6. *If $\sum_{i=1}^{\infty} p_i$ and $\sum_{i=1}^{\infty} | M_i N_i - M_{i+1} N_{i+1} |$ converge, and $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.*

Let t_j equal the sum of the positive terms in the first j terms of the series $\sum_{n=1}^{\infty} a_n$, and let r_j equal the sum of the negative terms in the first j terms of the series $\sum_{n=1}^{\infty} a_n$. Then

$$\sum_{n=1}^{j} |a_n| = t_j - r_j.$$

Since the $\lim_{j=\infty} t_j$ exists by Theorem 1, it follows that $\lim_{j=\infty} r_j = -\infty$. Now

$$\sum_{n=1}^{j} a_n = t_j + r_j.$$

Hence,

$$\lim_{j=\infty} \sum_{n=1}^{j} a_n = -\infty,$$

and the series $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$.

THEOREM 7. *If $\sum_{i=1}^{\infty} p_i$ converges, $\{M_i N_i\}$ is monotonically decreasing, and $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

This theorem follows from Theorem 2 in the same manner that Theorem 6 follows from Theorem 1.

SIMPLE EXAMPLES OF LIMITING PROCESSES IN PROBABILITY*

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An urn contains n balls numbered from 1 to n, but otherwise indistinguishable. A ball is withdrawn, its number is noted, it is replaced, and the contents of the urn thoroughly mixed. The process is repeated until some ball appears for the second time. We consider two problems:

(A) To find the average number A of drawings before a repetition occurs.

(B) To find the largest number B of drawings for which the probability of no repetition is still larger than $\frac{1}{2}$.

The explicit formulation of these numbers A and B presents no difficulty. To make up the average, A, the first drawing always contributes unity, the second contributes unity $(n-1)/n$ of the time, *etc.*; that is,

$$A = \frac{n}{n} + \frac{n(n-1)}{n^2} + \cdots + \frac{n!}{n^n}. \tag{1}$$

In the second problem we seek the largest integer, B, for which

$$\frac{n}{n}\,\frac{n-1}{n}\,\frac{n-2}{n}\cdots\frac{n-B+1}{n} > \frac{1}{2}. \tag{2}$$

One purpose of this paper is to determine asymptotic expansions for A and B. A second purpose is to show different types of student reaction to these problems, and the preliminary methods actually used by college students. Finally, it is hoped that the material of this paper will be useful to teachers of the calculus in showing to their classes (i) the importance of numerical calculation in ascertaining the probable nature of a solution, and (ii) the dangerous nature of limiting processes applied in the spirit of Eulerian mathematics.

The values A and B are not difficult of calculation, even for fairly large n, and they are especially easy if n is a power of 10. From (1) and (2) we thus obtain

(3)

n	A	B
10	3.660	4
100	12.210	12
1000	39.303	37
10000	124.999	118

These calculations are surprising in that both A and B are much smaller than our intuition would lead us to expect.

Now from the data in (3) a numerically-minded student may actually guess the exact form of the asymptotic formula for A. Here is a typical student argument. Since the values of A increase approximately 10-fold as n increases 100-

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fold, we should expect $A=\alpha\sqrt{n}$ as a first approximation, and

$$A = \alpha\sqrt{n} + \beta \tag{4}$$

as a strong approximation. Substituting the last two pairs of values of (3) in (4), and solving for α and β we obtain $\alpha=1.2533$ and $\beta=-0.329$. Noting from a table of constants that $\sqrt{\pi/2}=1.2533$, we guess that (4) may be written

$$A = \sqrt{\frac{\pi n}{2}} - \frac{1}{3}\,. \tag{5}$$

How does the formula (5) agree with the exact values in (3)? To three decimal places, we find

(6)

n	A	"Formula" A	Error
10	3.660	3.630	0.030
100	12.210	12.200	0.010
1000	39.303	39.300	0.003
10000	124.999	124.998	0.001

Thus without a semblance of mathematical proof, the formula (5) takes on an exceedingly high presumption of validity.

A different type of student thought and reaction is represented in the following development. We write (1) as

$$A = \frac{n!}{n^n}\left[1 + \frac{n}{1} + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!}\right]. \tag{7}$$

The bracket is, for large integral n, that portion of the expansion of e^n up to the point where the terms have obtained their maximum value. The next term in the expansion is equal to the last term of the bracket. The next following term is approximately equal to the next to last term in the bracket, and so on. Hence we guess that the bracket is approximately $e^n/2$. At first this appears to be very specious reasoning, but it turns out to be surprisingly near the truth. Let us first check it numerically. Call the bracket $S'(n)$, and let $S(n)=S'(n)+n^n/n!$. By actual computation we have

(8)

n	$S'(n)$	$\frac{1}{2}e^n$	$S(n)$	$\frac{1}{2}e^n-S'(n)$	$S(n)-\frac{1}{2}e^n$	Exponent
10	1.009	1.101	1.284	0.092	0.183	4
100	1.308	1.344	1.415	0.036	0.071	43
1000	9.768	9.850	10.016	0.082	0.166	433
10000	4.3917	4.4034	4.4269	0.0117	0.0235	4342

where all entries after the first column are to be multiplied by that power of 10 listed in the column called "Exponent." Hence it appears that the approximation is probably valid, and that an even better approximation is

$$S'(n) = \frac{1}{2}e^n - \frac{1}{3}\frac{n^n}{n!}\,. \tag{9}$$

Using Stirling's formula,

$$n! = n^n e^{-n}\sqrt{2\pi n} \tag{10}$$

in connection with (7) and (9), we again obtain (5).

The table (8) is fairly convincing evidence of (9); a more exact statement of (9) and a formal proof may be found in Polyai and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 80, No. 211. The validity of the application of Stirling's formula may be quite easily justified, and thus (5) is finally established. But it is no reflection on the American undergraduate to add that he finds the numerical evidence more convincing than the formal proof.

Turning next to problem (B), we rewrite (2) as

$$\frac{n!}{(n-B)!n^B} > \frac{1}{2}\,. \tag{11}$$

In (11) we replace the factorials by Stirling's formula, and if we replace B by x, where x is not restricted to integral values, we have

$$\frac{n^n e^{-n}\sqrt{2\pi n}}{(n-x)^{n-x}e^{-n+x}\sqrt{2\pi(n-x)}\,n^x} = \frac{1}{2}\,; \tag{12}$$

or, after cancellation and regrouping,

$$\frac{1}{e^x\left[1-\dfrac{x}{n}\right]^{n-x+1/2}} = \frac{1}{2}\,. \tag{13}$$

Further regrouping, with an eye to the future, gives

$$\frac{1}{e^x\left\{\left[1-\dfrac{x}{n}\right]^{-n/x}\right\}^{-x+x^2/n-x/2n}} = \frac{1}{2}\,. \tag{14}$$

From (3) we see that x is probably small compared to n. Replacing the brace in (14) by e, we have

$$\frac{1}{e^{x^2/n-x/2n}} = \frac{1}{2}\,. \tag{15}$$

For a first approximation it is certainly legitimate to disregard the term $-x/2n$, and we thus obtain the "formula"

$$x = \sqrt{n\log 2}\,. \tag{16}$$

The average undergraduate would certainly consider (16) above suspicion. His instructor, unless of a very cautious type, would describe this piece of Eulerian

reasoning as sloppy, but probably one which yields substantially the correct result. But numerical comparison of (3) and (16) is disconcerting.

(17)

n	B	(16) "formula"
10	4	2.63
100	12	8.33
1000	37	26.33
10000	118	83.26

This shows conclusively that the reasoning leading to (16) involves a serious error. One student pointed out that the results from (16) needed to be multiplied by about 1.4 to give the true values; and he suggested that a square root of 2 had been lost in the course of the work. Although a recheck of the technique in equations (11) to (16) failed to reveal the missing root 2, this remark turns out to be unexpectedly clairvoyant.

It is probably too much to expect an undergraduate to ferret out the error. The fact is that in equation (14) we make a certain small error in replacing the brace by e. Now the principal part of the exponent of the brace is $-x$, which is numerically large. This large exponent, taken in connection with the error in replacing the brace by e, is responsible for the error in (16).

A correct derivation may be obtained from (13) by taking logarithms, obtaining

$$-(n-x+\tfrac{1}{2})\log\left(1-\frac{x}{n}\right)-x=\log\tfrac{1}{2}. \tag{18}$$

In (18), expand the logarithm in a series. We thus obtain easily

$$x=\sqrt{2n\log 2} \tag{19}$$

as a first approximation, and

$$x=\sqrt{2n\log 2}+\frac{1}{2}-\frac{\log 2}{3} \tag{20}$$

as a strong approximation. It may be shown that for all n, B is either the largest integer in x as given by (20), or, in a few exceptional cases, it is the largest integer increased by unity. For n not greater than 1000, the only exceptions are for $n=2$, 10, 43, 83, 136, 253, 310, and 870.

Actual experiments to determine A and B are time-consuming. But for $n=10$, experimental tests may be carried on very rapidly as follows. In Andoyer's *Nouvelles tables trigonométriques*, vol. III, the values of the cosecant are given for every ten seconds of arc to 15 decimal places. Thus we find

$$\operatorname{cosec} 0^\circ\, 0'\, 10'' = 20626.48063\ \underline{2789863356}. \tag{21}$$

It is not unreasonable to assume that the last ten digits are "random"; the number (k) of digits in this last block before repetition is easily noted; in (21), $k=4$.

The first 3,000 entries from 0°0′0″ to 8°20′00″ were thus tabulated with the following result:

	k	Actual incidence	Most probable incidence
	1	302	300
	2	535	540
	3	647	648
	4	610	605
(22)	5	479	454
	6	254	272
	7	127	127
	8	38	43+
	9	8	10
	10	0	1

The actual average A is 3.646 as compared to the theoretical average 3.660 and the "formula" average 3.630. The sum of the actual incidences from $k=4$ up to $k=10$ is 1516, which is greater than 50%, and hence compatible with the value $B=4$. While a case of $k=10$ should occur on the average once in 2755 times, no case occurs in the first 3,000 entries. This is of course not surprising. The first instance of $k=10$ occurs in the 4981st entry,

$$\text{cosec } 13^\circ\ 50'\ 10'' = 4.18155\ \underline{5179203846}.$$

If in formula (5) we substitute $n=10$, and our experimental average $A=3.646$, we obtain the experimental approximation $\pi=3.166$.

A CONTINUED FRACTION RELATED TO SOME PARTITION FORMULAS OF EULER*

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The object of this paper is to show how a number of identities which are employed in the theory of partitions may be derived by means of a single continued fraction. We shall begin by discussing some general theory which will be needed.

1. An important class of continued fractions. If $t_1, t_2, t_3, \cdots$ are any numbers, then the infinite series

$$1 + t_1 + t_1t_2 + t_1t_2t_3 + \cdots, \tag{1.1}$$

and the continued fraction

$$\frac{1}{1} - \frac{t_1}{1+t_1} - \frac{t_2}{1+t_2} - \cdots \tag{1.2}$$

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ALEXANDRIA—SHRINE OF MATHEMATICS*

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The world, from the aspects of material life, is a place of change. All is dynamic, all is in flux. Fad and fancy are enthroned as the lodestars of our habits and tastes, and the guides of our customs are will-o'-the-wisps. Fashion beckons to us unceasingly to chase in its merry whirl. We find diversion and delight in the very substance of shadows, and all about us the newness of today is tomorrow's decay. Political and social readjustments crowd heavily and irresistibly upon each other, and we slough off overnight our prejudices of yesterday, to replace them eagerly by those of today.

Yet life's aspects are not all transitory, and especially those in the realms of thought often have much permanence in them. The ripples on the surface of the stream may dance with scattering abandon, and hasten distractingly hither and yon under any momentary gust. But the current beneath heeds them little. Externally we differ much from the sandalled citizens of classical Athens, and even from the buskined burghers of Shakespeare's time, but in the domain of the intellect the intervening ages dwindle, and we are much at one with them. Be our interests what they may, we can find in history that they reigned before. At the hour of prayer the devout Mussulman turns his face toward Mecca. The Christian and Jew think alike in hallowed terms of the golden city of Jerusalem. Babylon and Rome bask in the reverence of the antiquary, and Athens is a shrine of art. The mathematical scientist needs yield to none in the richness of his endowment from the past. He may find wellsprings of inspiration in the chronicles of many shrines, and of these the ancient stand of Euclid is not the least.

To all purposes, the ancient city of Alexandria was built to measure. The more it must be marvelled that it should ever have attained to, and even surpassed, an extremely high predestination. Yet it did so, and filled for many centuries a rôle pre-eminently important to the whole of human cultural history. On the southern shore of the eastern Mediterranean Sea, not far from the westernmost mouth of the great river Nile, a narrow ridge of limestone separates the sea from a lake named Mareotis. Not far offshore and lying like a breakwater parallel with it, is a long narrow island called Pharos. In the year 331 B.C. Alexander the Great saw in this spot unusual potentialities for the site of a city. He conceived upon it a great city which should serve at once two missions which he had set himself, namely, the spread of Hellenic influence over the world, and the return of the ancient land of Egypt to a former greatness and glory.

The time for the founding of a maritime city in this part of the world was extremely propitious. To destroy the sea power of Persia, the Macedonians had but just subjugated port after port in Asia Minor, and to eliminate Phoenicia had razed the old pre-eminent commercial city of Tyre. The new site, distant

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about four miles from the mouth of the Nile, was just removed on land from the miasmal swamps of the delta, and by sea from its many and shifting shallows and shoals. The Lake Mareotis afforded fresh water, and at the same time easy transit to the Nile by means of canals. The rich and populous interior of Egypt and of the African expanse beyond lay, therefore, directly accessible. With these advantages and superlative harbor facilities, the city became forthwith the prime center of the trade of the world, the commercial junction point of Asia, Africa, and Europe.

Upon the death of Alexander, Egypt fell under the governorship of his general Ptolemy. There was much mutual good fortune in this turn of events. For while Egypt, a fabulously fertile country, meant power and almost limitless wealth for Ptolemy, he in turn proved himself worthy of it. The distinctive culture of the country, with traditions extending into the interminable past, had always fascinated the Hellenic mind, and Ptolemy was not unreceptive to this charm. His rule, first as governor and later as king, was guided by intelligence and statesmanship, by a fine natural artistic taste, and by an appreciation of the dignity and substantial worth of intellectual attainment. His dynasty was to rule the land, now for better, then for worse, over a period of two hundred and fifty years.

From early times in the history of the Greeks there had existed so-called philosophical schools, which were in reality communities of scholars. Thales had founded one in Asia Minor, and Pythagoras had done so in southern Italy. These schools were frequently organized as brotherhoods, dedicated, as in the cases of the Pythagoreans, to the cult of the Muses. Their housings, therefore, came quite generally to be known as *museums*. Ptolemy conceived the ambition to establish such a museum at Alexandria. He envisaged the city as a center of Greek culture, not merely as a trading post, and with large revenues at hand he thought to make association with his school attractive by the then novel means of granting salary stipends, as well as board and residence, to prominent scholars of his choice. Not unnaturally this plan was an immediate success, and at about the year 300 B.C. the Alexandrian Museum had become an actuality. It included in its membership intellectuals of all sorts—poets, philosophers, grammarians, mathematicians, astronomers, geographers, physicians, historians, artists, and many others. The mathematician in this initial galaxy was the immortal Euclid.

The Greek study of mathematics had begun three centuries before this time. Initially it had been drawn largely from Egypt, in a state which was rich only in potentialities. It had been transformed in the interim, however, into a vast body of doctrine. Its abstractions had appealed to the Greek genius, and under the impingement of many able minds it had developed until the geometry of plane figures, and of the simpler and regular solids, was well advanced. Eudoxus had given a theory of proportion satisfactory even by our present standards. Algebraic equations through the quadratics had been solved, and the irrationalities arising in their solutions had been classified. Finally, the concepts of rigorous proof, accurate enunciation of assumptions, and the definition of terms had been fully evolved.

The transcendently famous accomplishment of Euclid was the composition of his *Elements of Geometry*, written in thirteen books, of which all have come down to us. In this work Euclid undertook to sift from the mass of material which then represented the mathematical knowledge of the day, those facts which appeared to be most fundamental. And these he built, as it were, brick-by-brick into a self-contained and unified scientific structure. How he succeeded is familiarly traditional. His accomplishment has ever since been a factor of importance. It has stood for ages as a work of art, stripped of irrelevancies, and carried through with an utmost economy of means. The conception was original and grand; and the execution was achieved with insight, discernment, consummate power and skill. In remaining alive to the present time, the work has survived the vicissitudes of passage through the eras of vigor and decadence of classical learning, through a thousand years of medievalism, and through some keenly critical centuries of modern times. Euclid wrote also on other subjects; optics, astronomy, and music for instance, but his *Elements* are supreme. They have been the subject of almost endless study and commentary, and all civilized languages have had their translations of them. They made of Euclid no less than a symbol of the attainable zenith in the realm of scientific exposition.

Ptolemy's constructiveness, which was manifested in his incorporation of the Alexandrian Museum, was matched again both in originality and grandeur by his founding of a public library. There had, of course, been libraries before that. The Phoenicians and Chaldeans and others had had them, as well as the Egyptians and the Greeks. But those had almost invariably been archives of records or repositories for valued writings rather than public libraries. The new Alexandrian Library by contrast was conceived in the modern sense, and was from the beginning intended to share the function of preservation of writings with that of public service. It was intended to make the world's literature, and the whole gamut of its recorded thought accessible to everyone. The Library was established almost simultaneously with the Museum and adjacent to it, and in time was to become monumentally important.

At the Museum the great tradition of Euclid passed on to his successors and through them to one even greater than he. For among the students of that time there appeared at Alexandria as a youth the all-overshadowing genius of Archimedes. Archimedes studied at the Museum for several years, and although the locale of his later great work was not Alexandria but Syracuse in Sicily, he carried through life the stamp of his early Alexandrian training. His thought trend has been recognized as typical of the Alexandrian school, and throughout his life he identified himself by continuous and intimate contact with the scholars of the Museum.

Meanwhile a fascinating personality had joined the Museum's staff. This was Eratosthenes, a mathematician of note, but also a universal genius, an original thinker as a poet, philosopher, historian, chronologist, and geographer. He was called to Alexandria in the first instance to direct the great Library, a post in virtue of which he was ex-officio a member of the Museum. Eratosthenes had

been born in 276 B.C., and had travelled much over the then known world. His contemporaries regarded him as a second Plato for learning. His mathematical works, unfortunately, have not survived to us, and the unimpressive items with which his name is still associated are wholly inadequate to give just evidence of his importance. That his contemporary reputation was not exaggerated, however, is certain, for many of his investigations were no less than pioneering and fundamental. In geography, for instance, his whole work reveals a magnificent and richly imaginative scientific approach. It seems that his predecessors in this field had concentrated their attentions largely upon separate lands and regions. Eratosthenes, on the other hand, sought at once to integrate the geographical knowledge of the entire world. Accordingly he collated to this end all available historical references, collected reports and descriptions by travellers, catalogued data on strange and remote places, observed the lengths of days, and studied the relevancy of physical observations and mathematical theory. Hardly less path-finding was his work in chronology. Prior to him the dating of historical events was in pretty thorough confusion. Where the Greeks reckoned times from the fall of Troy by Olympiads, others reckoned them by generations, by reigns, by dynasties, or what not. Eratosthenes sought order in this chaos, and his initial determination of reference epochs and correlations pointed the way in which his followers advanced.

The achievement for which Eratosthenes won most spectacular acclaim, however, was his measurement of the size of the Earth. Having heard that at the city of Syene near the first cataract of the Nile in upper Egypt, the sun stood at the zenith and illuminated the bottoms of deep wells at the time of the summer solstice, he compared this with observations at Alexandria. From the distance between the cities he calculated then in a manner which now seems familiar and elementary, the dimensions of the entire globe. We are told that Eratosthenes worked at Alexandria until in his old age he found himself engulfed in blindness, and until, because of this affliction, he ended his own life by starvation.

A century had now elapsed since the founding of Alexandria. The original Ptolemy had been succeeded by his son, the second Ptolemy, and he and his heir had in turn passed by in the succession. These had all been competent statesmen and liberal rulers. They had been sane in the conduct of their private lives, and publicly they had been generous patrons of the arts and sciences. The country had flourished under them, and Alexandria had grown into a metropolis populated now by almost a million people. The members of the Museum, while they were housed in sanctuary surroundings, could, therefore, none the less, issue forthwith into the by-ways of a great city, a maelstrom of human life.

Alexandria is situated in the latitude which with us lies in northern Florida and southern California. In summer there is a prevalence of northerly winds from off the Mediterranean. The climate is, therefore, close to a natural ideal, and partakes neither of the dry and shrivelling heat of the desert, nor of the humidity of the Nile delta. Its skies are prevalently bright and cloudless, its days breezy,

its nights brilliant. And its gardens were in olden times perpetually filled with flowers, while its orchards yielded richly, figs, lemons, almonds, apricots, pomegranates, dates, and spices. Men met this challenge of nature in the construction of the city. The streets of Alexandria were straight and wide. Midway of the strip of land between the sea and the lake, on which the city lay, and extending from east to west was a centrally important avenue. This was crossed at right angles by another such, which led from a Mediterranean harbor on the north to the lake harbor on the south. Throughout their lengths these streets were lined with colonnades of white stone, which gave needed shade and at the same time set off the many splendid buildings. Greek statues and Egyptian obelisks were all about the city, as were also temples and synagogues large and small. There were bazaars, baths, parks, market-places, a forum, and an immense gymnasium. There were also theatres and a hippodrome, a racecourse and many private mansions, and at the street crossings there were arcaded cisterns through which the city was supplied with fresh water. At one place a rocky hill of some height bore a building which was reputed to have been the finest in all Egypt, if not in the entire world. It was the Temple of Serapis, the chief deity of the city, a fused embodiment of an ancient god of the Nile with one of Greece. This temple played a notable part in the later unfolding of historical events.

Alexandria had three harbors. Upon the south, on Lake Mareotis, was the inland harbor. Through this and the lake and by canals to the Nile, the city fulfilled its assignments as the gateway to Egypt. Since the Nile was in turn connected by a canal across the desert with the Red Sea, the Alexandrian ships which departed from this port found their ways to Arabia and the east coast of Africa, to India, and even to China.

On the Mediterranean shore the topography had been decisive in Alexander's choice of the city's site. There, three-fourths of a mile offshore, lay the long island Pharos, and toward the ends of this, rocky promontories jutted out from the mainland. The lagoon between was thus excellently sheltered from the open sea. Even under the first Ptolemy this lagoon had been divided by a great causeway built to join the island with the shore, and thus two harbors had been created with entrances at opposite ends of the island. Of these harbors the western one was given over entirely to commerce. Its banks were lined with piers and warehouses and the general establishments of maritime merchants, and it was connected by canal with Lake Mareotis. The eastern harbor, or so-called Great Harbor, was similarly given over to trade along its inner reaches. Its outer shores, however, presented a very different aspect, in fact a superlative panorama. On the right as one entered this harbor, and on the very tip of the island, stood the world's original lighthouse. The African coast along there is poorly visible from the sea, and is infested with many dangerous shoals. Under the second Ptolemy, therefore, this beacon was constructed, and, moreover, upon a scale which was intended to outrival even the grandiose monuments of Egypt's remoter past. The lighthouse, which like the island was named Pharos, rose, therefore, from its base to a height exceeding that of the Great Pyramid.

It was constructed of alabaster and white marble, and so appeared as a conspicuous foil to the blue Mediterranean water. It has been ranked by common consent as one of the seven wonders of the antique world. Its light, which was fuelled with wood, had its rays concentrated by concave mirrors, until it could be discerned from many miles at sea.

On the promontory opposite this light stood the palace of the first Ptolemy. It overlooked a harbor inlet reserved for pleasure craft, and adjacent to it by land was the so-called royal section of the city. This was the conspicuously elegant part of Alexandria. Here each Ptolemy had in his turn added his own palace, and these structures stood as an array of models of the finest in Greek architecture. Splendid buildings were set off everywhere by groves and gardens. There was the mausoleum of Alexander the Great, the tombs of the kings, a national archive, a theatre and an amphitheatre, many exquisite small temples and a large temple to the sea god Poseidon. And here, close to the harbor, and connected by colonnades with the palaces stood the fine white buildings of the Great Library and of the Museum. Such were the environs in which the scholars of Alexandria lived. In a great basilica they ate their meals together, and had their dormitories and lecture halls. In groves of palms and under arcades adorned with classical sculptures they carried on their discussions with each other or with their students and disciples. Contemporary Alexandrian satire liked to depict them as costly birds fed and treasured in a golden cage.

The mathematical tradition established by Euclid and furthered by Eratosthenes and the young Archimedes passed onward to Apollonius. He had come to Alexandria in his youth, and was approaching manhood when Eratosthenes, a score of years older, came to the Museum. Like Archimedes before him, therefore, he had learned his mathematics from Euclid's successors, but unlike Archimedes he remained at the Museum to become himself a member of it. The leading work of Apollonius was in the field of the conic sections. These curves had been studied considerably before that time, and Euclid among others had written upon them. Apollonius, however, stripped their theory of all persisting irrelevancies, and fashioned it into essential generality and definitive completeness. His power in mathematics was extraordinary, to the extent that he became known in his time and thenceforth simply as *The Great Geometer*. His work on the conics, which was written in eight books, is still regarded as constituting a veritable culmination of the whole of classical pure geometry. It is ingenious, general, and extremely adroit; excellent in organization and exhaustive in scope. As an achievement it is so monumental, in fact, that it practically closed the subject to later thinkers. Mathematical genius found itself thereafter effectively unable to go further in this direction, and had need to seek for itself other outlets. Apollonius wrote also on other subjects. He was highly reputed as an astronomer, and contributed important theory on the eccentric and epicycular heavenly motions. Unfortunately his works have survived to us only in part. That on the conics is the best preserved. Of its eight books we have four in the original Greek and three others in Arabian translations, while one, the

last, is lost. Apollonius died in about 190 B.C.

The mathematical activities at Alexandria were exemplary of the Museum's general vigor, and especially of its scientific enterprise. In historical research the past of Egypt was ferreted out and recorded. Great advances were made in medical knowledge and technique, and also in biology and geography. The world was studied to its remotest-known corners with no stint of labor or expense. Expeditions were sent to India and China among other places to draw improved charts, correct misconceptions, and report upon customs and people. In an immense and magnificent zoölogical garden animals of all kinds were collected, and subjected to observation and intimate study.

Meanwhile the Library had been steadily active and had grown to vast proportions. Even under the first Ptolemy some fifty thousand rolls of manuscripts had been collected, and the succeeding kings had been no less enthusiastic. Manuscripts were officially sought throughout the world, and their acquisition was vigorously pressed. Mariners were generally commissioned to buy old writings upon any occasion, and by law all books brought to Alexandria had to be submitted to the Library for comparison with those already there and eventual copying. Especially trained staffs of scribes and artists were perpetually engaged in cataloguing and copying, or in ornamenting the rods upon which the manuscripts were rolled and the sheaths in which they were kept. Even in Apollonius' time the main collection, established near the Museum, had filled its housing there with a capacity of five hundred thousand rolls. The surplus had been made into a second library which was increasing rapidly, and which was placed in the Temple of Serapis at some distance in the city. These collections were both publicly accessible. They had spread the fame of Alexandria, drawn students in numbers to the city, and had inspired emulation to the extent that the world now possessed another similar, if smaller, library at the city of Pergamum in Asia Minor.

The era of good government, with its attendant intellectual golden age, which had extended from Alexandria's founding into the lifetime of Apollonius, was unfortunately not destined to be maintained. Indeed, the causes of decline were already rampantly active. The first three Ptolemys had been men of character, even in the face of their possession of utterly unrestrained personal authority and limitless wealth. Their successors were without that moral fibre, and in the royal palace, therefore, ideals and culture were fading. The fourth Ptolemy, who reigned during the latter part of Apollonius' time, was effeminate and a seeker after flattery. He was contemptuous of Egypt to the extent of disdaining to discharge the necessary affairs of state. Behaving often like a buffoon, he was in reality a monster, for he murdered his mother, and later his wife, his brother, and his friends. The decline of the dynasty was at hand, and, as might have been expected, the social standards of Alexandria's populace fell as those of royalty showed the way.

The people of Alexandria were a motley collection. The city had been founded as Macedonian, and had retained a Hellenic character in language

and customs. In the main, the nobles and the intellectuals were Greek. The populace, however, was a conglomerate. Native Egyptians outnumbered the Greeks, and were matched in numbers by the Jews. In smaller proportions people of all stamps and races had been drawn thither by the opulence and enterprise of commerce. Upon the streets aristocrats and citizens jostled freemen and slaves; and Persians and Ethiopians crossed paths with Syrians, Romans, and Arabs. And among all these the true Alexandrian was an amalgam, a quite characteristic person, we are told. Though he was hardly ever lazy, he rarely combined dependability with his industry. He was generally of an excitable and disorderly nature, ostentatious and noisy, and above all so sharp and shifty that the term *Alexandrian* came in other parts of the world to connote nothing less than scalawag. As a purpose in life he preferred revelry and merrymaking, matters into which he entered with such vigor and gusto, that he lost all sense of propriety and decorum whenever he entered a theatre or a racecourse. In such as these the seeds of decadence found fertile ground.

At the Museum the decline was only gradual. While proximity with the king's household, and the custom whereby the scholars were regarded as belonging to the royal retinue, were unwholesome, they were considerably offset by some other influences. The Museum continued free of essential restraint. As a rule the kings, even those who were weaklings or tyrants, affected to regard both the Museum and the Library with pride and ambition. With few exceptions they found themselves pleasing in their own eyes as patrons of the arts and sciences.

Into this epoch fell the life of the great scientific thinker Hipparchus. He was born perhaps in the very year in which Apollonius died. His coming to Alexandria, however, was much later, for he was on the staff of the Museum only in his maturer years. Hipparchus is best known as the father of scientific astronomy. He was, however, an important mathematician, and in fact exemplified in his type of thought the modified trend of the Greek mathematical genius. That genius now sought its way in the exploitation of geometry rather than in its further extension. Hipparchus is important to the astronomer for his improvements of instruments and refinements of observational technique. He is credited with the discovery of the precession of the equinox, with a determination of the inclination of the ecliptic, and with measurements of the parallax and some irregularities of motion of the moon. He improved the knowledge of the length of the year, and catalogued some thousand of the larger fixed stars. To the mathematician he is important above all as the founder of trigonometry. In this connection he calculated a table of chords in the circle, and so anticipated the table of sines. Generally he forged the then disjointly existing bits of this subject into an effective and powerful mathematical tool, and by his own skillful use of it attested to its potentialities. He is credited also with the invention of the latitude and longitude system of reference, with the conception of the paths of the sun and moon as eccentric orbits, and with important contributions to the theory of eclipses. He combined with great talents for observation the patience and persistence needed for a collation of his own results with

those of ancient record. His scientific imagination was active, and was such as to lead him almost always to effective theoretical formulations of his facts. As a creative scientist he was great enough to remain peerless in his field for seventeen centuries, to find an equal only in the European Kepler.

The time of Hipparchus' life at Alexandria was one of agitation and turbulence, which was often intense with anxiety, and sometimes even with fear. The Ptolemaic dynasty was no longer in mere decline, but had arrived at utter degradation. The then reigning king had gained his throne through the murder of his nephew, and although he was ostensibly interested in learning, murder had become such a commonplace with him as to assume often the proportions of massacre. Capricious and ungovernable in temper, he had become a general object of terror. The Museum suffered from the flight of many of its scholars, so many in fact that learning is said to have been noticeably stimulated in other parts of the world. This debased pattern of royalty was thereafter to become the usual one. Naturally social repercussions accompanied it. Class distinctions became accentuated. The rich surrounded themselves with oriental luxury and made increasingly ostentatious displays of their extravagances in clothes, in liveries and jewels, and in a use of cosmetics which grew into the fantastic. Their villas with elaborate gardens and vineyards dotted the beautiful shores and islands of Lake Mareotis, and rivalled each other within as seats of gluttony, vice, and hedonism.

The poor meanwhile became more and more degraded, ragged, and dirty. Many were unable to feed their children and so had perforce to abandon them outside the city walls where the waifs found adoption only as slaves. By consequence the underprivileged class became ever more bitter and restless. They collected easily into street crowds, and these all too often sought vent in brawls and bloodshed. Moral standards waned, and with them Egypt waned as a power among nations. She was to fall an easy prey to the rising might of Rome.

The worn-out dynasty of the Ptolemys was saved from a vicious and ignominious end by its last representative, Cleopatra. The crown descended upon her and her younger brother jointly in the year 51 B.C. She was seventeen, he was ten. The dissension foreseeable in a joint regency broke out, and Cleopatra was forced to flee Egypt. With the aid of an army gathered in Syria, she was engaging in the attempt to recover her share of the throne, when Caesar as newly-made master of the Roman world arrived at Alexandria. Caesar at once appointed himself the arbiter of the differences between Cleopatra and her king brother. It is familiar that Cleopatra knew in these circumstances how to win a decision in her favor. To the populace of Alexandria, however, Caesar was much less than welcome. They sensed in his presence the menace of Roman domination. There were threats and riots, and Caesar, whose Roman guard was but small, found it wise to garrison himself in the royal quarter of the city. The Egyptian fleet meanwhile rode the waters of the Great Harbor. Caesar saw in it potential jeopardy, should escape become necessary for him, and undertook, therefore, to see it destroyed. An incendiary crew was dispatched by night to set it afire.

Insofar as this mission was concerned, the expedition was a complete success. The fire, however, spread to the warehouses ashore, and breaking its bounds beyond that, ran its course through one of the great cataclysms of history. The flames found the Library adjoining the Museum, and when they had done, the contents of a half-million manuscript rolls were in ashes. What an ineffable tragedy! That great stock of writings, flower of a superb human culture, recorded thought of many of antiquity's greatest thinkers, was gone. It had taken effort and means over two and a half centuries to collect it—a half night sufficed for its obliteration. Euclid, Eratosthenes, Archimedes, Apollonius, Hipparchus and many others, wrote works that have not survived to us. Might not many of them now be ours if the events of that night had never occurred!

Soon after this, Cleopatra became the sole ruler of Egypt by virtue of her brother's death. Unlike her more immediate ancestors, she proved herself a capable ruler. She was shrewd, resolute, and fearless. To be sure she did not always in life avoid episodes savoring of the spectacular and romantic, and because of this her memory has often been made symbolical of all the glamour and seductive power which is the grace of her sex. All too easily her name conjures up the picture of the Egyptian queen, fabulously beautiful, mysterious as the orient itself, and redolent of all that is alluring and subtle. The picture, however, is an abstraction. Cleopatra was Greek rather than Egyptian. She was ambitious and intelligent, excellently educated, and generally capable.

The Museum in this time had fallen into a bad state. Life at Alexandria, with its vicissitudes in proximity with the decadent and tyrannical Ptolemys, was often precarious and hardly ever inviting. Great scholars elsewhere preferred to stay away. Though the surviving library in the Temple of Serapis was now the world's largest, and still assured to Alexandria primacy as a place for research, the loss of the Great Library had nevertheless been a climactic disaster. Cleopatra undertook the Museum's rehabilitation. She increased the royal patronage of the scholars, and sought to restore the dignity of their positions by personal attention, and by attendance at many of the Museum's functions. In the course of her travels, which were extensive, she sought contacts with men of learning, and in time succeeded in recruiting many of them to places at Alexandria.

Her greatest achievement in this connection, however, was a master-stroke overshadowing all others. Sometime before this date, the erstwhile library of Pergamum, which counted to some two hundred thousand rolls or more, had fallen into the hands of the Romans. In her ascendancy over Mark Antony, Cleopatra now persuaded him to present this library to her. The manuscripts were brought to the Museum. The Alexandrian library thus absorbed its former rival, and in doing so recouped its own great loss as it could not have done in any other way. Upon the self-inflicted death of Cleopatra in the year 31 B.C., her dynasty came to its end, and Egypt became a province of the Empire of Rome.

As might have been expected, its subjection to Rome placed Alexandria under some political strictures. On the other hand, it restored peace and order in

the city, with the upshot that much material prosperity which had been lost was recovered. The change, on the whole, was for the better. As for the Museum, that found itself emerging from its adversity on two accounts, its revenues were substantially increased, and its outworn quarters were renewed. In honor of the Emperor Augustus a new and magnificent temple was erected upon the site of the ill-fated Library. It was an impressive structure, richly ornamented and filled with works of art as the pomp of its occasion required, and so large that it was visible from the harbor and over much of the city. In this temple the Museum was given new halls and lecture rooms, and a stand was supplied for Cleopatra's library.

During the interim of something over a century, the character of the Museum had meanwhile been progressively, albeit slowly, changing. As initially founded, the institution was a community of scholars solely dedicated to the prosecution of original intellectual work. To be sure, the membership had from the very beginning been regularly commissioned with the tutoring in the royal household, and especially of the presumptive rulers. Aside from that, however, teaching had been a subordinate and quite informal activity. The relations of members with non-members were regarded as personal, and the rôles of master and pupil were assumed by private arrangement. This had gradually changed. With time, students had come to the Museum in increasing numbers. Presumably they had also come in earlier stages of their development. It became common for public lectures to be delivered by members, and out of these evolved series of lectures at which regularity of attendance was almost necessary. In the upshot, intellectual contacts became broader and easier. As appreciation of this spread, sojourns to Alexandria for study became general and popular over the world, and to have studied there carried some measure of prestige.

Thus the development of the Museum by slow degrees had proceeded, until at the epoch in point, namely the incipient Christian era, it resembled in almost all of its aspects the modern university. There were regularly scheduled courses of graded lectures, through which systematic higher instruction was imparted in a full gamut of academic subjects. There was a large body of students, more or less regularly enrolled. Meanwhile, the libraries also had been evolving and enlarging their range of functions. Whereas at first they had been devoted almost exclusively to the collection, preservation, cataloguing, and loaning of the manuscripts, they had by this time added to their activities some which were almost reminiscent of a publishing house. The art of copying manuscripts had been developed and was being prosecuted more and more. Caligraphers were now especially trained for this purpose, and a considerable staff of them was permanently engaged at the libraries in producing copies of the books to be found there to fill the demands for them from other places. The repute of Alexandria as the world's intellectual focal point was enhanced not a little by this activity. Teachers had need go there for training, and from there also came the books.

The first mathematician of more than transient importance in the Alexandria of Roman times was Menelaus. He was both a product and a member of the

Museum, and taught there until somewhat late in life when he was called as an astronomer to Rome. That was near the end of the first century. The significance of Menelaus lies mainly in his contributions to the development of trigonometry. He was apparently the first to divorce spherical trigonometry from subservience to its astronomical applications, and to enunciate its theorems with some completeness. Although the originals of his writings have not come down to us, his most important works have survived in Hebrew and Arabian translations.

The development of Greek trigonometry reached its culmination, however, in the hands of Claudius Ptolemy, who was a member of the Museum about a third of a century after Menelaus, namely in the period from about 125 A.D. to 160 A.D. This Ptolemy is not to be confused with the erstwhile kings of the same name. Little of a personal nature is known of him. Like Euclid, he lives in history as a mind and hardly at all as a man. Indeed, the parallelism bears carrying further. Euclid excelled as a selector and organizer and expositor of knowledge, and precisely so did Ptolemy. Each combined with admirable creative faculties of his own, a talent for the appreciation of the works of others, and a mind for discriminating essentials and discovering order in seeming chaos. Hence each became in his chosen subject the pre-eminent authority and teacher over a fabulous epoch of time.

Ptolemy's writings have essentially all survived. The most important of them is a work in thirteen books of mathematical and astronomical content. Ptolemy called it a mathematical syntax. Since Arabian times, however, it has been generally known as *The Almagest*. In its mathematical content *The Almagest* presents Greek trigonometry in the definitive form it was to retain, and over which no effective improvement was to be achieved for fully a thousand years. On the side of astronomy, it contains the exposition of a great theory of epicycular geocentric motion of the heavenly bodies which came to be universally known as the Ptolemaic system. To build this great structure Ptolemy harked back to the fundamental work of Hipparchus. With this he collated the works of many other investigators, added a systemization of the then existing mass of observational data, and brought to bear upon the whole the essential geometrical and trigonometrical facts. The work is a great masterpiece. No product of the genius of the entire classical era ever rivalled it for its profound influence upon human conceptions of the universe, and none achieved such unquestioned authority over so many centuries of time. Ptolemy's theory remained unchallenged until the advent of Copernicus. Even then its hold was so tenacious, that it was taught upon a par with its newer rival even in the universities of this country.

Aside from *The Almagest*, Ptolemy wrote important works also on a number of other mathematical subjects. In a work on optics he gave an incipient theory of the refraction of light. He wrote on sound, and in chronology carried forward from Eratosthenes. Finally, a work of his in eight books on mathematical geography is regarded as one of the greatest literary accomplishments of antiquity. In it he again based his deductions upon those of Hipparchus, and set forth a description of the then known world by the complete use of the latitude, longi-

tude system. He gave a discussion of the theory of map-making with different modes of projection, which was so exhaustive that maps were made by his prescriptions for many hundreds of years thereafter.

Thus far the Roman rule had been peaceful and unoppressive, at least insofar as Alexandria was concerned. The Museum had gone its way with little untoward interference from ruling officialdom. In the main the emperors had been friendly or indifferent, and even the misrule of a Caligula and a Nero had been tempered by the fact that among other of their aberrations they had included that of seeing in themselves artists and patrons of culture. Not until the year 215 was this status to suffer a change, and then, fortunately, the change was to be a transitory one. The emperor Caracalla, one of the notorious succession of short-lived army rulers, finding himself upon occasion in Alexandria, discovered that his person had been ridiculed and satirized by someone of its populace. His retaliation was immediate and ruthless. A goodly share of the city's young men were massacred, and the Museum was abrogated. The scholars were dispersed and fled, and some of the buildings were destroyed. An amende, however, was not long delayed. The tyrant came to an early death, and in revulsion to his many excesses the Roman Senate rescinded all his decrees.

For the Romans the era following upon the time of Ptolemy was one of intellectual stagnation which presaged the decline of the Empire. Greek learning, though it was still outwardly accorded some deference, was, in fact, sinking into unmistakable eclipse. Popular esteem had quite generally been transferred from the scholar to the actor and the dancer, and in the households of the rich the erstwhile places of the tutor and the philosopher were lapsing generally to favored retinues of entertainers. At Alexandria this trend was much delayed—was, in fact, still quite abeyant. In mathematics especially, fine genius was still to run a course, and this, moreover, through such dissimilar personalities as Heron, Diophantus, and Pappus.

In his salient characteristics as a thinker, Heron sought his master and guide in Archimedes. His genius was one in which an appreciation of theoretical abstractions shared place with a highly developed sense for mechanics, and even with a practical talent which sought its outlet in popular inventiveness. Heron's claim to greatness is not embodied in his inventions, even though these included the first steam engine, force pumps, air-driven heavy artillery and such, down the line to water clocks, self-regulating lamps, distorting mirrors, puppet theatres, and even robots in human form for pouring wine. These matters do, however, give to Heron's writings a flavor which is peculiarly their own, and which has been universally appreciated through the ages. Heron was a facile and clever technician. He was, however, also a wholly profound mathematical scientist. His works upon the subjects of mechanics, hydraulics, and pneumatics show that his many inventions were based upon a firm understanding of the facts of geometry, the relations governing the elastic properties of matter, and the mechanics of gases. He knew the principles of the simple machines and had much information on the centers of gravity of figures. In pure mathematics Heron's

work was concentrated upon the theory of mensuration. Lengths, areas, and volumes in great array, for plane and solid configurations, were ably and exhaustively discussed by him. Illustrations of their interrelations were given, moreover, by such applications as the determination of the distance from Alexandria to Rome by observations upon lunar eclipses, by directions for tunnelling through mountains so that excavators from opposite sides would properly meet, *etc.* It is a far cry from the initial austerity of Greek geometry to this genial and versatile work of Heron's. Yet Heron, like Euclid, dominated human thought in his field for much over a thousand years.

In Diophantus the Greek mathematical genius gouged for itself headlong a new and distinct channel. This it did, moreover, through ground which it had theretofore found peculiarly resistant. The channel is that of the arithmetization of mathematics. From the time of Pythagoras the mathematics of the Greeks had veered away from the purely arithmetical. Inherent difficulties in the Greek number concept had defied solution, and a fine sense of rigor had forbidden their being ignored. Moreover, the Greek scheme of numerical notation was an unfortunate one, and, in the upshot, algebraic problems, even to the solution of equations, had been cast into a clumsy and inflexible geometrical mold. With time the tradition of this had become rigid, until only the greatest thinkers were even aware of its restraints. With Diophantus came emancipation, as complete as it was abrupt. Where his predecessors had used arithmetical methods at best hesitatingly and sporadically, he used them exclusively and with confidence, power, and skill. The most important work of Diophantus is his *Arithmetica.* We have but six of its original thirteen books. It is a work of astonishing originality and ingenuity, its most interesting subject being that of indeterminate equations. These are still designated by us as Diophantine. Diophantus drew but sparingly from his predecessors, but he left a large endowment to his posterity.

In producing the geometer Pappus late in the third century, almost half a millenium after Apollonius, the Greek genius gathered itself once more to play in a favorite rôle. The especial field of Pappus was that of higher geometry, and to the theory of curves such as the quadratrix, the spirals, and the conchoid, he contributed many brilliant theorems. The writing for which Pappus is best known, however, is his so-called *Mathematical Collection*, composed of eight books. This was designed to be a synopsis of the best geometrical knowledge of the time, and its purpose was brilliantly accomplished. From the literary standpoint it is extraordinarily skillful, and scientifically it reveals a rare power of discriminating the finest works of the classical masters. The parts devoted to higher geometry contain many of Pappus' own discoveries. In the *Collection* we have essentially a compendium of mathematical gems which would otherwise not be known to us, since they were largely in works that are lost. We owe to it much of our understanding of ancient methods and of the history of their development. On the whole, fate was unkind to Pappus, for it denied him a life epoch worthy of his talents. In brilliance he would have been adequate to any

time, but he was the last exponent of an exhausted cause. As the last great geometer of antiquity he was robbed of importance, for he had no immediate successors, and when geometry again revived, his significance paled in the light of the great power of newer methods.

Mathematics is known to have remained in high favor at the Museum to the time of Pappus, for he drew many students about him. Along with most other subjects, however, it was now soon to be neglected. The attention of the time was focusing itself with utter completeness upon philosophy and especially upon theology. The stage was being set for the great religious upheaval. From the very time of Alexander's conquest, Egypt had been a religious melting-pot. The Paganism of the Greeks, impinging upon the native cult of the gods of the Nile, had mixed and fused with it into an amalgam. This was heavily alloyed by Hebraism, because of the many Jews in the country, and took up also traces of other oriental cults with which trade inevitably brought contact. To this was then added Christianity, the first Christian community having been founded at Alexandria by St. Mark hardly a dozen years after the Crucifixion.

In those early times the teachings of Christ found a ready acceptance in Egypt as they did elsewhere in the world. The Roman Empire, though outwardly brilliant and glorious, was nevertheless one in which the lot of the multitude beyond the pale of wealth and power was not happy. A far-flung officialism repressed all popular social and economic aspirations and held general education to a sadly low level. There was a large slave population, and for this especially hope and inspiration were remote. The pristine teachings of Christ, concerned as they were with brotherhood, self-renunciation, and salvation, preached a refreshing significance into the lives of the downtrodden. Among them, therefore, converts came easily. And as with time Christianity descended from its earlier sublime heights to the ground where ideals were bartered for material vantages, its membership spread from confines to the less favored, and came to include persons of means and even some with broader influence and power. This was its status at the time of Menelaus, near the end of the first century.

The new religion thereafter emerged into prominence rapidly, and therewith into rivalry with the older faiths of Pagan and Jew. In the inevitable struggle it found itself weak in its simplicity and in the uncodified state of its beliefs. Its opponents were agile, sophisticated, and learned, and it became clear that they were to be confuted only by more learning. Christianity to be defensible required a rationalization of its creed and a thorough-going intellectualization of its doctrine. It needed scholars, and to meet the need a Christian school of higher education was founded at Alexandria in openly professed rivalry to the Museum. This was at about the time during which Ptolemy was writing his *Almagest*. The ensuing period was the formative one for Christian ceremonial and dogma, and was, therefore, in its way, transcendently important. Able thinkers at the Christian school, in full possession of the best of Hellenic thought, fashioned the ritual and molded theology. At the Museum, meanwhile, the phil-

osophical and un-Christian doctrines of Neoplatonism had come into being. Thus Heathen and Christian scholars gravitated to opposite poles, and before long confronted each other figuratively with daggers drawn.

In the impending conflict the masses of Alexandria were far from holding themselves aloof. As their allegiances dictated they championed their respective causes with ever-growing fervor and ever-dwindling restraint. Street riots and uprisings with bloodshed became common, and upon occasion grew to such proportions that many of Alexandria's fine buildings came to destruction. In the beginning the Roman authorities were against the Christians, mainly for their uncompromising refusal to acknowledge divinity to the emperor. There was a period of unspeakably cruel and inhuman persecution, in virtue of which the era of Diophantus and Pappus stands in Christian annals as an epoch of the darkest martyrdom.

The adoption of Christianity by the Emperor Constantine, early in the fourth century, had the prime effect of reversing the rôles, and of placing the lot of the persecuted upon the Pagans. The Museum was especially affected, since it was placed thereby into subservience to the local Christian Patriarch. Of the scholars, some accepted Christianity and some fled. The majority, however, clung to their faiths and their posts through many lamentable vicissitudes. With the descent of the Roman throne to the Emperor Theodosius even this situation changed for the worse. Finding themselves in an intolerable position with their lives endangered, the Museum scholars fled their historic quarters, and took up an embattled position in the stronghold of their faith, namely, in the great Temple of Serapis. The maneuver was a wholly tragic one. In fanatical excitement, infuriated Christians laid siege to the Temple, and, when resistance had been broken, they consigned to destruction not only that magnificent building, but all it contained. An incomparable and priceless collection of works of Greek and Egyptian art thus fell prey to a senseless vandalism. And the immense library which was housed in this temple, which consisted of over three hundred thousand rolls of manuscripts, was destroyed. A second mortal blow had thus been delivered to the written record of antique culture. Again the painfully accumulated marks of centuries of genius had been heedlessly wiped away.

Nominally the Museum still continued in existence. The days of creative scholarship, however, were past. In the place of the earlier great sages of the Christians there were now only pedantic theologians, wallowing in morasses of tortuous arguments and fantastic refinements of dogmatism. In the sciences, as in art and literature, fresh creation had given way to a self-effacing and sterile deference to the great masters of the past. At the Museum mathematics was taught now by Theon and his daughter Hypatia. Theon wrote an excellent commentary upon the *Almagest* of Ptolemy, and is still prominently known for his recension of Euclid's *Elements*. Hypatia, on her part, wrote upon the first six books of the *Arithmetica* of Diophantus, and it is believed that the survival of these books as against the loss of the others is largely ascribable to the excellence of her commentary. Hypatia was the last of the Alexandrian mathematicians

and probably the last of the Museum's scholars. Destiny made her a tragic figure. Believing steadfastly in the gods of ancient Greece, she refused to apostatize even in peril. Popular resentment of this was heightened because of her high position, and, it is said, because of her great personal beauty. At all events she was finally set upon by a vulgarized and enraged mob, and torn limb from limb in the streets of Alexandria.

As an intellectual center, Alexandria was now extinguished. The buildings of the Museum were unoccupied and fell into ruin and decay. For a time Cleopatra's library, which was now supervised by a calligraphic guild, continued to supply the world with manuscript copies. But then even this lapsed, and when in the year 640 Alexandria fell to a new conqueror, who carried the bright torch of Mohammedanism, the library had long been closed. Its contents were so covered with dust that its importance was for some time overlooked. What wealth lay there neglected! But even this residue was not destined to survive. The Arabian governor, perplexed over the disposition of so immense a hoard of writings, referred the matter to Mecca. His reply was, that if the manuscripts contained anything contrary to the writings of Mohammed they were pernicious, and if they did not contain such they were superfluous. In either case their destruction was decreed, and they were removed from the face of the Earth. The destruction which the Pagan had accidentally begun, and the Christian had fanatically carried forward, the Mussulman thus in his ignorance finished.

The narrative reaches its close. Almost a millenium had passed since in the freshness of youth Alexandria had kindled two lights. The one of these was material, and sent forth its guiding rays from a Wonder of the World, a gleaming structure which rose high above the Mediterranean waters and the rocky African shore. The other light was immaterial, and shone only for those who sought it, from thirteen books inscribed upon shabby rolls of papyrus. Time has dealt differently with these lights. Many centuries ago the Earth shuddered and shook down the wonderful Pharos, and the restless sea has erased all traces of it. But the light of Euclid still shines, not only in the place of its inception, where the waters of the Nile issue upon the sea, but in all lands and wherever on the face of the Earth men continue to seek guidance from the light of the human intellect.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

INTEGRATED VERSUS TRADITIONAL MATHEMATICS*

J. S. GEORGES, Wright Junior College, Chicago

The proper approach to the analysis of the educational functions of mathematics seems to be a recognition of man's relations to the civilization he has created, and an evaluation of the contributions which mathematics has made to the maintenance and the advancement of that civilization.

If we attempt to deal with man in his relations to civilization, we are forced to acknowledge the fact that man is a supreme reality. To understand him we must understand his activities, the activities which are motivated by these relations to civilization. His activities raise certain problems which involve principles of ethics, of social institutions, of economics, of philosophy and religion, of politics, and of industry and science. To meet the social needs of the individual, education must in some measure find answers to all of these problems. Solution of isolated problems in special fields of endeavor may or may not contribute to an understanding of man's relations to civilization. On the other hand, an analysis of these relations in terms of all the subjects which constitute the curriculum is apt to result in a coördination of the educational efforts of the various subjects toward a satisfactory general solution.

In its relations to man's activities, education so far has had a twofold function. This is manifested in the distinctions between specialization and general education. Specialization is primarily industrial education in purpose and method. General education can be humanistic both in purpose and method, but need not be that. When general education is not primarily humanistic, it ceases to be general. The distinction between these two types of education needs to be pointed out. They are real and distinct, and do not depend upon the particular name we use in describing an educational system.

The two types of education may supplement each other, but cannot substitute for each other. If mathematics is to train only specialists it may succeed in producing craftsmen but not educated men, in supplying trained specialists but ignorant citizens, in giving expert scientists but not socialized human beings. It is true that in each subject the individual receives a training in some particular capacity, but the question of educational importance is, is that the only capacity which that subject can train? Must mathematics be concerned only with the development of the mathematical faculty of man, or is it possible for it to make contributions toward the development of the scientific faculty,

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the historical faculty, the esthetic faculty, and so on? Of the many common human faculties, it is quite reasonable to assume that mathematics, under favorable learning situations, can contribute a great deal to the development of some of them, and less to the development of others. But these contributions must be definite, must be attainable in actual learning situations as actual learning products.

The educational objectives of mathematics, however defined and however determined, are the contributions of the subject to the general education of man in his relations to civilization. The merit of the claim that mathematicians make concerning the unique educational functions of mathematics must be judged in terms of the attainments of those objectives. A private institution may set up any type of curriculum it desires, and if it can sell the idea to enough students, it may undertake to attain its aims by the best methods at its disposal. A public institution cannot do that. When an institution or educational system seeks to serve society educationally, then society has the right to examine those proposed objectives as to whether or not they are desirable, and if found desirable, whether or not they are attainable, and if attainable, whether or not they are worth the effort and the expense.

We have indicated above that the contributions of mathematics to the development and maintenance of civilization are significant and undeniably so.We have called attention to the fact that in the development of the common attributes of man, that is, in the humanistic or general education, mathematics can and does contribute noticeably. With such definite educational claims in behalf of mathematics, we should find mathematics a favorite subject among the curricula makers of the nation. But far from it! Everywhere we find mathematics on the defensive, struggling to maintain its very existence. Thus, we are faced with a fallacy. How can a subject which has done so much toward elevating man to the standard of the civilization he now possesses be discarded by those who claim to be the embodiment of that civilization?

In trying to find an answer to these questions, we are forced to examine the courses which are responsible for this attitude of the educators and of the general public. These are the so-called traditional courses in mathematics: algebra and plane geometry in the high school, college algebra, trigonometry, and analytic geometry in the junior college. These courses may manifest the spirit of mathematics, but may not singly interpret that spirit and make it shine forth in the brilliance that it deserves. We ask the teachers of mathematics the following relevant questions:

1. Have the traditional courses in mathematics met the educational needs of the non-specializing student?
2. Have the traditional courses in mathematics met the educational needs of the specializing students, other than those specializing in physical sciences and mathematics?
3. Have the traditional courses in mathematics met completely the educational needs of the students specializing in physical sciences?

4. Can the instructional materials of the traditional courses in mathematics be reorganized and modified to meet the educational objectives of mathematics?

We who love mathematics must answer these questions truthfully in order that we may find a satisfactory solution to this complex problem. To see that the picture of the educational importance of mathematics is dark and gloomy, we need only to look at it with discerning eyes. In a civilization whose genesis and maintenance depend upon mathematical thought and analysis, we find mathematics sadly neglected. We find our children going through high schools without being required to have upon their diplomas the stamp of approval of mathematics. We find in our public junior colleges, devoted to the education of our young men and women along broad and general educational lines, that not one day of a mathematical study is required.

The traditional courses in mathematics have had their opportunities to present the unique educational functions of mathematics in general education. If they have failed, it is because they have not presented mathematics, but compartmental trainings which have little or no connection with the problems of life. Mathematics is a method of thinking. Does the student of high school algebra acquire this method of thinking? Algebra is a symbolic language and as such is mathematics. But how many of our high school graduates who have had algebra have learned this language, which makes accessible to them the literature of the sciences? Is it any wonder that college algebra is not a favorite subject among the non-specializing students? Geometry is a method of reasoning, its only justification besides the recognition of a few spatial relationships, and as such geometry is mathematics. But how many high school graduates have acquired the rigorous and logical thinking which geometry is supposed to endow? Is it any wonder that analytic geometry finds no friends among the non-specializing students?

Consider those courses in their new setting in the junior college. They can develop, under favorable circumstances and sympathetic teaching situations, an intelligent attitude toward order, toward laws, and toward relationships, a scientific attitude, an understanding of transformations, an appreciation of invariants, an intelligent attitude toward the principle of representation, and an intelligent attitude toward special cases and generalizations. These and other desirable attitudes which should be acquired by the students in their general education can be developed by mathematics. But can they be developed by college algebra, by analytic geometry, by trigonometry of an average college? Their past records do not show it.

The non-specializing student, even if he were coerced into taking one semester of mathematics, has his choice between college algebra and trigonometry. Are these courses organized for the non-specializing student? They are not. The student knows it. No, if we are to give the student in general education one semester of mathematics, it cannot be algebra. He cannot be sold on the idea. He is less enthusiastic about trigonometry.

When we come to analyze the problem of the specializing students, other than those specializing in the physical sciences and mathematics, the traditional courses offered are just as non-appetizing to them. The pre-medic student, for example, is required by the professional school he is to attend to take one or two semesters of mathematics. Should he be subjected to the same treatment that the engineering student is? Why are the mathematical needs of the student of biology different from those of the student of chemistry? Does the student of social subjects have any mathematical needs? The future specialist, whatever his field, will be or should be an educated man. As pointed out above, the educated man has certain obligations to the civilization he enjoys, and mathematics can and does contribute definitely to the development of the educated man.

In college the future specialist has an opportunity to take certain elective courses. It would be interesting to determine how many students who are not required to take college algebra, elect it for the fun of it. And if they did, what has college algebra to offer them educationally? Usually the offerings of college algebra are a sequence of topics, inherently related of course, but actually without any unifying principle. We are speaking of the traditional course in algebra as it is organized at the present and not of algebra that can be organized as a unified course.

The class records of our junior college show definitely that the administrators, the curricula makers, and the teachers of non-mathematical subjects are pronounced in their opinions that the traditional mathematics courses do not constitute an integral part of the curriculum of general education. What is the opinion of the teachers of college mathematics regarding the matter? In a questionnaire sent out by the Committee on the Improvement of Science and Mathematics in General Education, appointed by the American Association for the Advancement of Science, to 250 universities, colleges, and teacher-training institutions, the following question was asked of the teachers of college mathematics: "Do you consider that the conventional college courses in mathematics, as represented by a majority of current text-books, are satisfactory for the non-specializing student?" The answers were: yes, 51; no, 130; uncertain, 24. This is the judgment of the teachers of mathematics. In the opinion of 60% of these teachers, the traditional courses in mathematics fall short of the general educational objectives.

It is interesting to note the reasons for such an evaluation of the educational values of the traditional course by the teachers who teach them. To the question: "Do you believe that one of the most significant contributions that mathematics should make for the non-specializing student is to develop the ability to do critical thinking?", the answers were: yes, 168; no, 27; uncertain, 3. Thus, in the opinion of the majority of the teachers of mathematics, the traditional courses: (1) are not satisfactory for the non-specializing student; and (2) fail to contribute to the education of the non-specializing student the attainment of general objectives, such as critical thinking.

Moreover, these teachers are interested in the solution of the problem, that

is, the determination of the contributions of mathematics to the general education of both non-specializing and specializing students. The following problem was proposed to them for consideration: "The clarification of a point of view for teachers concerning the place of mathematics in general education." One hundred and seventy voted it as very important; twenty-five, as having some importance; and four, as having no importance.

Whether or not the traditional courses in mathematics have met or have not met the general educational needs of students specializing in mathematics, physical sciences, and engineering, depends upon what is meant by the educational needs. Does a student trained in manipulations and processes acquire along with the development of skills and abilities in those processes an understanding and appreciation of the fundamental concepts of mathematics? We may reasonably assume that the student who is to specialize in mathematics and become a future mathematician, either as a research worker or as a teacher, will be imbued with the spirit of mathematics and its manifestations in life. However, the problem is not so simple in the case of the future engineer, physicist, chemist, *etc.* While the reported studies of the teachers in these fields, who endeavor to show that the students do not know their mathematics, must not be taken too seriously, we must, nevertheless, be concerned enough to take an inventory of our educational success. How many students taking one year of college mathematics of the traditional type develop the habit of critical thinking, of functional thinking? How many have an understanding of the fundamental notions of function, transformation, invariance, and limit? The answer can be found by the teachers in the senior college, and their judgment is far from favorable.

We in the junior college are in a position to report that the training in traditional courses offered in high school does not develop quantitative thinking. The high school graduate who is supposed to have had developed the art of rigorous thinking will do well if he thinks at all mathematically. Of appreciations of the significance of mathematical concepts and processes in life and the world he is totally ignorant.

As to whether or not the instructional materials of traditional mathematics can be reorganized and modified to meet the unique educational objectives of mathematics, we can state that the majority of the college teachers of mathematics are interested in the problem and that several institutions have experimentally developed certain so-called general or integrated mathematics courses along the lines of mathematical analysis. To the suggestion that: "The traditional courses in college algebra should be replaced by a mathematics course which attempts to unify algebra, trigonometry, analytic geometry, and elements of calculus," the report of the Committee showed the following replies: yes, 100; no, 53; uncertain, 42.

It is interesting to note that, if the majority of the college teachers of mathematics want such a reorganization, then why have they not solved the problem? The following report is enlightening. To the suggestion that: "The emphasis placed on 'pure research,' as a basis for advancement, has retarded the develop-

ment of a real concern about and research upon teaching problems related to introductory courses in mathematics," the answers were: yes, 110; no, 50; uncertain, 41.

To the question: "Do you feel that if the traditional courses were modified for the non-specializing students, they would be in danger of becoming superficial?", the answers were: yes, 98; no, 62; uncertain, 41. While the judgments of the teachers of mathematics, as reflected by their answers, are not conclusive regarding this matter, we might question: "Why should they become superficial?" Is presenting mathematical analysis, as a course of instruction, reducing mathematics to superficiality? We do not think so. Neither do the students who take such courses. An impartial comparison of such courses with the traditional courses will show that they are far from being superficial. On the contrary, they are richer in content and conducive to motivation and inspiration of the student.

We, at Wright College, have endeavored to present a course in mathematical analysis which utilizes the principles, processes, and concepts of mathematics from algebra, trigonometry, analytic geometry, and calculus, and believe that we are using the right methods of procedure to solve the problem of determining the general educational objectives of mathematics. The attainment of these aims is our present problem. Whether we succeed or not, we feel gratified that we are a part of a movement to undertake to fulfill the unique functions of mathematics in general education, and to strive to develop a sense of appreciation for mathematics in its relation to the educated man's responsibilities to society and civilization.

Mathematics: A Study Guide for Teachers, a publication of the University of Oregon Curriculum Laboratory, is now available in mimeographed form and may be secured from the University of Oregon Coöperative Store, Eugene, Oregon, for 25 cents. This study guide includes a discussion of the place of mathematics in general education, present status, and procedures for planning and developing a twelve-year program.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A SUBSTITUTE FOR THE LAW OF TANGENTS

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In the process of finding the remaining parts of a triangle when two sides and the included angle are known, the Law of Tangents,

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b},$$

is ordinarily used to find the remaining angles A and B. The third side c is then found by the Law of Sines. This method has the disadvantage that the angles are found by use of the relation $A+B+C=180°$, and so this equation cannot be made to serve as a check on the work. The following solution does not have this disadvantage.

We use the formula

$$\cot A = b/(a \sin C) - \cot C,$$

which is easily obtained from

$$b/a = \sin B/\sin A = \sin (A + C)/\sin A,$$

and the corresponding one for B,

$$\cot B = a/(b \sin C) - \cot C.$$

Since the same three quantities, b/a, $\sin C$, and $\cot C$, appear in both formulas the computations are only slightly longer than when using the Law of Tangents. Moreover, in computing c from

$$c \sin A = a \sin C,$$

only one new function, $\sin A$, is involved.

Note by the Editor. The two main sources of error in the solution of a triangle are the uses of the logarithmic and trigonometric tables, and the additions (and subtractions) involved in the problem. It is interesting to compare Mr. Fertig's method with the standard one on this basis. The solution by means of the Law of Tangents requires eight uses of the tables and nine additions; in Mr. Fertig's solution the corresponding numbers are ten and seven. Hence the complexities of the two methods are about equal. The use of the Law of Sines as a check on the Law of Tangents solution would require two additional uses of the tables and two more additions. R. J. W.

EXTENSION OF A THEOREM OF SERVAIS ON PERFECT NUMBERS

G. F. CRAMER, Tulane University

A perfect number is an integer N such that $F(N)=S(N)/N=2$, where $S(N)$ is the sum of all of the divisors of N. A multiply perfect number of multiplicity r is one such that $F(N)=r$, where r is an integer >2. Such a number is called a P_r. It is not known whether there are any odd perfect or multiply perfect numbers, but a great many even ones are known.

Servais* proved that, if there is an odd perfect number having exactly n distinct prime factors, then the smallest of these factors does not exceed n. His method can be used, with slight changes, to prove the following:

THEOREM. *Given a rational number $A>1$ and an integer $n>1$. If there is an odd integer $N=\prod_{i=1}^{i=n} p_i^{\alpha_i}$, with $p_i<p_{i+1}$ for all $1\leqq i<n$, having $F(N)=A$, then $p_1<(A+n-1)/(A-1)$.*

Let us consider the function $F(p^x)=(p^{x+1}-1)/p^x(p-1)$, where $x>0$ and $p\geqq 3$. It is easy to show that $F(p^x)$ increases with increasing x and that $\lim_{x\to\infty} F(p^x)=p/(p-1)$. Hence it is clear that $F(p_i^{\alpha_i})<p_i/(p_i-1)$. From the fact that $F(N_1N_2)=F(N_1)F(N_2)$ whenever $(N_1, N_2)=1$, it follows that

$$A=F(N)<\prod_{i=1}^{i=n}\frac{p_i}{p_i-1}.$$

Since $p_1<p_2<\cdots<p_n$, where the p_i are odd primes, we see that $p_2>p_1+1$, $p_3>p_1+2$, and, in general, $p_k>p_1+k-1$ for all integral k such that $1<k\leqq n$. Then $p_k-p_1-(k-1)>0$, and, adding $p_kp_1+(k-2)p_k$ to each side, we obtain the inequality $p_kp_1-p_1+(k-1)p_k-(k-1)>p_kp_1+(k-2)p_k$. Factoring each side, we find that $\{p_1+(k-1)\}(p_k-1)>p_k\{p_1+(k-2)\}$, and this gives $(p_1+k-1)/(p_1+k-2)>p_k/(p_k-1)$. From this set of $n-1$ inequalities we obtain, by multiplication, the new relation

$$\prod_{i=1}^{i=n}\frac{p_i}{p_i-1}<\frac{p_1}{p_1-1}\cdot\frac{p_1+1}{p_1}\cdot\frac{p_1+2}{p_1+1}\cdots\frac{p_1+n-2}{p_1+n-3}\cdot\frac{p_1+n-1}{p_1+n-2}.$$

The right side of this reduces to $(p_1+n-1)/(p_1-1)$ and hence we may write $A<(p_1+n-1)/(p_1-1)$. From this it follows at once that $(A-1)p_1<A+n-1$ and $p_1<(A+n-1)/(A-1)$.

When $A=2$, our theorem becomes the theorem of Servais, while other positive integral values of A yield similar theorems about odd multiply perfect numbers having exactly n distinct prime factors. Our hypothesis stated, however, only that $A>1$ is rational, not necessarily integral.

COROLLARY. *If there is an odd P_r with $r>2$, then it must have more than $2r-2$ distinct prime factors.*

The corollary follows from the inequality $3\leqq p_1<(r+n-1)/(r-1)$.

* Dickson, History of the Theory of Numbers, vol. 1, p. 26.

ON APPROXIMATE CUBATURE

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Since there are relatively few multiple integrals which can be evaluated in terms of elementary functions, approximate methods are of importance in applied mathematics. Much less attention appears to have been given to approximate cubature, however, than to approximate quadrature. The purpose of this note is to give a simple and direct discussion of a prismoid formula or Simpson rule for n-fold integrals. The formula for $n=2$ has been published by Woolley (*Mechanics Magazine*, 1851, p. 262 and *Transactions of the Institution of Naval Architects I*, 1860, p. 17) and others (*Encyklopädie der Mathematischen Wissenschaften*, vol. 2, part 3, first half, p. 136). This case and the case $n=3$ might well be mentioned in a course in calculus along with the case $n=1$.

A necessary and sufficient condition for the simple prismoid formula,

$$\int_p^q f(x)dx = (q-p)/6[f(p) + 4f(\overline{p+q}/2) + f(q)], \tag{1}$$

to be exact for all real p and q and for $f(x)$ four times differentiable is that $d^4f/dx^4=0$, *i.e.*, $f(x)=ax^3+bx^2+cx+d$.

An intuitive induction based on (1) and the corresponding formula for a double integral leads to the formula

$$\begin{aligned} &\int_{p_n}^{q_n} \cdots \int_{p_1}^{q_1} f(x_1, \cdots, x_n)dx_1 \cdots dx_n = \\ &\qquad \frac{\prod_1^n (q_i - p_i)}{3(2)^n}\left[\sum f(r_1, \cdots, r_n) + 2^{n+1}f\left(\frac{p_1+q_1}{2}, \cdots, \frac{p_n+q_n}{2}\right)\right], \end{aligned} \tag{2}$$

where $\sum f(r_1, \cdots, r_n)$ stands for the sum of the 2^n terms obtained when each r_i is replaced by p_i or q_i.

Regarding (2) as an identity in the p_i and q_i, we differentiate with respect to q_i and then set $q_i=p_i$ for $i=1, \cdots, n-1$. This yields

$$\begin{aligned} \int_{p_n}^{q_n} f(p_1, \cdots, p_{n-1}, x_n)dx_n = \frac{q_n-p_n}{6}\Big[&f(p_1, \cdots, p_{n-1}, p_n) \\ &+ 4f\left(p_1, \cdots, p_{n-1}, \frac{p_n+q_n}{2}\right) + f(p_1, \cdots, p_{n-1}, q_n)\Big]. \end{aligned} \tag{3}$$

After differentiating (3) four times with respect to q_n, we set $q_n=p_n$, and recalling that $(p_1, \cdots, p_n)$ is any set of values for $x_1, \cdots, x_n$, we have $\partial^4 f/\partial x_n^4=0$. Similarly, for $i=1, \cdots, n-1$, we have $\partial^4 f/\partial x_i^4=0$. Therefore, if (2) is exact, f is a polynomial of at most degree three in each of the variables.

We now confine attention to terms of the form $\prod_1^n x_i^{m_i}$, ($m_i=0, 1, 2,$ or 3). If

(and only if) formula (2) is an identity in the p_i and q_i for such a term, we call the combination of exponents $(m_i, \cdots, m_n)$ an admissible n-tuple. From the symmetry of (2) any permutation of an admissible n-tuple is admissible. We wish to know exactly which n-tuples are admissible.

THEOREM. *If $(m_1, \cdots, m_{n-1})$ is an admissible $(n-1)$-tuple with $m_1+\cdots+m_{n-1}\geqq n$, a necessary and sufficient condition for $(m_1, \cdots, m_{n-1}, u)$ to be an admissible n-tuple is that $u=0$ or 1.*

Proof. To prove the theorem, integrate $\prod_1^n x_i^{m_i}$ getting

$$A_n \equiv \prod_1^n (q_i^{m_i+1} - p_i^{m_i+1})/(m_i+1),$$

and use (2), getting

$$B_n \equiv \frac{\prod_1^n (q_i - p_i)}{3(2)^n}\left[\sum (r_1^{m_1} \cdots r_n^{m_n}) + 2^{n+1}\prod_1^n \left(\frac{p_i+q_i}{2}\right)^{m_i}\right].$$

The sufficiency is clear when we observe that if $A_{n-1}=B_{n-1}$ for a given $(n-1)$-tuple, we will have $A_n=B_n$ for the n-tuple obtained by adjoining a zero or one to the given $(n-1)$-tuple.

To show the necessity we suppose $A_n=B_n$ with m_n replaced by u. Then either $u=0$ or u must satisfy the equation

$$3(2)^{m_1+\cdots+m_{n-1}+u-1} = [2^{m_1+\cdots+m_{n-1}+u-(n+1)} + 1](u+1)\prod_1^{n-1}(m_i+1), \tag{4}$$

obtained from $A_n=B_n$ by setting $p_i=1$, $q_i=0$, $(i=1, \cdots, n)$. Similarly, from $A_{n-1}=B_{n-1}$, we obtain (5), an equation analogous to (4). Dividing (5) by (4) we have

$$2^u = \left[1 + \frac{2^M(2^{u-1}-1)}{1+2^M}\right](u+1), \tag{6}$$

where $M\equiv m_1+\cdots+m_{n-1}-n$ is $\geqq 0$ by hypothesis. By trial, (6) is found to hold for $u=1$, but not for $u=2$ or 3; and for $u\geqq 4$, the right member is $\geqq 2^u+2 > 2^u$.

COROLLARY. *An n-tuple consisting of zeros and (or) ones or an n-tuple consisting of a 2 or a 3 together with $(n-1)$ zeros and (or) ones is admissible. These are the only admissible n-tuples except for permutations.*

The corollary is evident from the theorem and the fact that the admissible 1-tuples are 0, 1, 2, and 3.

Admissible 2-tuples are hence (0, 0), (0, 1), (1, 1), (2, 0), (2, 1), (3, 0), and (3, 1); and we can state that a necessary and sufficient condition for $f(x, y)$ to be integrated exactly by (2), $\partial^4 f/\partial x^4$ and $\partial^4 f/\partial y^4$ assumed to exist, is that $f(x, y) = ax^3y+bxy^3+B(x, y)$, where $B(x, y)$ is a binary cubic.

There are $(3n+1)$ admissible n-tuples.

In practice it may be desirable to subdivide a given "rectangular" range of integration or to approximate a non-rectangular range by a sum of "rectangular" cells. This amounts to introducing a general Simpson rule.

Confining attention to double integrals for a moment, it may be remarked that simple cubature formulas for other than rectangular ranges would be useful. Some plane figures can not be approximated satisfactorily by rectangles without using a very large number of them. As an example of a formula for a triangular range we have

$$\tag{7} \begin{aligned} \int\int f(x, y)dx\,dy = \frac{S}{11}\Big[f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) \\ + 8f\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)\Big], \end{aligned}$$

where (x_i, y_i), $(i=1, 2, 3)$, are the vertices and S is the area. Properties of the centroid show that (7) is exact for $f(x, y)=ax+by+c$. Furthermore, if the formula is to be exact for all triangles, and if certain derivatives of f are assumed to exist, then f must have the above form. The formula often gives good approximations when f has other forms and when the triangle is curvilinear.

A formula similar to (7) and similar to (2) with $n=2$ can be used for an integral over a parallelogram. We use ordinates at the four corners and 8 times the ordinate at the centroid. This is exact for $f(x, y)=ax+by+c$.

The method here outlined for obtaining cubature formulas by starting with the prismoid formula (1) also can be applied to other quadrature formulas of the Newton-Cotes type.

SOLID ANGLES

J. W. Cell, North Carolina State College

The concept of a solid angle is seldom introduced in the modern course in trigonometry for engineering students. Yet, to name two applications, this geometrical idea is convenient in studying light and in studying solenoid electrical coils. This topic can be introduced naturally in spherical trigonometry or at the same time that the term radian is defined in plane trigonometry. Even a brief discussion of the concept is preferable to its complete omission and the basic definition is easy. In the following paragraphs we outline some of the facts about solid angles.

Definition. *Let C be a curve bounding an area A on a unit sphere. By the solid angle subtended by A we mean the figure composed of all the rays issuing from the center of the sphere and passing through points of C. The solid radian measure of this angle is defined to be the area of A.*

A polyhedral angle is a well known example of a solid angle. From the defini-

tion it follows that the maximum value for a solid angle is 4π solid radians, since the surface area of the unit sphere is 4π square units.

THEOREM 1. *The volume of a spherical sector is given by*

$$V = R^3\theta/3,$$

where θ is the solid angle at the vertex of the spherical sector.

For a unit sphere we see that $\theta = 3V$. This fact can be compared with the corresponding property for a plane angle, namely that $\phi = 2A$, where A is the area of the plane sector with vertex angle ϕ.

THEOREM 2. *The surface area of a zone on the surface of a sphere is given by $S = R^2\theta$, where θ is the solid angle which the zone subtends at the center of the sphere.*

If we recall that the surface area of a zone is given by $S = 2\pi RH$, where H is the altitude of the zone and R is the radius of the sphere, then another theorem follows from the preceding one.

THEOREM 3. *The solid angle at the vertex of a right circular cone with altitude h, radius r, and slant height s, is*

$$\theta = 2\pi(1 - h/s) = 2\pi\left(1 - \frac{h}{(h^2 + r^2)^{1/2}}\right) = 2\pi(1 - \cos\alpha),$$

where α is the plane angle between the axis of the cone and an element on the surface of the cone.

From the formula for the surface area of a spherical triangle, we obtain the following:

THEOREM 4. *The solid angle which is subtended by a spherical triangle with angles A, B, and C (each measured in radian measure) is given by*

$$\theta = (A + B + C) - \pi,$$

the spherical excess in radian measure.

THEOREM 5. *The solid angle subtended at the vertex of a pyramid, with rectangular base (sides a and b) and with its vertex h units above one corner of the base, is given by*

$$\theta = \tan^{-1}\frac{ab}{h(a^2 + b^2 + h^2)^{1/2}}\cdot$$

One could prove this theorem by considering the surface area intercepted, between the sides of the pyramid, on a unit sphere with center at the vertex of the pyramid. One could then divide the surface area into two right spherical triangles and apply Napier's rules to determine the angles. From this theorem we obtain immediately the following:

THEOREM 6. *The solid angle at the vertex of a right pyramid with square base (altitude h and side of base 2a) is given by*

$$\theta = 4 \tan^{-1} \frac{a^2}{h(h^2 + 2a^2)^{1/2}} = 4 \sin^{-1} \frac{a^2}{a^2 + h^2} \cdot$$

Problem 1. Given that the number of lumens illumination on a plane area, caused by a point source of light, is equal to the candle power of the light multiplied by the solid angle which the plane area subtends at the light. Determine the number of lumens illumination:

a. On a circular area of radius 6 feet if the light is 60 candle power and is 8 feet directly above the center of the area. (Ans. 48π lumens).

b. On a square area of side 6 feet if a light of 100 candle power is placed 10 feet above the center of the area. (Ans. $400 \sin^{-1} (9/109)$ lumens).

Problem 2. Compare the total amounts of illumination on the floor of a room which measures 20 feet by 20 feet and has a ceiling height of 20 feet if:

a. One light of 1000 candle power is placed directly over the center of the floor and in the ceiling. (Ans. $4000 \sin^{-1} (1/5)$ lumens).

b. One light of 250 candle power is placed at each upper corner of the room. (Ans. $500\ \pi/3$ lumens).

c. One light of 250 candle power is placed at the middle of each side of the ceiling. (Ans. $2000 \tan^{-1} (1/3)$ lumens).

Problem 3. Timbie and Bush in *Principles of Electrical Engineering*, third edition, page 368, show that for any point on the axis of a circular solenoid,

$$H_z = \frac{2\pi NI}{L}\left\{\frac{L/2 + y}{\{r^2 + (L/2 + y)^2\}^{1/2}} + \frac{L/2 - y}{\{r^2 + (L/2 - y)^2\}^{1/2}}\right\},$$

where N is the number of turns of wire in the length of the coil L, r is the radius of the solenoid, I is the current which the coil carries, H_z is the magnetic flux in the direction of the axis of the solenoid, and y is the distance from the center of the axis of the solenoid to the point in question.

A solenoid, for the purpose of this problem, can be thought of as a coil of wire wound uniformly around an oatmeal box, with but a single layer of wire.

a. Show that this formula can be stated in the simpler form,

$$H_z = (NI/L)(\theta),$$

where θ is the solid angle which the solenoid subtends at the point in question.

b. Give the special values for the point at the middle and at one end of the solenoid. What do these become if the solenoid is long compared to its diameter?

Remark. In problems such as the preceding one it is sometimes more convenient to determine the solid angle subtended by the opening and then to subtract this from the proper value. Thus, to determine the solid angle subtended by a circular cylinder at the middle of one end, it is easier to compute the solid angle subtended by the opening (the solid angle at the vertex of a right circular cone) and subtract this value from 2π.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Plane Trigonometry. Revised edition. By R. W. Brink. New York and London, D. Appleton-Century Co., 1940. 12+226+110 pages. \$2.00.

Introduction to the Calculus, Part I. By S. Beatty and J. T. Jenkins. Toronto, University of Toronto Press, 1938. 650 pages. \$5.50.

Stencils for Solving $x^2 \equiv a$ (mod n). By R. M. Robinson. Berkeley and Los Angeles, University of California Press, 1940. 14 pages+272 Hollerith cards.

Mathematical Logic. By W. V. O. Quine. New York, W. W. Norton and Company, Inc., 1940. 13+348 pages. \$4.00.

Darstellende Geometrie, Erster Teil. Elemente; Ebenflächige Gebilde. By R. Haussner. Fünfte, unveränderte Auflage. (Sammlung Göschen, Band 142.) Berlin, Walter de Gruyter and Co., 1940. 207 pages. RM 1.62.

Vierstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen in zwei Farben zuzammengestellt. By H. Schubert. Neue, verbesserte und vermehrte Auflage von R. Haussner. (Sammlung Göschen, Band 81.) Berlin, Walter de Gruyter and Co., 1940. 181 pages. RM 1.62.

College Algebra. By H. T. Davis. New York, Prentice-Hall, Inc., 1940. 13+423 pages. \$2.50.

The Special Theory of Relativity. By H. Dingle. (Methuen's Monographs on Physical Subjects.) London, Methuen and Co., Ltd., 1940. 7+94v pages. 3/6 s.

REVIEWS

College Mathematics. A First Course. By W. W. Elliott and E. R. C. Miles. New York, Prentice-Hall, 1940. 6+396 pages. \$3.00.

Many texts have appeared in recent years that are intended to provide a one-year course in the mathematics thought essential to a general education. The book under review is such a text. It presents in one volume what the authors consider the essentials of algebra and plane trigonometry, and an introduction to plane analytic geometry and calculus.

The subjects have been presented in their traditional order. The first 151 pages constitute a fairly complete course in college algebra beginning with a rapid review of elementary algebra and a warning about zero denominators. Among the omissions are the subjects of partial fractions, inequalities, and determinants (although second and third order determinants are included in the part on analytic geometry).

Trigonometry is treated in the second part of seventy pages. Happily the student will not be handicapped by "opposite over hypotenuse, *etc.*" as a first

definition and by finding sin 0° by a limiting process. Also, the authors leave undefined the undefined and do not even have a lazy 8 in their tables. However, the bright student may make himself known by pointing out places where the authors have overlooked the trouble that may be caused by a denominator being zero. Objection might be raised to the phrase "x is *the* angle whose sine is y" in the context in which it appears, and must be raised to the phrase "we choose *them* for *the* principle value of $\sin^{-1} x$."

The part on analytic geometry is intended only as an introduction to the subject, and seventy-nine pages is sufficient for the purpose. Polar coördinates and parametric equations are discussed, but the straight line and conic section form the major part of the work. The definition of the graph for a function is, however, unnecessary as it was given (and in a better way) in the part on algebra.

An acceptable introduction to calculus might be given in seventy pages if extreme care were taken. Over a page of this part is devoted to evaluating the limit as x approaches zero of $x/\sin x$ although only two related problems are given, and only the calculus for algebraic functions is discussed. The procedure is formal and the reasoning even more intuitive than seems advisable. It is implied that if at a point Δy is positive, then the derivative is positive. Continuity of the derivative is required to show that the derivative is zero at a maximum or minimum point. Integration is introduced and used only as the inverse of differentiation and is applied only for finding areas and volumes.

An instructor will do well to consider Elliott and Miles if he wishes to map out a trip, following a time-honored trail with many scenic bypaths eliminated, from the foothills of elementary algebra to a point on Mt. Calculus where the difficult climb to the peak begins. If, however, he wishes to make the trip in a year he will have to stick to the main trail as it takes two years to visit all points of interest listed in this guide book.

J. F. RANDOLPH

The Development of Mathematics. By E. T. Bell. New York and London, McGraw-Hill Book Company, 1940. 13+583 pages. \$4.50.

This book was written in response to a "request . . . for a broad account of the general development of mathematics." *Men of Mathematics* was a history of mathematics in terms of its men. This book is a history of the evolution of mathematical ideas; in the author's words, an "endeavor to portray mathematics as the constantly growing, human thing that it is, advancing in spite of its errors and partly because of them."

After a "General Prospectus" the author devotes six chapters to the history of mathematics up to 1687, chiefly in chronological order. From that time the order is primarily ideational. Fully half of the book is devoted to mathematics of the nineteenth and twentieth centuries.

Readers of books by this author are so numerous that it is scarcely necessary for the reviewer to comment on his refreshing style flavored with the spice of

many a debunking quip. But the reviewer cannot refrain from quoting one sentence which might stir up interesting controversy: "To say that a particular problem has lost its importance for modern mathematics may therefore be merely a rationalized confession of incapacity."

In the opinion of the reviewer, aside from its value as a pleasant-to-read history of the evolution of mathematics, this book is especially worth while in its careful implantation of a few very fundamental ideas about mathematics as a whole. One of these applies to the evolution of a concept of proof: what was proof for one age ceased to be a proof for the next. Another of these ideas is that mathematics derives its interest and its usefulness from its strivings toward the unknown: "Without fertilization by creative new ideas, each [mode of mathematical thought] was doomed to sterility" and "In mathematics of all places, finality is a chimera. Its rare appearances are witnessed only by the mathematical dead." Such ideas are not only mentioned in the beginning but are so woven into the book that no reader can escape them.

It is to be devoutly hoped that many non-mathematicians, as well as mathematicians, will read this book. B. W. JONES

Plane Trigonometry. Revised edition. By R. W. Brink. New York and London, D. Appleton-Century Co., 1940. 12+226+110 pages. $2.00.

The first edition of this book appeared in 1928; it was reviewed in this MONTHLY by Professor Harriet E. Glazier (vol. 36, 1929, pp. 92–94). The present one has been so extensively revised that it is really a new book. The treatment is rather full. It is felt that the approach to a new concept can best be made by an ample explanation of the idea involved. The progress is very gradual, and amply illustrated by long lists of examples. Answers to the odd-numbered exercises are given; the others can be obtained upon request by teachers.

Chapter II, Trigonometric functions of an acute angle, seems unnecessarily detailed, since the general definitions were given in the preceding chapter. The discussion of logarithms includes graphs, exponential and logarithmic equations, and change of base, all starred for possible omission. A commendable feature of numerical computation is a clear discussion of the accuracy of computed results.

Since projection is among the starred subjects, it could not be used to prove the addition theorems. The older procedure is followed, requiring eight pages for these and the laws of sines and cosines; by the vector method, a single page would suffice for all cases and be in keeping with present tendencies.

A starred chapter on De Moivre's theorem and its consequences is a good addition in a longer course.

The tables include five-place tables of logarithms, logarithmic and natural trigonometric functions, and various others, including squares, square roots, and reciprocals.

The make-up of the page is excellent; the figures are well drawn; the proof reading has been carefully done. VIRGIL SNYDER

Bibliography of Mathematical Works Printed in America through 1850. By L. C. Karpinski. With the coöperation, for Washington Libraries, of W. F. Shenton. Ann Arbor, University of Michigan Press; London, Humphrey Milford, Oxford University Press, 1940. 26+697 pages. $6.00.

While the author entitles this work *Bibliography*, it contains such a wealth of additional material so admirably presented that it is difficult to write a review of it with any kind of restraint. A so-called bibliography may be merely a check-list; or it may include with the first edition of each work a statement which attempts to give its purpose, its special characteristics and its contents, even its history. This bibliography strikes a happy mean between these two. As a check-list of the publications it aims to cover, it is outstandingly complete, but in several ways; it reveals the aims of the authors of mathematical works in America through 1850, and so becomes a source-book for the study of the history of American mathematics up to that time. A perusal of the author's "Introduction" will reveal his purpose in this respect.

Professor Karpinski in his preface expresses generous appreciation of the wide and varied services at his command in the preparation of the work but the reviewer and all who examine it now as well as the host who, for generations to come, turn to it for help and guidance must recognize the master mind which conceived and directed its destinies to this fine end.

The most conspicuous way in which the nature of each work is covered is the reproduction of titles and also of other pages by zinc line engravings, a tremendously impressive feature. Indeed, "bibliographers may regard this work as introducing a new method in their field, *i.e.*, the far wider use of zinc line engravings." It is not only new but fascinating in the vitality that it gives to the whole book. With each figure, the legend provides a brief statement of the outstanding feature of the work or one of its contents. For example, Fig. 46. "Parts of two pages from the *Noticia breve . . . arithmêtica pràctica* showing fractions, 'quebrados,' and the rule of double false position as treated by Padilla, 1732." (p. 53)

The Bibliographical List of Books, Pamphlets, and Broadsides is arranged by centuries, the Sixteenth, the Seventeenth, the Eighteenth, and the period from 1801 to 1850 inclusive.

The value of the bibliography is greatly enhanced by the complete indexes. Besides the usual "General Index," there are "Topical Indexes." These cover works dealing with the following topics: Algebra; Analytic Geometry and Conic Sections; Arithmetic; Calculus; Games and Puzzles; Geometry and Mensuration; History, Biography, and Philosophy; Mechanics and Optics; Practical Navigators; Surveying; Trigonometry; Tables, Ready Reckoners, Logarithms, the Slide Rule, and Mathematical Instruments; General Works. In addition, there is an "Index on Non-English and Canadian Works" which covers "Choctaw, Dutch, French, German, Hawaiian, Portuguese, Spanish and Latin, Canadian"; and an "Index of Printers and Publishers." Such an array of indexes is sufficient evidence of the historical value of *Bibliography of American Works* . . . . More-

over, the completeness of this study would not be indicated unless attention were called to the fact that the bibliographer has gone into the matter of "Entry Titles—Works whose Publication is Problematical," and also into those of "Encyclopedias and Encyclopedic Reference Works" and "Journals and Newspapers with Mathematical Articles."

It is a satisfaction always to find new light thrown on one's studies. The reviewer found two instances of this character and there are doubtless many throughout the book. In particular, an algebra, Mahan, Jason H.: *Key to the hitherto impenetrable secret . . .* , 1847 (p. 499) is added to the *Bibliography of Algebra Textbooks* (1936) although the "Index Works Dealing with Algebra" (p. 658) seems to include unjustifiably some dozen or more titles.

The book-making must be commended. The arrangement of the engravings, never far removed from the place of the title of the book and its editions, is a feature which alone must have involved infinite care and patience. This care characterizes all the set-up with results that are most satisfying.

It is a temptation to linger over items of interest, such as attempts to sell works through giving them amusing titles, and mathematical recreations of the nature of *Chinese Philosophical and Mathematical Trangrams*; in fact, to emphasize further that the work under review is invaluable first of all as a bibliography but equally as a source of history, suggestions for the classroom and for clubs, and above all for the reader who is after something new in any line of writing. Libraries must own this *Bibliography* but so should departments of mathematics in secondary schools and colleges.

LAO G. SIMONS

Wiley Trigonometric Tables. New York, John Wiley and Sons; London, Chapman and Hall, 1940. 81 pages. $0.75.

The compilers have tried to produce a set of tables which is most appropriate for a trigonometry course, as accurate as possible, and, by choice of type and arrangement, of maximum convenience to the user.

Six tables are included: squares and square roots; selected constants and their logarithms; natural logarithms of numbers, entries from 1.0 to 10.0 by tenths, logarithms to five places; five-place common logarithms from 1 to 1000, with proportional parts; five-place trigonometric tables, including S and T corrections for small angles; four-place tables of natural trigonometric functions at intervals of $10'$, including angles in radians.

In the logarithms of numbers, the spacing is in sets of three with cross lines marking every tenth row. The same arrangement is used in the tables of logarithmic trigonometric functions. Here the pages are crowded, as more entries must be made on each page.

The entire book is compact, the outside dimensions of the bound volume being $8\frac{1}{2}$ by $5\frac{1}{2}$ inches.

VIRGIL SNYDER

Mathematics of Accounting and Finance. By C. H. Langer and T. B. Gill. Chicago, Walton Publishing Co., 1940. Parts I, II, 971 pages+64 pages of tables +283 pages of problems+42 pages.

Part I begins with a comprehensive review of arithmetic, extending through the first 136 pages of the text. There is a commendable effort to define all terms used, and to follow in logical order the processes employed in arithmetic operations with integers, fractions, and decimals. Many helpful hints are given for checking additions and subtractions, and the well known rules for casting out nines and elevens are developed. These last methods of checking can hardly be regarded as practically useful; but for their intrinsic interest, they may well deserve a place in a treatment as detailed as the one given in the present volume. The section ends with a chapter on approximate numbers. There are some pronounced errors in this chapter. The definition of the term "absolute error" is not clear, and the principles relating to approximations in addition are not entirely significant. In paragraph 221, the following statement occurs: "The absolute error of the difference of two numbers which differ from their true values by certain absolute errors is equal to the difference of their absolute errors." This statement is false, unless one assigns a special meaning to the clause "which differ from their true values by certain absolute errors." And in that case, the principle in itself is not as important as the one which is better known, namely: *The absolute error in the sum or difference of two numbers is less than or equal to the sum of the absolute errors in the numbers.* The chapter contains a number of helpful illustrations, but in the space devoted to the topic more might have been accomplished if the authors had treated the subject in a later section of the text, after the student had become familiar with algebraic notation. The definition of *significant figures* is given in a footnote in a later part of the text, but the important relations connecting the relative errors in numbers with the number of significant figures which may be retained after various operations on them, is not even touched upon. If carefully developed, the subject should not be too difficult for the average student; and considering its practical importance, it deserves better treatment than it has received in most books on algebra. It is hoped the present authors will include a comprehensive treatment of this subject in a later edition. In other respects, the section on arithmetic is presented in an interesting manner. Exception may be taken to the author's definition of an axiom as "a truth so evident as not to require proof."

The rest of Part I is devoted to algebra and its application to various problems in accounting. The fundamental operations are developed and applied to problems in simple interest, percentage, discounts, and averages. In addition, there are chapters on alligation, ratio and proportion, and simultaneous linear equations. A working knowledge of determinants of the second and third orders is developed, and illustrated with pertinent problems in accounting. The volume ends with a chapter on the method of successive trials—applied to solve simultaneous linear equations in two and three unknowns. Although these illustrations undoubtedly give the student a fair idea of the nature of the method of succes-

sive approximation, one may question the desirability of presenting the method in connection with applications to which it is not particularly well suited.

Part II deals with the binomial theorem, powers and roots of a number, the quadratic equation, logarithms, progressions, and the application of these principles to more advanced problems in accounting and finance. The chapters dealing with compound interest, bonds, annuities, depreciation, and building and loan associations are ably presented and illustrated. One might wish the authors had included a chapter on short-term loan companies, and the various installment plans in retail credit, in view of the importance of the latter in the modern financial structure. An unusual feature of the second volume is an exposition of 71 pages on how to use the slide rule.

The large amount of space devoted to the topics covered invites comparison of the text with standard books on algebra. It must be admitted that the mathematical treatment does not always compare favorably with the corresponding discussions in standard texts. On pages 142–143, for instance, the authors state the associative and commutative laws for addition and multiplication, but fail to state the distributive law. The latter is given much later, in a footnote on page 163, preceded by various other "laws," such as the "law of signs in multiplication" and the "law of exponents for multiplication." The word "law" is nowhere defined, and it is not clear whether it is meant to be synonymous with a theorem or an assumption. Although the omission of some proofs in an elementary text is inevitable, it is expected that a clear distinction be drawn between fundamental assumptions and theorems accepted without proof.

The treatment of determinants is sketchy, but probably developed sufficiently for the practical applications illustrated. In the discussion of the binomial theorem (Part II), the authors give a proof of the theorem for the first three terms only; but the derivation is labeled as a general proof. On page 523, the method of extracting the square root of $(1-x)$ is correct only if x is less than one. This fundamental restriction is omitted; and there is of course no discussion of the magnitude of the error in stopping with any term in the process. The definition of a *series* on page 675 is incorrect.

A few shortcomings of lesser importance were noted.

Both volumes contain large sections of problems, but no answers are given to any of them. Eight-place tables of compound interest, present value, amount and present value of an annuity, as well as five-place logarithmic tables are given in Part II. Both books contain good indexes and tables of contents, and they are printed in commendable format.

GERTRUDE BLANCH

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

CLUB PUBLICATIONS

1. *Log-In-Rhythm.* Copies of the 1938, 1939, and 1940 editions of this magazine published by the *Mathematics Club* of *Hunter College* reached us recently and these contain a variety of features which should prove of interest to other clubs. The 1938 staff, under the editorship of Esther Luttan, listed the replies received from a questionnaire sent to a number of professors of mathematics at Hunter College and elsewhere. Some of the questions were:

1. In what particular branch of mathematics are you most interested?
2. On what are you now working?
3. What are the prospects of women in your field?
4. What do you consider the most important (preferably recent) mathematical discovery?
5. What advice would you give to a mathematics major who expects to enter into your field of work?

One query, "What classroom boner* is outstanding in your memory?", gave rise to replies such as:

"Δx is the excrement of x."

"$136,040 correct to four significant figures is $1360."

"An eclipse is a comic section."

"Arc tan x = arc sin x/arc cos x."

"An examination paper once revealed that the 'Ahmes papyrus was broken in half and has fragments of an American papyrus in the cracks.'"

The 1939 issue features interviews by Vivian Fruchtbaum and Justine Schmertz with Professor Richard Courant of New York University, Mr. E. C. Molina of the Bell Telephone Laboratories, and Professor Arnold Dresden of Swarthmore College. Interviews in 1940 were made with Professor R. C. Archibald of Brown University and Professor Helen Walker of Teachers College, Columbia University, by Tecla Combariati, Doris Newman, and Bianca Rivoli. Each of the magazines contains short articles by students which include helpful bibliographies. Some of these are: Topology, Louise Miller; Statistical sunspots, Shirley Orlinoff; Student curriculum in 200 A.D., Doris Newman; Pyramids, Beulah Pomerantz; Build up your own geometry, Gwenevere Freedman; Omar Khayyam, Bianca Rivoli; New York State Savings Bank Insurance, Bertha Link; Sonya Kowaleski and Emmy Noether, Wilma Szabo; and Extra-sensory perception, Esther Luttan and Jeannette Miller.

Clubs interested in exchanging publications or in securing copies of the last four issues at ten cents a copy for club or departmental files should address: Editor-in-Chief, *Log-In-Rhythm*, Department of Mathematics, Hunter College, New York, N. Y.

2. *Math Mirror.* The eighth in the annual series published by the *Mathematics Clubs* of *Brooklyn College*, this one lives up to the reputation established by its predecessors. Student articles include: Time and relativity, Herbert Fusfeld; On the equation $f^{(n)}(x)=x$, Richard Bellman; Solids, Melvin Weiner; General root theory, Frank Harary; and On sums of powers of primes, Richard Bellman. Professor Courant of New York University, guest speaker at one of the *Math Society* meetings, spoke on the problem of inscribing a triangle of minimum perimeter in a given acute triangle, showing a solution not based on the calculus; the address is reported in the *Math Mirror* by Peter Chairulli. The Pi Mu Epsilon Contest questions, and a list of twelve problems which will test the mathematical abilities of any college student, are also to be found in this publication.

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* *Editorial Note.* This department will welcome for publication other favorite classroom boners from clubs, departments, and readers. E. H. C. H.

Clubs interested in exchanging copies or purchasing previous issues for their files should address: Professor H. F. Mac Neish, Faculty Advisor, Brooklyn College, Brooklyn, New York.

PICTURES OF FAMOUS AMERICAN MATHEMATICIANS

Are there any readers of the MONTHLY who have collected a file of pictures of famous American mathematicians? A number of years ago one graduate student at the University of Chicago obtained some good snapshots of Professors Moore, Wilczynski, Slaught, Dickson, Lunn, Bliss, and MacMillan; these negatives were widely circulated among the graduate students, and pictures obtained are valued by them now. Kappa Mu Epsilon chapters have chosen as titles for officers the names of American mathematicians such as Birkhoff, Bôcher, Cajori, Carmichael, Dickson, Einstein, Hedrick, Moore, Benjamin Pierce, Rietz, Slaught, Smith, and Townsend. Would clubs or individuals be interested in setting up a collection of photographs or snapshots of our American mathematicians? How about a "Mathematics Hall of Fame"!

CLUB REPORTS, 1939-40

Mathematics Club, University of Alberta

Seven meetings were held during the year at which the following papers were presented: Sir Isaac Newton, a symposium by Anna Malanchuk, Sybil Fratkin, and Eoin Whitney; Rocket flight to the moon, by Dr. J. W. Campbell; A mathematical shortcut, by Marjorie Stockwell; Forest mensuration, by Neil German; Lobachevskian geometry, by George Kokotailo; Counting by twelves, by Jack Turner; and Rotations, permutations, and impossibilities, by Dwight Williams.

The Square Circle, Woman's College of the University of North Carolina

At the December meeting three seniors presented a program on the development of the calculus, Eleanor Cashwell speaking on Forerunners in the early calculus, Ethel Whitley summarizing the Medieval history of the calculus, and Mary Brown speaking on Modern forerunners of the calculus. Another meeting was devoted to the topics: Thomas Jefferson and mathematics, by Shirley Elliott; History of mathematical absurdities, by Frances Baer; and The human elements in mathematics, by Juanita Miller. Dr. H. Kimmel completed the year's program with a talk on New number systems. Officers were: President, Dorothy Koehler; Vice-President, Mary Rives; Secretary, Bobbie Lee Clegg; Program Chairman, Editha Morris.

Junior Mathematical Club, University of Chicago

The Club's program for the year included papers by experts in various fields distinct from pure mathematics. Professor T. O. Yntema of the Cowles Commission spoke on Mathematics in economic research, Professor H. T. Davis of Northwestern University spoke on A mathematical interpretation of the business cycle, Dr. A. S. Householder spoke on Mathematical biophysics and the central nervous system, Dr. Thorkill Jacobson discussed the Beginnings of chronology in Mesopotamia, Mr. Anatol Rapoport gave an illustrated lecture on the Foundations of harmony, Mr. R. Dubisch presented A mathematical approach to aesthetics, and Professor M. J. Andrade discussed the Analogies between mathematical language and the natural languages. Papers by graduate students on pure mathematics included: Continued fractions, by Morris Bloom; Infinite products, by Herman Meyer; and Dirichlet series, by Melcher Fobes. The club prize for the best paper was presented to William Karush for his paper on Linear inequalities. Talks of a different character were given by Professor G. A. Bliss who spoke to the Club on the History of the Chicago mathematics department, Mr. W. C. Carter and Dr. O. F. G. Schilling who described respectively Mathematical pedagogy in England and on the continent, and by Arthur Wormser of the Berlin Polytechnic Institute who described his new Folding geometry.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 457. *Proposed by V. Thébault, San Sebastián, Spain.*

Construct three circles which have one common point and which are such that each touches two sides of a given triangle, the six points of contact being concyclic.

E 458. *Proposed by J. L. Brenner, University of Minnesota.*

Prove that in any power of the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

two elements in the main diagonal will be the same. Show that the same result holds for any matrix (a_{rs}) in which $a_{r1}=a_{2r}$, $a_{1r}=a_{r2}$, $(r>2)$, and $a_{11}=a_{22}$.

E 459. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Show that the altitudes and ex-radii of any triangle satisfy the following relations:

$$\sum \frac{h_a^2(r_b+r_c)}{r_br_c(h_a+2r_a)} = 2, \qquad \sum \frac{r_br_c}{(r_b+r_c)(h_a+2r_a)} = \frac{1}{2}\,.$$

E 460. *Proposed by Henry Scheffé, Oregon State College.*

Let $s(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$ be the sum of the first n terms of the harmonic series. A well known expression for $s(n)$ which does not formally involve the sum of n terms is the integral

$$\int_0^1 \frac{u^n-1}{u-1}\,du.$$

It is desired to write $s(n)$ in the form $s(n)=f^{(n)}(0)$. Find an expression for $f(x)$ in terms of the integral of an elementary function.

SOLUTIONS

E 421 [1940, 318]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres having a point in common, construct a secant through this common point, meeting the spheres again in points P, Q, R, S, so that we shall have, both in magnitude and in sign,

$$PQ:PR:PS = u:v:w,$$

where u, v, w are given.

Solution by the Proposer.

Let L be the point common to the four given spheres (A), (B), (C), (D). Then the required secant meets the sphere (A) in the point P whose powers for the spheres (B), (C), (D) are in the ratios

$$PQ\cdot PL : PR\cdot PL : PS\cdot PL = PQ : PR : PS = u:v:w.$$

Hence P belongs to a sphere (X) coaxal with (B), (C), and again P belongs to a sphere (Y) coaxal with (C), (D). (See N. A. Court, *Modern Pure Solid Geometry*, p. 185, art. 585.) Thus P is determined as the common point, other than L, of the three spheres (X), (Y), (A), and the required secant is LP.

Also solved by L. M. Kelly.

E 422 [1940, 318]. *Proposed by J. E. Trevor, Cornell University.*

Let s_n, $(n<10)$, be the sum of the first n terms of the series

$$1 + 12 + 123 + 1234 + \cdots,$$

and let S_n, $(n<10)$, be the sum of the first n terms of

$$1 + 21 + 321 + 4321 + \cdots.$$

Find closed expressions for s_n and S_n, and deduce the linear relation

$$10S_n - (9n-2)s_n = n(n+1)(n+2)/2.$$

Solution by E. A. Nordhaus, University of Wisconsin Extension.

Let $i=1, 2, \cdots, n$ and $j=1, 2, \cdots, n-1$ be indices of summation. By grouping corresponding digits, we have

$$s_n = \sum t_{n+1-i}10^{i-1}, \quad \text{where} \quad t_r = 1 + 2 + 3 + \cdots + r = r(r+1)/2.$$

Thus

$$2s_n = \sum (n+1-i)(n+2-i)10^{i-1} = (n+1)(n+2)A - (2n+3)B + C,$$

where

$$A = \sum 10^{i-1} = (10^n - 1)/9, \qquad B = \sum i\cdot 10^{i-1}, \qquad C = \sum i^2\cdot 10^{i-1}.$$

To evaluate B, we note that

$$\begin{aligned} 9B &= \sum i\cdot 10^i - \sum i\cdot 10^{i-1} = n\cdot 10^n + \sum j\cdot 10^j - 1 - \sum (j+1)10^j \\ &= n\cdot 10^n - 1 - \sum 10^j = n\cdot 10^n - A. \end{aligned}$$

By a similar device, we find $9C=n^2\cdot 10^n-2B+A$. Hence

$$162s_n = 200A - n(9n+29), \quad \text{where} \quad A = (10^n-1)/9.$$

Again grouping corresponding digits, we obtain

$$S_n = \sum i(n+1-i)10^{i-1} = (n+1)B - C,$$

whence $81S_n=10(9n-2)A+11n$.

The desired linear relation is obtained by eliminating A from these two formulas.

Also solved by R. K. Allen, Nathan Newman, E. P. Starke, Alan Wayne, and the proposer.

E 423 [1940, 318]. *Proposed by C. W. Trigg, Los Angeles City College.*

Squares are constructed on the sides of a right triangle ABC. Denote the centroid of the square on BC and exterior to ABC by A', and the centroid of the square on BC and "interior" to ABC by A''. Use corresponding notation for the centroids of the other four squares. Show that:

(i) the centroids of ABC, $A'B'C'$, $A''B''C''$ coincide;

(ii) $A'B'C'$ and $A''B''C''$ are never equilateral;

(iii) two vertices of $A'B'C'$ (or $A''B''C''$) fall on an altitude of $A''B''C''$ (or $A'B'C'$), and the third vertex falls on the side to which that altitude is drawn;

(iv) one side of $A'B'C'$ (or $A''B''C''$) and the altitude to that side are equal; the foot of this altitude divides the side into segments proportional to the legs of the right triangle;

(v) the sum of the areas of $A'B'C'$ and $A''B''C''$ equals one-half the area of the square on the hypotenuse;

(vi) the difference of the areas of $A'B'C'$ and $A''B''C''$ equals twice the area of ABC.

Solution by D. L. MacKay, Evander Childs High School, New York.

Let the vertices of the triangle be $A(b, 0)$, $B(0, a)$, $C(0, 0)$. Then those of the derived triangles are

$$\begin{array}{lll} A'(-a/2,\ a/2), & B'(b/2,\ -b/2), & C'(\{a+b\}/2,\ \{a+b\}/2), \\ A''(a/2,\ a/2), & B''(b/2,\ b/2), & C''(\{b-a\}/2,\ \{a-b\}/2). \end{array}$$

(i) By taking one-third the sum of the respective coördinates, we obtain $(b/3, a/3)$ as the centroid of each triangle.

(ii) Triangle $A'B'C'$ cannot be equilateral, since

$$A'C'^2 - A'B'^2 = \{(2a+b)^2+b^2\}/4 - (a+b)^2/2 = a^2/2.$$

In like manner, triangle $A''B''C''$ is not equilateral.

(iii) The points C', B'', A'' all lie on the internal bisector of angle ACB. This is an altitude of triangle $A'B'C'$, since $A'CB'$ is the external bisector. The equation of $A'B'$ is $x+y=0$, which is satisfied by the coördinates of C''. The proof for triangle $A''B''C''$ is similar.

(iv) We have $A'B'^2=(a+b)^2/2=C'C^2$. Also, $B'C=b/\sqrt{2}$ and $CA'=a/\sqrt{2}$.

(v–vi) The areas of triangles $A'B'C'$ and $A''B''C''$ are easily seen to be $(a\pm b)^2/4=(c^2\pm 2ab)/4$. Thus their sum and difference are $c^2/2$ and ab.

Also solved by W. B. Clarke (who made an elegant drawing) and the proposer.

E 424 [1940, 319]. *Proposed by V. Thébault, Le Mans, France.*

If the number $123\cdots(n-2)n$, in the scale of $n+1$, is multiplied by a two-digit number, the sum of the two digits being n, show that the digits of the product are all alike, save that the final digit may be zero.

Solution by R. K. Allen, Montpelier, Vermont.

In the scale of $n+1$, a two-digit number whose digit-sum is n is divisible by n, and the other factor is either a one-digit number or 10. Accordingly, we multiply first by n, obtaining

$$\begin{aligned} 123\cdots(n-2)n\times n &= 123\cdots(n-2)n0-123\cdots(n-2)n \\ &= 111\cdots 11, \qquad (n \text{ digits}), \end{aligned}$$

and then by the other number.

Also solved by H. T. R. Aude, W. E. Buker, H. D. Larsen, Nathan Newman, E. P. Starke, and the proposer. Larsen used the identity

$$1+\sum_{r=1}^{n-1} r(n+1)^{n-1-r}=\{(n+1)^n-1\}/n^2.$$

E 425 [1940, 395]. *Proposed by W. C. Rufus, University of Michigan.*

A man sold cows at a price per head equal to the number. With the proceeds he bought calves at \$10 each, and a pig with the remainder (less than \$10). If a pig cost one-sixth as much as a cow, how many calves did he buy?

Solution by Daniel Finkel, New York.

Let x be the price of the pig in dollars (an integer less than 10). Then the price of each cow was $6x$, and we have $(6x)^2\equiv x \pmod{10}$. Therefore x is even, say $x=2y$; and

$$72y^2\equiv y \pmod 5, \quad \text{where} \quad 0<y<5.$$

We deduce in turn,

$$72y\equiv 1 \pmod 5, \qquad 2y\equiv 1 \pmod 5, \qquad y=3, \quad x=6,$$

and it follows that the number of calves was 129.

Also solved by R. K. Allen, W. E. Buker, A. C. Cohen, Jr., M. L. Constable, E. G. H. Comfort, J. W. Dappert, Wm. Douglas, F. C. Hall, D. F. Johnson, H. D. Larsen, Yetta V. Maizlish, P. W. A. Raine, Janet M. Rung, Hazel E. Schoonmaker, E. P. Starke, Albert Tornlof, C. W. Trigg, I. M. DeLong, and the proposer.

Editorial Note. In the original version of the problem, the number of calves was given to be odd. Almost all the solvers pointed out that this restriction was superfluous.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3984. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

The two points P, Q are symmetric as to the circumcenter of a triangle, the isogonal conjugates of P, Q are P', Q', and M is the midpoint of $P'Q'$; prove that $PQ\cdot P'Q'=4R\cdot HM$, where H is the orthocenter and R is the circumradius of the triangle.

3985. *Proposed by Charles M. Stein, New York City.*

Prove that if $f(x)$ is analytic within and on a circle C having x as center and containing y, then

$$\cdots\left(1+\frac{y-x}{n}D\right)\cdots\left(1+\frac{y-x}{2}D\right)\left(1+\frac{y-x}{1}D\right)f(x)=f(y),$$

where $D=d/dx$.

3986. *Proposed by V. Thébault, Le Mans, France.*

The six points α, β, γ, α', β', γ' are taken respectively on the edges BC, CA, AB, DA, DB, DC of the tetrahedron $ABCD$; and the radical planes of the circumsphere of $ABCD$ and the spheres $(A\beta\gamma\alpha')$, $(B\gamma\alpha\beta')$, $(C\alpha\beta\gamma')$, $(D\alpha'\beta'\gamma')$ cut the planes of the faces BCD, CDA, DAB, ABC in the lines Δ_a, Δ_b, Δ_c, Δ_d. Show that these four straight lines are in the same plane if $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent, and conversely.

3987. *Proposed by V. Thébault, Le Mans, France.*

The spheres S_a, S_b, S_c, S_d have their respective centers at the vertices of the tetrahedron $ABCD$, and radii whose squares are one-half the sum of the squares

of the sides of the face opposite to the vertex considered; and the spheres S'_a, S'_b, S'_c, S'_d have centers at A_1, B_1, C_1, D_1, symmetrics of A, B, C, D with respect to G the centroid of the tetrahedron, and radii A_1A, B_1B, C_1C, D_1D. Let C_a be the intersection of S_a and S'_a, and similarly for C_b, C_c, C_d. Prove that: (1) The four circles C_a, C_b, C_c, C_d lie upon the same sphere Σ with its center at G and passing through the intersection of the Longchamps sphere with the anticomplementary sphere of the circumsphere of $ABCD$. (2) Show that Σ is the Monge sphere for the Steiner ellipsoid circumscribing $ABCD$.

Note. The Longchamps sphere is orthogonal to the spheres S_a, S_b, S_c, S_d. See N. A. Court, *L'Enseignement Mathématique*, Geneva, 1930, pp. 31–34; V. Thébault, *loc. cit.*, 1937, pp. 81–89. The anticomplementary sphere of the sphere $(ABCD)$ is the circumsphere of the tetrahedron formed by planes through the vertices of $ABCD$ parallel to the respective opposite faces.

SOLUTIONS

3912 [1939, 239]. *Proposed by J. R. Musselman, Western Reserve University.*

If G be the centroid of the triangle ABC, and L', M', N' be the symmetrics of A, B, C respectively as to G, show that the circles $AM'N'$, $BN'L'$, and $CL'M'$ meet on the circumcircle of ABC at the Steiner point of the triangle ABC.

Generalization by R. Goormaghtigh, Bruges, Belgium.

The two problems 3911, 3912 are special cases of two theorems given by the proposer in *Mathesis*, 1939, pp. 51–52; the following theorem is a generalization containing these various properties:

If L', M', N' be the symmetrics of A, B, C respectively as to the homologous point in the medial triangle to any point P on an equilateral hyperbola (H), circumscribed to ABC, the circles $AM'N'$, $BN'L'$, $CL'M'$ meet on the circumcircle at the homologous point to the center of (H) in the antimedial triangle.

Let $A'B'C'$ be the antipedal triangle of P as to ABC; M' and N' are the images of P in the midpoints of AC and AB, and the circle $AM'N'$ is equal to $BPCA'$. If $\overline{A'A_1}=\overline{PA}$, then A_1 is on the circle $AM'N'$, the second extremity of the diameter passing through A.

Further, let A_0 be the image of A through the circumcenter of ABC, then the second intersection S of the circles ABC, $AM'N'$ is the point where A_1A_0 meets again the circle ABC.

If α, β, γ, ξ are the abscissas of A, B, C, P on the equilateral hyperbola (H) having as equation $xy=1$, the coördinates of A' are

$$\beta+\gamma+(\beta\gamma\xi)^{-1}, \qquad \beta^{-1}+\gamma^{-1}+\beta\gamma\xi;$$

therefore those of A_1,

$$(1) \qquad \alpha+\beta+\gamma+\beta^{-1}\gamma^{-1}t, \qquad \alpha^{-1}+\beta^{-1}+\gamma^{-1}-t, \qquad t=(1-\beta\gamma\xi^2)\xi^{-1},$$

are linear in t, and the locus of A_1 is a straight line when P describes the hyperbola (H).

When P is on A, A_1 is on A_0; when $t=0$, A_1 is, from (1), the point homologous to the center of (H) in the antimedial triangle to ABC; being also the second intersection of A_1A_0 with the circle ABC, this point is nothing but S, and this proves the theorem.

The theorem of 3911 will then be obtained when (H) is the Jerabek hyperbola, counter-point conic of the Euler line, having as center the point homologous, in the medial triangle, to the Feuerbach point of the tangential triangle, P being the circumcenter of ABC.

The theorem of 3912 will be obtained when (H) is the Kiepert hyperbola, the counter-point conic of the circumdiameter passing through the Lemoine point, having as center the point homologous, in the medial triangle, to the Steiner point, P being the centroid of ABC.

Solved also by Frank Ayres, Jr. and O. J. Ramler.

Editorial Note. Ramler's solution of 3911 and Ayres's solution of 3912 used the same kind of coördinates as in the solution of the latter above. Ramler used barycentric coördinates for 3912.

The theorems in these two problems are related to the theorem in the solution of 3797 [1938, 486] which says that, if a rectangular hyperbola (S) passes through the vertices of a triangle ABC and a point P, and if A_0, B_0, C_0, L, M, N are respectively the midpoints of the sides of ABC and the midpoints of AP, BP, CP; then the circumcircles of $A_0B_0C_0$, A_0MN, B_0NL, C_0LM intersect on the conic (K) circumscribing the hexagon $ANBLCM$ in S, the center of (S). If P is at the centroid G of ABC, the conic (K) becomes the Steiner ellipse for $A_0B_0C_0$ and (S) is then the isogonal conjugate of OK with respect to ABC, where O and K are the circumcenter and symmedian point for ABC. The center S in this case is called the Steiner point for $A_0B_0C_0$. This is easily seen to give the solution of 3912.

In 3911, the symmetric of O in BC is easily seen to be also the symmetric L' of A with respect to N. Also, (O) is the inscribed circle for the tangential triangle of ABC; then the theorem in 3838 [1939, 602] completes the solution of this problem.

Also, the theorem by Goormaghtigh follows from the reference to 3797. For, if $A_2B_2C_2$ is the antimedial triangle of ABC and P, P_0, P_2 are homologous points for ABC, $A_0B_0C_0$, $A_2B_2C_2$, then A_0 is the midpoint of AA_2 and AP, A_0P_0, A_2P_2 are parallel and their midpoints L, L_0, L_2 are collinear with G, $A_0P_0=AP/2$; and hence L_2 is the symmetric of A with respect to P_0. The circumcircles $(A_0B_0C_0)$, (A_0MN), *etc.*, are homologous to (ABC), (AM_2N_2), *etc.* The first set of four circles intersect in S, the center of (S), the rectangular hyperbola through A, B, C, P, and the second set intersect in S_2, the center of (S_2) through A_2, B_2, C_2, P_2. Hence S and S_2 are homologous with respect to ABC and $A_2B_2C_2$.

3913 [1939, 301]. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

If $f(x)=e^{kx}\tan x$, $k\geqq 0$, discuss the effect of the value of k on the location of maximum, minimum, and inflection points.

Solution by F. Underwood, University College, Nottingham, England.

Whatever the value of k, the curve $y=f(x)$ consists of an infinite number of branches, such that for each, y varies from $-\infty$ to $+\infty$ as x varies from $n\pi-\pi/2$ to $n\pi+\pi/2$, where n is any integer. Within each branch interval, y varies continuously and vanishes at its midpoint. We have

$$f'(x) = e^{kx}(k \tan x + \sec^2 x),$$

$$f''(x) = e^{kx} \sec^2 x(k^2 \sin x \cos x + 2k + 2 \tan x).$$

When $f'(x)=0$, $\sin 2x=-2/k$; and this gives real values of x for $k\geqq 0$ if and only if $k\geqq 2$. When this condition is satisfied, let $\sin 2\alpha=2/k$, $0<\alpha\leqq\pi/4$; then, if $k>2$, there is a maximum for $x=n\pi-\pi/2+\alpha$ and a minimum for $x=n\pi-\alpha$, and an inflection point between the two. For $k=2$ there are no maxima nor minima, but a horizontal inflection tangent at $x=n\pi-\pi/4$.

For $0\leqq k<2$, there are no maxima nor minima, but there is still a single point of inflection on each branch for all values of $k\geqq 0$. For, if we set $t=\tan x$, the equation $f''(x)=0$ gives $2t^3+2kt^2+(2+k^2)t+2k=0$, and this equation is easily shown to have only one real root for the values of k considered here.

Solved also by the proposer.

Editorial Note. The proposer showed, among other additional facts, that, as k increases from 0, the inflection abscissa decreases from $n\pi$ until $k=2$, when it is at $n\pi-\pi/4$, and then increases approaching $n\pi$.

3914 [1939, 301]. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

A dog is tied to a rope of length L, which is fastened on the other side of a smooth topped fence of height h at a point a units from the top, $L>h+a$. Discuss the shape and dimensions of the region over which he can roam; and find its area.

Solution by A. K. Waltz, Clarkson College, Potsdam, N. Y.

Let RS be the top of the wall; A, the fixed point of the rope; B and O, the projections of A on RS and the bottom of the wall; and let rectangular axes be chosen with the origin at O, the x-axis along the wall bottom, and the y-axis on the level ground on the dog side. When the dog is at P with the rope taut crossing the wall at Q, let N be the projection of P on RS, $t=NP$. Then, if ABQ is rotated about BQ into $A'BQ$ in the continuation of plane PQN, the points A', Q, P must be collinear and $A'P=L$. It then follows that the coördinates of P are $x=\pm[L^2-(a+t)^2]^{1/2}$, $y=[t^2-h^2]^{1/2}$. The boundary curve is symmetric with respect to the y-axis with the ordinate $[(L-a)^2-h^2]^{1/2}$ at the origin, cutting orthogonally the x-axis at $\pm[L^2-(a+h)^2]^{1/2}$, and concave in the direction perpendicular to the wall. The polar equation is

$$\rho^2 = K + 2a^2 \sin^2\theta - [(K + 2a^2 \sin^2\theta)^2 + 4a^2h^2 - K^2]^{1/2},$$

$$K = L^2 - h^2 - a^2 > 0.$$

By use of the substitution $z=2a^2\sin^2\theta+K$ the area may be expressed in the form

$$\int_0^{\pi/2}\rho^2 d\theta = \frac{\pi}{2}(a^2+K) - \frac{1}{2i}\int_\beta^\alpha \frac{(\gamma\delta+z^2)dz}{\sqrt{(\alpha-z)(\beta-z)(\gamma\delta+z^2)}},$$

$$\alpha = 2a^2+K, \qquad \beta = K, \qquad \gamma = (K^2-4a^2h^2)^{1/2},$$

$$\delta = -\gamma, \qquad \alpha > \beta > \gamma > 0 > \delta.$$

This type of integral is discussed in Hancock's *Theory of Elliptic Functions*, vol. 1, pp. 184, 185. By lengthy and tedious transformations it may be reduced to integrals of the type

$$\int \frac{\sin^4\phi}{\cos^4\phi}\left(\frac{d\phi}{\Delta\phi}\right); \qquad \int \frac{\sin^2\phi}{\cos^4\phi}\left(\frac{d\phi}{\Delta\phi}\right); \qquad \int \frac{1}{\cos^4\phi}\left(\frac{d\phi}{\Delta\phi}\right);$$

$$\int \frac{dx}{x^2\sqrt{X}}; \qquad \int \frac{dx}{x\sqrt{X}}; \qquad \Delta\phi = \sqrt{1-k^2\sin^2\phi},$$

and X is a quadratic form. By use of a reduction formula, page 60 in Hancock's *Elliptic Integrals*, the first three of these are expressible in elliptic integrals of the first and second kind, while the last two are found in elementary tables. The area may then be found by use of tables.

3915 [1939, 301]. *Proposed by V. Thébault, Le Mans, France.*

Two numbers have each eight figures, the last of which is seven in each one; and the first eight figures of their product are identical as well as the last eight. Find the two numbers.

Solution by Wang Chih Yi, Yenching University, Peiping, China.

From the problem it is clear that the product must be divisible by 11,111,111 and its last eight figures must all be nine. After dividing, the quotient is of the form x00,000,009. Now since $10\equiv 3$, $30\equiv 2$, $20\equiv 6$, $60\equiv 4$, $40\equiv 5$, $50\equiv 1 \pmod 7$, if we set $x=6$, this number 600,000,009 is divisible by 7, and the quotient is 85,714,287. Hence the two numbers are 85,714,287 and 77,777,777.

There are other solutions; for example, (i) 99,086,757 and 67,281,107; and (ii) 91,417,827 and 72,925,237.

Solved also by J. W. Clawson, E. P. Starke, and the proposer.

Editorial Note. All of the solvers obtained the first result above; Starke obtained a second solution for $x=2$ giving 68645357×32372507, and he remarked that x cannot be 9, for then the product is $10^{16}-1$ which requires more than eight digits in one of its factors. The proposer considered also the case where the product has 15 digits, each a 9, which may be written $27\times 37\times 1001001001001$; and, if the last factor can be decomposed into two suitable factors a, b, each terminated by unity, then $27a$ and $37b$ would furnish a solution.

3916 [1939, 301]. *Proposed by V. Thébault, Le Mans, France.*

Find a sequence of three figures such that if the sequence is regarded first

as a number in a system with a certain unknown base, and then as a number in the duodecimal system (base 12), the value of the number in the second case is one-half of that in the first. Show that there are only two such unknown bases for the given problem.

Solution by Kwan Chao Chih, Yenching University, Peiping, China.

Suppose a, b, c (not all zero) is a sequence of three figures. In an unknown base m, the number is

$$am^2 + bm + c, \tag{1}$$

and in the duodecimal system, it is

$$12^2a + 12b + c. \tag{2}$$

Since the latter is given in the problem as one-half of the former, we have

$$(288 - m^2)a + (24 - m)b + c = 0. \tag{3}$$

Now from (2) we know that a, b, c are all non-negative integers and less than 12. Thus, (3) holds true if the coefficients of a and b are not both positive; consequently, $m > 16$. Furthermore, $m < 19$. For even in the case $m = 19$ we have

$$5b + c = 73a. \tag{4}$$

There exists no solution of (4) for a, b, c all less than 12, unless $a = b = c = 0$. Hence we can only have either (i) $m = 17$ or (ii) $m = 18$, as the unknown base.

We can further tabulate all the possible sequences of the three figures a, b, and c as follows:

(i) $m = 17$, $a = 7b + c$

$a = 1$	2	3	4	5	6	7	8	9	10	11	7	8	9	10	11
$b = 0$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
$c = 1$	2	3	4	5	6	7	8	9	10	11	0	1	2	3	4

(ii) $m = 18$, $36a = 6b + c$

$a = 1$	1	2
$b = 6$	5	11
$c = 0$	6	6

Solved also by Michael Aissen, J. W. Clawson, E. P. Starke, and the proposer.

3917 [1939, 302]. *Proposed by R. A. Johnson, Brooklyn College.*

Determine the values of n for which the sum of the squares of n positive integers in arithmetic progression is the square of an integer, and show how to determine such progressions.

Remarks by J. Barinaga, Madrid University, Spain.

This problem was proposed by A. Boutin in the *L' Intermédiaire des Mathématiciens*, vol. V, 1898, p. 75, Question 1252; and solved partially by A. Palm-

ström in the same volume, pp. 205–208. The problem does not appear to have been solved completely at this date. In Dickson's *History of the Theory of Numbers*, vol. 2, p. 322 are several references. It would be interesting and perhaps original to extend the known results for ordinary arithmetic progressions to those of higher order, that is to sequences (u_i) where $\Delta^k u_i = \text{const.}$, $k > 1$.

Partial Solution by E. P. Starke, Rutgers University.

Let the integers of the progression be $a, a+d, a+2d, \cdots, a+(n-1)d$. Then the sum of their squares is easy to compute and we have

$$\begin{aligned} a^2 + (a+d)^2 + \cdots + [a+(n-1)d]^2 \\ = na^2 + n(n-1)ad + n(n-1)(2n-1)d^2/6 = S^2, \end{aligned} \tag{1}$$

where we have written S^2 according to the hypothesis. If we put $x = 2S$ and $y = 2a + (n-1)d$, we reduce (1) to the more useful form

$$x^2 = ny^2 + (n+1)n(n-1)d^2/3. \tag{2}$$

We are to determine values of n such that (2) has solutions in integers.

Suppose first that $n = k^2$ is a perfect square. Then (2) becomes $(x-ky)(x+ky) = (k^2+1)k^2(k^2-1)d^2/3$, for which rational solutions are easily found for any d and any factorization of the right member. The terms of the resulting progression can be made integral by an obvious multiplication. For example, when $n = 49$ we see that (2) is $(x-7y)(x+7y) = 39200d^2$. Try $d = 1$ and put $x - 7y = 28$, $x + 7y = 1400$ to find $x = 714$, $y = 98$. Again, if we put $d = 1$ and $x - 7y = 100$, $x + 7y = 392$, we have $x = 246$, $y = 146/7$; hence $x = 7 \cdot 246$, $y = 146$, $d = 7$ is a solution of (2).

For other values of x, (2) is a so-called Pellian equation which cannot have solutions unless its coefficients are satisfactory quadratic residues. A couple of examples will elucidate the meaning and application of this necessary condition. If $n = 6$, then (2) becomes $x^2 = 6y^2 + 70d^2$. Since 6 is a non-residue mod 7, there is no solution for $x^2 \equiv 6y^2 \pmod 7$ unless y is a multiple of 7. But then x is a multiple of 7, and hence $x^2 - 6y^2$ is a multiple of 49, so that d is a multiple of 7. Upon dividing (2) by 7^2, we have an equation of the same form connecting $x/7$, $y/7$, and $d/7$. If $y/7$ is still a multiple of 7, this process may be repeated until we have, in place of y, a number prime to 7. Then the proof that (2) has no solution for $n = 6$ is complete. Again if $n = 7$, equation (2) becomes $x^2 = 7y^2 + 112d^2$. Thus x is a multiple of 7; put $x = 7z$ to get $7z^2 = y^2 + 16d^2$. Since -16 is a non-residue mod 7, $y^2 \equiv -16d^2 \pmod 7$ has no solution if d is prime to 7. If d is a multiple of 7, an argument analogous to the above shows that still there is no solution.

The values of n, not squares and not greater than 100, which remain after consideration of quadratic residues are $n = 11, 23, 24, 26, 33, 47, 50, 52, 59, 73, 74, 88, 96$. For each it can be shown that infinitely many progressions exist which satisfy the demands of the problem. However, no general form of solution for n appears.

The complete solution of $x^2 - Cy^2 = \pm H$, for possible values of H, is due to

Lagrange. (See Chrystal's *Algebra*, vol. 2, pp. 478–485.) In simple cases the identity

$$(3) \qquad | ux \pm cvy |^2 - C | uy \pm vx |^2 = (x^2 - Cy^2)(u^2 - Cv^2)$$

is quite useful. For example, if $n=52$, we have from (2), $x^2=52y^2+53\cdot 17\cdot 52d^2$. Try $d=1$ and put $x=26z$ to obtain $13z^2=y^2+17\cdot 53$. Using (3) we can construct a solution if we know solutions of $y^2-13z^2=\pm 17$, $y^2-13z^2=\mp 53$. Of course solutions of $y^2-13z^2=\pm 1$ are found directly from the convergents of the expansion of $\sqrt{13}$ in a continued fraction. We find easily by trial, $y_1=15$, $z_1=4$ is a solution of $y^2-13z^2=17$; $y_2=8$, $z_2=3$, of $y^2-13z^2=-53$; whence $y=36$, $z=13$ is a solution of $y^2-13z^2=-17\cdot 53$. But since a is positive, $y=2a+(n-1)d>51$. Hence take (10, 3) for $y^2-13z^2=-17$; (8, 3) for $y^2-13z^2=-53$; (18, 5) for $y^2-13z^2=-1$, to obtain $y=276$, $z=77$. Then we find $a=225/2$, $S=1001$. For an integral solution, take $a=225$, $d=2$, $S=2002$. Other combinations and other values of d produce other solutions.

In a similar manner the following progressions were obtained:

$(n, a, d, S)=(4, 13, 6, 46)$, (9, 2, 3, 48), (11, 18, 1, 77), (16, 27, 2, 172), (23, 7, 1, 92), (24, 1, 1, 70), (25, 27, 2, 265), (26, 25, 1, 195), (33, 7, 1, 143), (36, 17, 6, 822), (47, 539, 1, 3854), (49, 25, 1, 357), (50, 7, 1, 245), (52, 9, 4, 910), (59, 22, 1, 413), (64, 31, 2, 808), (73, 442, 1, 4088), (74, 70266, 1, 604765), (81, 73, 6, 3087), (88, 4152, 1, 39358), (96, 52, 1, 1012), (100, 47, 2, 1570).

It should be remembered that there are infinitely many progressions for each of these values of n and that those here indicated are not necessarily the simplest.

Editorial Note. G. W. Wishard considered the case of consecutive integers and found for $n=2$ the values of $a=3$, 20, 119, 696, 4059, 23660, *etc.*; for $n=11$, $(a, S)=(18, 77)$, (38, 143), (456, 1529), (854, 2849), (9192, 30503); for $n=23$, $(a, S)=(7, 92)$, (17, 138), (1351, 6582).

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

At the Philadelphia meeting of the American Association for the Advancement of Science in December, G. T. Whyburn, University of Virginia, was elected Vice-President and Chairman of Section A; Dunham Jackson, University of Minnesota, Secretary of Section A; and C. C. MacDuffee, Hunter College, member of the Sectional Committee.

Professor A. A. Albert of the University of Chicago will deliver a William Lowell Putnam lecture on mathematics at Harvard University on March 20, 1941. He will also give lectures at Princeton University, Brown University, and the Institute for Advanced Study during the latter part of March.

Dr. W. B. Caton has been made head of the department of mathematics at Athens College, Athens, Alabama.

Dr. J. R. Jenness of the College of the Ozarks has been made head of the department of physics and mathematics at Parsons College, Fairfield, Iowa.

Dr. S. A. Jennings of the University of British Columbia has been appointed lecturer.

Assistant Professor A. N. Lowan of Yeshiva College has been promoted to an associate professorship.

Professor J. C. Morehead of the Carnegie Institute of Technology has retired with the title emeritus.

Associate Professor H. A. Ruger of Teachers College, Columbia University, has retired.

C. S. Sutton of Tufts College has been appointed to an assistant professorship at The Citadel.

The following appointments to instructorships are announced:
Georgia School of Technology: Dr. D. T. McClay
Long Island University: Mr. E. A. Knobelauch
Purdue University: Dr. R. A. Leibler
Agricultural and Mechanical College of Texas: Dr. R. E. Basye
Tri-State College, Angola, Indiana: Dr. M. G. Moore
University of Virginia: R. C. Morrow
University of Wisconsin at Milwaukee: W. L. G. Mitchell

Professor W. C. Graustein of Harvard University was instantly killed in an auto accident on January 22, 1941. He had been on the mathematics faculty at Harvard since 1919, serving as chairman of the department from 1932 to 1937. At the time of his death he was chairman of the board of tutors of mathematics and assistant dean of the faculties of the university. He was vice-president of the Mathematical Association during 1929, 1930, and 1940.

HIGH SCHOOL TEACHER PREPARATION

A committee of the North Central Association has recently made a careful study of the educational procedures prevailing at twelve representative liberal arts colleges, and now is inviting the faculties of all 188 independent liberal arts colleges in this area to attend one of ten regional conferences and discuss their problems and experiments in high school teacher preparation. The problems will be discussed primarily as they affect college teaching in the content areas. In addition to the sectional discussions, each conference will have an address by a high school superintendent on "What we expect from the colleges," and one by an educational psychologist on the topic "Learning problems at the college level." The conferences are scheduled as follows:

February 7, 8—Chicago, Illinois
February 14, 15—Springfield, Missouri
February 21, 22—Toledo, Ohio
February 28, March 1—Delaware, Ohio
March 7, 8—Greencastle, Indiana
March 14, 15—McPherson, Kansas
March 21, 22—Lincoln, Nebraska
March 28, 29—St. Paul, Minnesota
April 4, 5—Cedar Rapids, Iowa
April 18, 19—Galesburg, Illinois

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-fourth summer meeting of the Mathematical Association of America will be held at the University of Chicago the week of September 1, 1941, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The Association will have two sessions on Monday and a session on Wednesday afternoon; at the latter, one of the hour lectures in the series, "Trends in Research," will be given by Professor G. A. Bliss on "The Calculus of Variations" under the auspices of the Association, and the retiring presidential address will be given by Professor W. B. Carver.

At the time of the meetings the University of Chicago will be engaged in the celebration of the fiftieth anniversary of its foundation, the theme of which has been designated by the University as "New Frontiers in Education and Research." As an important part of the expression of its interest in this celebration, the Department of Mathematics of the University has collaborated with the Society and the Association in arranging special programs which it is hoped will give their meetings unusual significance for future mathematical education and research.

The Colloquium lectures of the Society are to be delivered by Professor Oystein Ore of Yale University. By invitation of the Committee on Gibbs Lecturers, Professor Sewall Wright of the Department of Zoology of the University of Chicago will give the Josiah Willard Gibbs Lecture on "Statistical Genetics and Evolution."

The addition to the regular programs of the Society consists of a Conference on Algebra and a Conference on the Theory of Integration as well as four one-hour lectures designed for the mathematical groups as a whole rather than primarily for specialists in the fields of the lectures. The hour lectures will be given by leaders in their respective branches of mathematics on Trends in Research, and it is intended that the lectures shall be descriptive of recent progress and of the directions which it is likely that future research in the fields may take. An hour lecture on "Abstract Spaces" will be given by Professor M. H. Stone of Harvard University; another on "Analytic Number Theory," by Professor H. A. Rademacher of the University of Pennsylvania; a third on "The Calculus of Variations," by Professor G. A. Bliss of the University of Chicago; and finally one on "Topology," by Professor Solomon Lefschetz of Princeton University.

Each conference will consist of seven forty-minute addresses followed by organized discussion. The conference speakers will report on results of research in which they themselves have been particularly interested. The speakers in the Conference on Algebra will be Professors Garrett Birkhoff of Harvard University, Nathan Jacobson of the University of North Carolina, N. H. McCoy of Smith College, John von Neumann of the Institute for Advanced Study, J. F. Ritt of Columbia University, Oscar Zariski of Johns Hopkins University, and Dr. O. F. G. Schilling of the University of Chicago. The speakers in the Conference on the Theory of Integration will be Professors Garrett Birkhoff, Salomon

Bochner of Princeton University, Nelson Dunford of Yale University, T. H. Hildebrandt of the University of Michigan, R. L. Jeffery of Acadia University, G. B. Price of the University of Kansas, and Norbert Wiener of Massachusetts Institute of Technology.

Registration headquarters and accommodations for members of the Society and Association and their families and guests will be provided in the modern college dormitories of the University, Judson Court and Burton Court. A special rate of $3.00 per day for room and meals with half rate for children under 12 has been arranged. A joint dinner of the mathematical organizations will be held on Thursday evening, September 4. The usual opportunities for recreation will be provided, and excursions to various points of interest will be arranged. The University of Chicago will have on view a number of exhibits relating to various features of its work.

W. D. CAIRNS, *Secretary-Treasurer*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 2, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN Pittsburgh, May 3.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May.
MICHIGAN, Ann Arbor, March 15,
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3 or 4.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, April, 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, March 28–29.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit, May 3.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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CONTENTS

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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THE TWENTY-FIFTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-fifth annual meeting of the Mathematical Association of America was held at Baton Rouge, Louisiana, Wednesday and Thursday, January 1–2, 1941, in conjunction with the meetings of the American Mathematical Society and the National Council of Teachers of Mathematics. About five hundred twenty were in attendance at the meetings, including the following two hundred thirteen members of the Association:

E. F. ALLEN, Oklahoma A. and M. College
A. A. AUCOIN, University of Houston
H. G. AYRE, Western Illinois State Teachers College
W. L. AYRES, University of Michigan

R. W. BABCOCK, Kansas State College
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R. W. BRINK, University of Minnesota
J. C. BRIXEY, University of Oklahoma
FOSTER BROOKS, Kent State University
H. E. BUCHANAN, Tulane University
R. S. BURINGTON, Case School of Applied Science
J. E. BURNAM, Hardin-Simmons University
L. E. BUSH, College of St. Thomas
W. E. BYRNE, Virginia Military Institute

W. D. CAIRNS, Oberlin College
Sister M. PATRICIA CALLAGHAN, Fontbonne College
C. C. CAMP, University of Nebraska
MILDRED E. CARLEN, Brown University
W. B. CARVER, Cornell University
J. E. CASE, St. Louis University
E. W. CHITTENDEN, University of Iowa
ANNIE M. H. CHRISTENSEN, Louisiana State University
H. C. CHRISTOFFERSON, Miami University
L. M. COFFIN, Coe College
L. W. COHEN, University of Kentucky
NANCY COLE, Sweet Briar College
J. B. COLEMAN, University of South Carolina
E. G. H. COMFORT, University of Arkansas
H. H. CONWELL, Beloit College
LENNIE P. COPELAND, Wellesley College
W. A. CORDREY, Southeastern Louisiana College
N. A. COURT, University of Oklahoma
R. W. COWAN, University of Alabama
W. H. H. COWLES, Pratt Institute
H. M. COX, University of Nebraska
G. F. CRAMER, Tulane University
FORREST CUMMING, University of Georgia
J. C. CURRIE, Northeast Junior College of Louisiana State University
D. R. CURTISS, Northwestern University
J. H. CURTISS, Cornell University

R. D. DAUGHERTY, Kansas State College
W. M. DAVIS, Illinois Institute of Technology
D. S. DEARMAN, Mississippi Southern College
L. J. DECK, Muhlenberg College
L. L. DINES, Carnegie Institute of Technology
R. T. DONNELL, Castles Heights Military Academy
J. L. DORROH, Ouachita College
L. T. DUNLAP, Pennsylvania State College
W. L. DUREN, JR., Tulane University

E. D. EAVES, University of Tennessee
P. D. EDWARDS, Ball State Teachers College
H. J. ETTLINGER, University of Texas
G. C. EVANS, University of California

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B. F. FINKEL, Drury College
H. T. FLEDDERMANN, Loyola University
ELIZABETH FREAS, Louisiana State University
THORNTON C. FRY, Bell Telephone Laboratories
GORDON FULLER, Alabama Polytechnic Institute
H. K. FULMER, Georgia School of Technology

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R. E. GASKELL, University of Alabama
B. E. GATEWOOD, Louisiana Polytechnic Institute

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Dorothy McCoy, Belhaven College
Dora McFarland, University of Oklahoma
Sister Presentation McNelly, Our Lady of the Lake College
A. C. Maddox, Louisiana State Normal College
Anna E. Many, Sophie Newcomb College
W. A. Martin, Alabama Polytechnic Institute
Sister Mary Felice, Mount Mary College
Sister Mary of Mercy, Incarnate Word College
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B. A. TUCKER, Southeastern Louisiana College

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R. W. WAGNER, Oberlin College
R. K. WAKERLING, Texas Technological College
R. J. WALKER, Princeton University
J. A. WARD, Delta State Teachers College
L. E. WARD, University of Iowa
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H. A. WRIGHT, Transylvania College
J. T. C. WRIGHT, Appalachian State Teachers College

R. C. YATES, Louisiana State University

The mathematicians stayed in the new women's dormitories of Louisiana State University and ate in the dining room there and at the Faculty Club nearby. The lobby and parlor of Evangeline Hall, the central dormitory, afforded convenient meeting places for all and were constantly utilized. President and Mrs. P. M. Hebert gave a tea Monday afternoon, assisted by the ladies of the department of mathematics. Many participated in the excursion Tuesday afternoon to St. Francisville, where two ante-bellum plantations were visited, thanks to the arrangements generously made by the University for permission to visit these two old and interesting homes. Major J. P. Cole and his colleagues provided automobile transportation wherever needed. A number of visitors took advantage of the nearness of New Orleans to see the Sugar Bowl game Wednesday afternoon, as well as to visit New Orleans before or after the meetings.

Fully three hundred attended the joint dinner Wednesday evening. Professor F. A. Rickey was the lively toastmaster, and, after a hearty welcoming speech by President Hebert and musical numbers, speeches were made by Professors H. E. Buchanan and C. V. Newsom, and by Miss Mary A. Potter, president of the National Council. At the conclusion of the program Professor G. C. Evans exhibited an elegant silver coffee set and a handsomely illuminated tribute to the services of Dean R. G. D. Richardson on the occasion of his retirement from the secretaryship of the American Mathematical Society after twenty years of service. On the motion of Professor Winger, the company present adopted a resolution of thanks to the administration and the department of mathematics of Louisiana State University, to Professor Sanders and his colleagues for the thoroughness of preparations for our comfort, to Major Cole and his assistants for the fine arrangements for the excursion.

The American Mathematical Society held sessions on Monday afternoon and evening, Tuesday morning, and Wednesday afternoon for the reading of papers. Hour addresses were given Monday afternoon by Dr. Leo Zippin on "Topology of rigid motions" and Tuesday morning by Professor Saunders Mac Lane on "Extensions of groups." On Wednesday morning Professor G. C. Evans gave his presidential retiring address on "Surfaces of minimum capacity." Professor Marston Morse was elected president of the Society for 1941–42; and Professor J. R. Kline has been appointed secretary of the Society in succession to Dean R. G. D. Richardson.

The National Council of Teachers of Mathematics held sessions on Monday, Tuesday, and Wednesday mornings and Monday afternoon. In addition, a general meeting was held Monday evening at which, after an address of welcome by Professor S. T. Sanders and a response by President Mary A. Potter, Professor W. B. Carver spoke on "For the million or for a few?" A session was held Tuesday evening at Southern University with addresses by Miss Potter, Professor H. C. Christofferson, and Miss Ethel Harris Grubbs. The extensive program, arranged under the direction of Professor F. L. Wren, will be reported in full in the *Mathematics Teacher*. An exhibit of mathematical models and material, prepared by various high schools and colleges, showed great interest and ability on the part of many students.

The Mathematical Association held a joint session with the National Council Wednesday, and a separate session Thursday morning, the program having been arranged by the program committee under the chairmanship of Professor Marie J. Weiss. The program follows, together with abstracts of some of the papers numbered in accordance with their place on the program:

Joint Session of the Association with the National Council

1. "The teaching of mathematics in the junior college" by Dr. Virginia Modesitt, Wright Junior College, Chicago.

2. "What are the administrative and guidance uses of mathematics examinations?" by Professor H. M. Cox, University of Nebraska.

3. "The professional interests of mathematical instructors in junior colleges" by Professor D. R. CURTISS, Northwestern University.

1. The teaching of mathematics in a junior college presents many problems peculiar to the junior college, the solution of which may however have a bearing on the problems of teaching the first two years of mathematics in the four-year college.

In the first place, the needs and interests of the students in the public junior college present a wider variety than is to be found in most four-year colleges. Moreover, the junior college is so recent a development in the educational field that the courses which are offered must be justified, not on the basis of tradition, but because they are of value to the student. And, in the third place, many of our students take only one semester of mathematics. Hence, an attempt must be made to give them a more general idea of mathematics than can be given in any one of the traditional first-semester freshman mathematics courses.

For these reasons the regular freshman course taught at Wright Junior College in Chicago is so designed that at the end of one semester the student has had some introduction to the methods of all the subject-matter usually included in a traditional freshman course. This is possible through the use of function as a unifying basis for the course. The meaning of function, graph of a function, tabular representation of a function, concept of rate of change of a function are introduced in the first chapter. Succeeding chapters discuss particular types of functions, linear, quadratic, polynomial, trigonometric, exponential, implicit quadratic, polar representation, parametric representation, and functions representing surfaces. This arrangement makes it possible to present in a one-year course meeting five times a week the material which is usually presented in the three traditional freshman courses. It also makes it possible for the student at the end of one semester to have a better idea of both the subject-matter and the basic methods of modern mathematics than could be obtained from a one-semester course in algebra, for example.

There is a great deal of work yet to be done on other problems which arise in connection with the teaching of junior college mathematics. There is the problem of what to do with students who do not have enough high school mathematics to be able to take the regular course. There is the problem of developing a second-year course which will carry on the methods and concepts developed in the first year. There is the problem of evaluating the work done. At this time it can only be said that we feel that we are working in the right direction toward the solution of our problems.

2. A résumé of this paper will appear in an early issue of the MONTHLY.

3. This paper will appear in the April, 1941, issue of the MONTHLY.

SEPARATE SESSION OF THE ASSOCIATION

1. "A program for college geometry" by Professor A. W. TUCKER, Princeton University.

2. "Some aspects of differential geometry in the large" by Professor G. A. HEDLUND, University of Virginia.

3. "On groups of homeomorphisms" by DEANE MONTGOMERY, Smith College.

1. This is a program for a course in synthetic geometry which meets the following specifications:

1) has no prerequisite other than plane euclidean geometry,
2) includes some non-euclidean and projective geometry,
3) takes account of the Erlanger Programm of Felix Klein,
4) can be given, if desired, in one term (about 40 hours).

It is the outgrowth of several years' experiment in teaching a one-term sophomore course on *Modern Geometric Concepts* at Princeton University. The principal feature of the program is the systematic factorization of general circular and projective transformations into products of simple transformations: reflections, inversions, and polar reciprocations with respect to circles. Thus the study of properties preserved by the conformal and projective transformation groups is reduced to the classical "generalization" by inversion and polar reciprocation. Affine and non-euclidean geometries are developed as sub-geometries of projective and conformal geometry.

2. Classical differential geometry is to a large extent concerned with the infinitesimal properties of a surface. In differential geometry in the large the surface is considered in its entirety under the assumption that the surface satisfies some completeness property such as that of Hopf and Rinow. The simplest such surfaces are those for which the Gaussian curvature K is constant, or, as these are termed, Clifford-Klein space forms. These space forms may be realized by identification of points congruent under a properly discontinuous group of rigid motions of the sphere, the euclidean plane, or the hyperbolic plane, depending on whether $K>0$, $K=0$, or $K<0$, respectively.

In the case $K>0$, the surface is of the topological type of the sphere or of the projective plane. The geodesics are all periodic in either case.

In the case $K=0$, if the surface is closed it is of the topological type of the torus or of the Klein bottle. In either case the universal covering surface U is the euclidean plane, and in U the geodesics have the property that there is a unique geodesic passing through a given point and orthogonal to a given geodesic. Recently Professor Morse and the speaker have been able to show that the property just stated, imposed on the geodesics on a surface of the topological type of a torus or Klein bottle, is sufficient to insure that $K\equiv 0$.

In the case K a negative constant, if the surface S is closed it may be orientable of genus greater than one or non-orientable of genus greater than two. There exist transitive geodesics. In the universal covering surface of S there is a unique geodesic joining two given points. Recently Professor Morse and the speaker have been able to show that the latter property alone is sufficient to imply the existence of transitive geodesics.

3. This paper will appear in an early issue of the MONTHLY.

Meetings of the Board of Governors

Members of the outgoing and of the incoming Board met on Wednesday afternoon and Thursday noon, respectively.

The following thirty-eight persons and three institutions were elected to membership on applications duly certified:

To Institutional Membership

Kansas State Teachers College, Emporia, Kansas

The University of Tennessee, Knoxville, Tennessee

North Texas State Teachers College, Denton, Texas

To Individual Membership

J. E. Alman, A.M.(Claremont) Teacher, Lincoln School of Teachers Coll., Columbia Univ., New York, N. Y.

J. D. Bankier, M.A.(Queen's Univ.) Fellow, Rice Inst., Houston, Texas

J. K. Baumgart, A.M.(Michigan) Instr., Cumberland Coll., Williamsburg, Ky.

W. J. Combellack, A.M.(Colby) Instr., Northeastern Univ., Boston, Mass.

A. B. Cunningham, Ph.D.(West Virginia) Instr., Undergrad. Centers, Pennsylvania State Coll., Dubois, Pa.

H. P. Eichert, A.B.(New York Univ.) Engineering asst., Consolidated Edison Co. of New York, Brooklyn, N. Y.

J. R. Ellis, M.S.(New Mexico) Instr., Booker T. Washington High School, Tulsa, Okla.

E. D. Hellinger, Ph.D.(Göttingen) Lecturer, Northwestern Univ., Evanston, Ill.

J. F. Heyda, Ph.D.(Illinois) Instr., Michigan State Coll., East Lansing, Mich.

T. J. Higgins, A.M.(Cornell) Instr., Elec. Eng. Dept., Purdue Univ., West Lafayette, Ind.

Harriet Howard, A.M.(Boston Univ.) Dir., Women's Residence Halls, New York State Coll. for Teachers, Albany, N. Y.

E. L. Kaplan. Student, Carnegie Inst. of Tech., Pittsburgh, Pa.

L. M. Kelly, A.M.(Boston Univ.) Instr., Univ. of Missouri, Columbia, Mo.

C. H. Kendall, A.B.(Union Coll.), LL.B.(Buffalo) Lecturer, Univ. of Buffalo, Buffalo, N. Y.

R. B. Kleinschmidt, M.S. in C.E.(Pennsylvania) Instr., Civil Eng., Undergrad. Centers, Pennsylvania State Coll., Pottsville, Pa.

E. R. Kolchin, A.B.(Columbia) Lecturer, Barnard Coll., Columbia Univ., New York, N. Y.

O. E. Lancaster, Ph.D.(Harvard) Asst. Prof., Univ. of Maryland, College Park, Md.

A. T. Lonseth, Ph.D.(California) Instr., Iowa State Coll., Ames, Iowa

Dorothy Manning, Ph.D.(Stanford) Instr., Wells Coll., Aurora, N. Y.

Ralph Mansfield, M.S.(Chicago) Instr., Chicago Teachers Coll., Chicago, Ill.

J. J. McGlade, A.M.(Montclair State T.C.) Instr., Coll. of Paterson, Paterson, N. J.

R. J. Michel, Ph.D.(Missouri) Head of Dept., Southeast Missouri State Teachers Coll., Cape Girardeau, Mo.

M. G. Moore, Ph.D.(Illinois) Instr., Tri-State Coll., Angola, Ind.

Jenaro Moreno, Prof., Inst. Pedagogico, Santiago de Chile, Chile

G. L. Paine, M.S.(Oklahoma) Capt., Head of Dept., Oklahoma Milit. Acad., Claremore, Okla.

R. J. Pitts, A.M.(Michigan) Instr., Fort Valley State Coll., Fort Valley, Ga.

Ruth E. Porter, A.M.(Oregon) Teacher, Math. and Physics, Senior High School, Albany, Ore.

Moses Richardson, Ph.D.(Columbia) Instr., Brooklyn Coll., Brooklyn, N. Y.

Arthur Rosenthal, Ph.D.(Munich) Research fellow and lecturer, Univ. of Michigan, Ann Arbor, Mich.

N. E. Rutt, Ph.D.(Pennsylvania) Prof., Louisiana State Univ., University, La.

G. A. Sedlak, Instr., Cudahy Voc. School, Cudahy, Wis.

Andrew Sobczyk, Ph.D.(Princeton) Instr., Oregon State Coll., Corvallis, Ore.

C. F. Stephens, M.S.(Michigan) Instr., Prairie View State Coll., Prairie View, Texas

B. M. Stewart, Ph.D.(Wisconsin) Instr., Michigan State Coll., East Lansing, Mich.

G. A. WHETSTONE, Ph.D.(Washington) Instr., Amarillo Coll., Amarillo, Texas
LOUISE A. WOLF, Ph.D.(Wisconsin) Asst. Prof., Univ. of Wisconsin Exten. Div., Milwaukee, Wis.
JACK WOLFE, Ph.D.(New York Univ.) Instr., Brooklyn Coll., Brooklyn, N. Y.
C. T. YORK, A.M.(Texas) Instr., West Texas State Teachers Coll., Canyon, Texas

The Secretary announced the names of the Regional Governors, elected by the membership of various regions for a two-year term:

Region 3 (Delaware, New Jersey, Pennsylvania, and West Virginia), F. W. OWENS;

Region 4 (District of Columbia, Maryland, and Virginia), E. J. MCSHANE;

Region 6 (Arkansas, Louisiana, and Mississippi), S. T. SANDERS;

Region 8 (Illinois, Indiana, and Michigan), W. C. KRATHWOHL;

Region 9 (Iowa, Minnesota, and Wisconsin), CORNELIUS GOUWENS;

Region 11 (Oklahoma and Texas), H. J. ETTLINGER;

Region 14 (California and Nevada), H. M. BACON.

The financial report for the year 1940 was presented and accepted. It had been previously examined by President Carver for the Finance Committee and by Professor Brink.

T. C. Fry was nominated as the representative of the Association on the National Research Council for a three-year term beginning July 1, 1941, in succession to F. D. Murnaghan.

The invitations of Lehigh University to meet there next December, and of the University of Colorado for the summer of 1943, were accepted with thanks.

It is a pleasure to announce that Professor L. R. FORD was elected Editor-in-Chief of the MONTHLY for a five-year term beginning with the issue for January 1942.

At the recommendation of the Executive Committee, the Board (1) appropriated fifty dollars for expenses of the War Preparedness Committee in 1941; (2) referred to a committee the provision of money for the Chauvenet Prize, since the prevailing rate of interest does not provide sufficient funds for an award every third year; (3) recommended to Section officers that suitable attention be paid to including on Section programs papers attractive to teachers of first- and second-year college mathematics; (4) voted to appoint a conference committee of the Association which shall be ready to confer and to coöperate with similar committees from other national bodies in conferences on education; (5) voted to approve the establishment, with the aid of Association funds, of Herbert Ellsworth Slaught Memorial Papers, expository papers of forty pages or more, probably as supplements to the MONTHLY. The Board also considered measures appropriate to junior college interests.

The Board elected B. H. BROWN, Dartmouth College, Second Vice-President for the two-year term 1941–1942.

The following associate editors of the MONTHLY were elected for the year 1941, as recommended by Professor Moulton. Professor Langer is to be in charge of the Slaught Memorial Papers.

W. B. Carver	M. R. Hestenes	R. G. Sanger
H. S. M. Coxeter	E. H. C. Hildebrandt	D. E. Smith
Otto Dunkel	C. A. Hutchinson	Virgil Snyder
B. F. Finkel	R. E. Langer	R. J. Walker
L. R. Ford	J. R. Musselman	Marie J. Weiss
Orrin Frink, Jr.		M. E. Wescott

Annual Business Meeting

The annual business meeting and election of officers was held Thursday morning, January 2, 1941. The Secretary announced the names of those who had been elected to membership at the meeting of the Board. He reports here the deaths of the following members:

P. E. Baur, Associate professor of mathematics and drawing, Baldwin-Wallace College. (December 7, 1939)

J. M. Colaw, Attorney at law, Monterey, Virginia. (February 26, 1940)

Col. C. P. Echols, Professor of mathematics, U. S. Military Academy. (May 21, 1940)

Annie L. M. Fitch (Mrs. Edward), Clinton, New York. (September 12, 1940)

W. W. Hartshorn, Teacher, Union High School and Junior College, Brawley, California. (June 6, 1940)

Charles Hopkins, Instructor in mathematics, Tulane University. (September 15, 1939)

G. A. Knapp, Professor of mathematics, Maryville College. (November 4, 1940)

C. A. Lindemann, Professor of pure mathematics, Bucknell University. (April 28, 1940)

Maj. T. E. Naish, Royal Engineers, retired, Penticton, Lake Okanagan, Canada. (August 10, 1939)

Beulah Russell, Associate professor of mathematics, College of William and Mary. (February 23, 1940)

George Rutledge, Professor of mathematics, Massachusetts Institute of Technology. (September 21, 1940)

A. W. Smith, Professor of mathematics, Colgate University. (February 11, 1940)

G. H. Taber, Vice-President, retired, Gulf Oil Corporation, Pittsburgh, Pennsylvania. (December 10, 1940)

H. G. Titt, Professor of mathematics, Huron College. (August 21, 1940)

F. N. Willson, Professor emeritus of mathematics, Princeton University. (November 15, 1939)

C. R. Wilson, Assistant professor of mathematics, Rutgers University. (July 24, 1940)

The results of the election of officers were as follows:

President for 1941–1942: R. W. Brink, University of Minnesota.

First Vice-President for the partial term through 1941: R. E. Langer, University of Wisconsin.

Report of the Secretary-Treasurer as Treasurer, December 12, 1940

Receipts

Balance Dec. 12, 1939		$6,862.65
1939 indiv. dues	$ 382.40	
1940 indiv. dues	7,707.65	
1940 instit. dues	617.40	
1940 subscriptions	1,001.71	
Initiation fees	246.00	
Advertising	590.50	

Expenditures

Publisher's bills (Oct. '39–Oct. '40)	$ 6,450.09
Reprints	368.29
President's office	11.80
Editor-in-chief's office	847.59
Printing *Register*	493.48
Committee on tests	29.30
Committee on membership	148.43

RECEIPTS (continued)

Reprints	281.49	
Sale copies of MONTHLY	377.46	
Sale First Carus Mon.	16.25	
Sale Second Carus Mon.	15.00	
Sale Third Carus Mon.	21.25	
Sale Fourth Carus Mon.	16.25	
Sale Fifth Carus Mon.	50.32	
Sale Archibald's Outline of Hist. of Math.	192.40	
Annals subscriptions	5.00	
Duke Journal subscriptions	6.00	
Math. Reviews subscriptions	268.98	
Sale Rhind Papyrus	103.10	
Drury Coll. int. Hardy Fund	120.00	
Refund expense Joint Commission	23.75	
Refund ins. back copies MONTHLY	6.17	
Sale Bethlehem Steel Bonds	2,098.50	
Int. Genl. End. Fund	594.31	
Int. Carus Fund	148.75	
Int. Chace Fund	201.49	
Int. Chauvenet Fund	15.00	
Int. Houck Fund	68.75	
Int. current funds	137.74	15,313.62
Total 1940 receipts to date		$22,176.27
Total expenditures		16,850.93
Balance to end of 1940 business		$ 5,325.34
Received on 1941 business		850.75
Book Balance Dec. 12, 1940		$ 6,176.09

EXPENDITURES (continued)

Expense Joint Commission		47.50
Secretary-Treasurer's office		
Postage	$ 447.09	
Bond	11.26	
Office expense	126.26	
Express, tel., *etc.*	75.97	
Clerical work	2,452.39	
Printing	172.74	
Bank charge	34.62	3,320.33
Annals subvention		200.00
Duke Journal subvention		250.00
Math. Reviews subvention		500.00
Expense of sections from init. fees		128.08
Columbus meeting		185.00
Dartmouth meeting		87.30
Paid *Annals* subscriptions		10.00
Forwarded *Annals* subscriptions		5.00
Forwarded *Duke Journal* subscriptions		6.00
Forwarded *Math. Reviews* subscriptions		255.98
Expense acct. *Math. Reviews*		16.40
Sust. memb. in Amer. Math. Soc.		100.00
Refund subscriptions		6.75
Insurance back copies MONTHLY		20.90
Storage back copies MONTHLY		30.00
Paid back copies MONTHLY		35.40
Paid B. F. Finkel int. Hardy Fund		120.00
Library expense		21.94
Transfer to Carus Fund		243.75
Transfer to Chace Fund		261.82
Transfer to Houck Fund		100.43
Expense Carus Fund		121.26
Paid *Eudemus* int. from Chace Fund		227.22
Paid for Columbus and Southern Ohio Electric Bonds		2,181.00
Paid accrued int. on this		16.61
Canadian periodical tax		3.28
Total expenditures		$16,850.93
Checking account		38.62
Oberlin Savgs. Bk. acct. restricted		666.40
Peoples Banking Co. savgs. acct.		1,799.60
Cleveland Trust Co. savgs. acct.		1,671.47
Youngstown Sheet & Tube Co. 4% First Mort. Bds. Ser. C 1961		2,000.00
Bank balance Dec. 12, 1940		$ 6,176.09

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance December 12, 1939		$7,181.54
Receipts: Sales	$ 119.07	
Interest	193.69	312.76
		$7,494.30
Expense acct. Carus Fund	$ 121.26	
Cost above par, Firestone Bond	52.36	173.62
		$7,320.68
Certificate of deposit	$1,640.26	
C. & O. $3\frac{1}{2}$% Refunding Mortgage Bonds Series D, 1996	2,000.00	
U. S. Treasury $3\frac{1}{8}$% Bond of 1946–49	1,000.00	
HOLC 3% Bond 1944–52	1,000.00	
U. S. Savings Bonds	150.00	
Firestone Tire & Rubber Co. $3\frac{1}{2}$% Deb., 1948	1,000.00	
Cash in bank, restricted, certificate of participation	397.60	
Cash in bank, unrestricted	132.82	
Balance December 12, 1940		$7,320.68

Arnold Buffum Chace Fund

Balance December 12, 1939		$8,060.45
Receipts: Sale Papyrus	$ 103.10	
Interest	215.71	318.81
		$8,379.26
Expenditures: Paid *Eudemus* int. from Chace Fund	$ 227.22	
Cost above par, N. Y. Steam Corp. Bond	63.75	290.97
		$8,088.29
U. S. Treasury $3\frac{1}{8}$% Bonds 1946–49	$2,000.00	
HOLC 3% Bond 1944–52	1,300.00	
U. S. Savings Bonds	1,125.00	
Montana Power Co. $3\frac{3}{4}$% First Mortgage Bonds 1966	1,000.00	
North American Co. 4% Debenture Bond 1959	1,000.00	
One half Shawinigan W. & P. Co. $4\frac{1}{2}$% First Mortgage Bond 1970	500.00	
N. Y. Steam Corp. $3\frac{1}{2}$% First Mortgage 1963	1,000.00	
Cash in bank, restricted, certificate of participation	17.60	
Cash in bank, unrestricted	145.69	
Balance December 12, 1940		$8,088.29

Jacob Houck Memorial Fund

Balance December 12, 1939		$7,686.10
Receipts: Interest		242.84
		$7,928.94
U. S. Treasury Bonds	$4,000.00	
$1\frac{1}{2}$ Shawinigan W. &. P. Co. $4\frac{1}{2}$% First Mortgage Bonds 1970	1,500.00	
Gatineau Power Co. $3\frac{3}{4}$% First Mortgage Bond 1969	1,000.00	
Peoples Banking Co. savings acct.	1,428.94	
Balance December 12, 1940		$7,928.94

CHAUVENET PRIZE FUND

Balance December 12, 1939		$ 602.94
Interest		15.00
		$ 617.94
HOLC 3% Bond 1944–52	$ 500.00	
Cash in bank	117.94	
Balance December 12, 1940		$ 617.94

LIFE MEMBERSHIP FUND

Liability on life memberships as of January 1, 1940	$ 892.10
To be transferred to current funds, surplus	24.71
Liability on life memberships as of January 1, 1941	$ 867.39

GENERAL ENDOWMENT FUND

Balance December 12, 1939		$18,200.00
U. S. Treasury $3\frac{1}{4}$% Bonds 1944–46	$1,000.00	
U. S. Treasury $3\frac{1}{4}$% Bonds 1943–45	1,000.00	
HOLC 3% Bonds 1944–52	5,500.00	
Land Trust Certificate, Hotel Cleveland Site	700.00	
Montana Power Co. $3\frac{3}{4}$% First Mortgage Bonds 1966	2,000.00	
Texas Power & Light Co. 5% First Mortgage Bond 1956	1,000.00	
C. & O. $3\frac{1}{2}$% Refunding Mortgage Bond Series C 1996	1,000.00	
Pennsylvania R. R. Co. $3\frac{3}{4}$% Genl. Mortgage Bonds Series C 1970	2,000.00	
Cols. & So. Ohio Elec. $3\frac{1}{4}$% First Mortgage Bonds 1970	2,000.00	
Oberlin Savings Bank savings account	2,000.00	
Balance December 12, 1940		$18,200.00

Of the funds on hand, indicated in the first division of this financial report, $132.82 belongs to the Carus Monograph Fund, $145.69 to the Arnold Buffum Chace Fund, $117.94 to the Chauvenet Prize Fund, while $867.39 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date of January 1, 1941.

When the accounts were closed December 12, 1940, there remained on the total business for 1940 the following items:

BILLS RECEIVABLE

1940 individual dues	$200.00
Advertising	80.00
Reprints	30.00
	$310.00

BILLS PAYABLE

Publisher's bills (Nov., Dec. '40)	$1,200.00
Editor-in-chief's office	130.00
Committee on membership	50.00
Secretary-Treasurer's office	270.00
Subsidy *Duke Journal*	50.00
Subsidy *Math. Reviews*	500.00
War Preparedness Committee	50.00
Carus Monograph Fund	132.82
Chace Fund	145.69
Chauvenet Prize Fund	117.94
Life membership fund	867.39
Init. fees due to sections	990.00
Math. Reviews subscriptions	13.00
	$4,516.84

If to the balance on 1940 business shown in this report, \$5,325.34, there be added the estimated bills receivable, \$310.00, and there be subtracted the estimated bills payable, \$4,516.84, there results an estimated final balance at the close of 1940 business of approximately \$1,100. This represents an estimate of such accumulated savings of the Association as have not been transferred to general endowment. The corresponding estimate of a year ago was \$920.

W. D. CAIRNS, *Secretary-Treasurer*

AN ANALYTICAL STUDY OF PERSPECTIVE DRAWINGS OF QUADRIC SURFACES

NEIL LITTLE, Purdue University

1. Introduction. Consider any curve λ in space and any conical surface whose apex is at some point P not on λ and each of whose elements passes through some point on λ. If this conical surface is intersected by a plane π, the drawing of the resulting curve of intersection is known as a perspective drawing of λ. If a cylindrical surface is used instead of a conical surface, the drawing is called non-perspective, being orthographic if the elements of the cylinder are perpendicular to the plane π and oblique otherwise. These latter were discussed by the author in a previous paper,* and also by Roever in a subsequent paper.†

In this paper, P will be taken as being the point p units from the origin on the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \tag{1}$$

If l, m, and n are direction cosines, the coördinates of P are (pl, pm, pn). Two positions of the plane π will be considered. First, π will be taken as a vertical plane through the origin perpendicular to the horizontal line drawn from P to the z-axis. Drawings with π in this position will be called *vertical perspective* drawings. Second, π will be taken as the plane through the origin normal to the line (1). Such drawings will be called *normal perspective* drawings.

The following transformation, used in both the papers referred to above, will again be useful:

$$\begin{aligned} x &= \frac{Xln - Ym + Zl\sqrt{l^2+m^2}}{\sqrt{l^2+m^2}}, \\ y &= \frac{Xmn + Yl + Zm\sqrt{l^2+m^2}}{\sqrt{l^2+m^2}}, \\ z &= Zn - X\sqrt{l^2+m^2}. \end{aligned} \tag{A}$$

* An analytic study of the non-perspective picturization of quadric surfaces, this MONTHLY, vol. 44, p. 292.

† W. H. Roever, Analytic treatment of perspection with bearing on picturization, this MONTHLY, vol. 45, p. 278.

By this transformation the line (1) becomes the Z-axis, the Y-axis remains in the original xy-plane, and the X-axis is in the plane determined by the line (1) and the original z-axis (Fig. 1). From the point P, the X- and Y-axes will appear

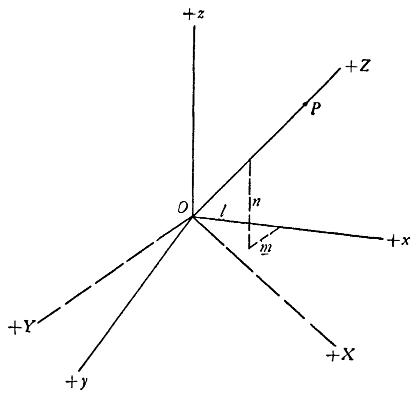

FIG. 1

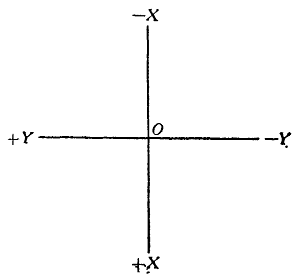

FIG. 2

as in Figure 2. The perspective drawings of curves in space will be found by the use of these lines as coördinate axes. We shall refer to the perspective drawing of any point or line in space as the *representative* of that point or line.

2. Vertical perspective drawings. The equation of π is $my+lx=0$. If the values of x and y as given in the equations of transformation (A) are substituted in this equation, the result may be written

$$Z = -\frac{nX}{\sqrt{l^2+m^2}}.$$

If this value of Z is substituted back into equations (A), the results are

$$\text{(B)} \qquad x = \frac{-Ym}{\sqrt{l^2+m^2}}, \qquad y = \frac{Yl}{\sqrt{l^2+m^2}}, \qquad z = -\frac{X}{\sqrt{l^2+m^2}}.$$

2.1. Representative of point (α, β, γ). Equations of the line through (α, β, γ) and $P(pl, pm, pn)$ are

$$\text{(2)} \qquad \frac{x-\alpha}{x-pl} = \frac{y-\beta}{y-pm} = \frac{z-\gamma}{z-pn}.$$

Applying transformation (B), two independent equations in X and Y are obtained. Their solution is

$$\text{(3)} \qquad \begin{aligned} X &= \frac{p\sqrt{l^2+m^2}\,[n(\alpha l+\beta m) - \gamma(l^2+m^2)]}{p(l^2+m^2) - (\alpha l+\beta m)}, \\ Y &= \frac{p\sqrt{l^2+m^2}\,(\beta l - \alpha m)}{p(l^2+m^2) - (\alpha l+\beta m)}. \end{aligned}$$

These are the coördinates of the representative of the point (α, β, γ).

These equations enable one to compute the coördinates of the representative of any given point in space, but not conversely. For a complete exposition of this problem in non-perspective drawing, with an obvious extension to the case at hand, see the paper by Roever referred to above.

2.2. Representatives of the coördinate axes. The representative of any line is the intersection of π with the plane containing P and that line. The equation of the plane containing P and the x-axis is $ny-mz=0$. Making transformation (B) and simplifying gives

$$mX + nlY = 0. \tag{4}$$

This is the equation of the representative of the x-axis. In similar fashion, the equations of representatives of the y- and z-axes are, respectively,

$$lX - mnY = 0, \tag{5}$$

and

$$Y = 0. \tag{6}$$

The representatives of the coördinate axes then appear as in Figure 3. This is identical to their appearance in an orthographic drawing.

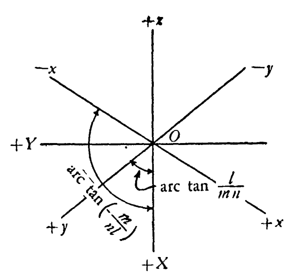

FIG. 3

2.3. Representatives of curves in space. If a given curve is the trace of a surface in a plane parallel to a coördinate plane, the curve may be represented by two equations one of which contains two variables, the other only one. If the equation containing two variables can be solved for one of them in terms of the other, then equations (3) may be used to express X and Y in terms of some one of the variables x, y, or z. Such equations would then be parametric equations of the curve in question, x, y, or z being the parameter. In particular, parametric equations of the trace of any quadric surface in a plane parallel to a coördinate plane can thus be found.

If a curve in space is the intersection of two quadric surfaces, parametric equations of the representative of the curve can be found from the equations of two of the projecting cylinders of the curve, by use of the equations (3) as before.

The drawing of a quadric surface in perspective can then be done by determining parametric equations of whatever traces one chooses and then drawing the representatives of those traces.

2.4. Curves of visibility. As in oblique and orthographic drawings, curves of visibility are needed. The curve of visibility of a particular surface is the intersection of the plane π with the enveloping cone of the surface, apex at P. The equation of the enveloping cone of the quadric surface $F(x, y, z)=0$ with apex at (α, β, γ) is*

$$\begin{aligned}&4F(\alpha, \beta, \gamma)F(x, y, z)\\ (7)\qquad &= \left[(x-\alpha)\frac{\partial F}{\partial \alpha} + (y-\beta)\frac{\partial F}{\partial \beta} + (z-\gamma)\frac{\partial F}{\partial \gamma} + 2F(\alpha, \beta, \gamma)\right]^2.\end{aligned}$$

The equation of the curve of visibility of any quadric surface can be found by making the transformation (B) on equation (7). The general equation is very long and cumbersome and will not be given here. The curve of visibility of $ax^2+by^2+cz^2=1$ is

$$\begin{aligned}&(ap^2l^2 + bp^2m^2 - 1)cX^2 + 2p^2lmnc(b-a)XY\\ (8)\qquad &+ \left[abp^2(l^2+m^2)^2 + (am^2+bl^2)(cp^2n^2-1)\right]Y^2 - 2cpn\sqrt{l^2+m^2}\,X\\ &+ 2(b-a)plm\sqrt{l^2+m^2}\,Y - (l^2+m^2)(ap^2l^2 + bp^2m^2 + cp^2n^2) = 0,\end{aligned}$$

and that of $ax^2+by^2+z=0$ is

$$\begin{aligned}&X^2 + 4plm(a-b)XY - 4\left[abp^2(l^2+m^2)^2 + pn(am^2+bl^2)\right]Y^2\\ (9)\qquad &+ 2p\sqrt{l^2+m^2}\,(n + 2apl^2 + 2bpm^2)X - 4p^2lmn(a-b)\sqrt{l^2+m^2}\,Y\\ &+ p^2n^2(l^2+m^2) = 0.\end{aligned}$$

2.5. Vanishing points. The equation

$$(10)\qquad pny + (a-pm)z - apn = 0$$

is the equation of any plane through P parallel to the x-axis, a being a constant. Every line in space parallel to the x-axis lies in some one such plane. The intersections of such planes with π are the representatives of lines parallel to the x-axis. Subjecting (10) to transformation (B) gives

$$(11)\qquad plnY - (a-pm)X - apn\sqrt{l^2+m^2} = 0.$$

If (11) is solved simultaneously with the equation of the representative of the x-axis, the point of intersection is found to be $\left(-pn\sqrt{l^2+m^2},\ pm\sqrt{l^2+m^2}/l\right)$. Since this result is independent of a, it follows that the representatives of all lines in space parallel to the x-axis meet at the point found. In similar fashion it can be shown that the representatives of lines in space parallel to the y-axis intersect at $\left(-pn\sqrt{l^2+m^2},\ -pl\sqrt{l^2+m^2}/m\right)$. Representatives of lines in space parallel to the z-axis are themselves parallel, so have no point of intersection. Such points of intersection of the described sets of lines are called *vanishing points*. To the draftsman, making a perspective drawing by mechanical means, their location is of prime importance.

* *Cf.*, J. R. T. Bell, Coördinate Geometry of Three Dimensions, 1912, art. 136.

3. Normal perspective drawings. The equation of π is now $Z=0$. Substituting $Z=0$ in the equations of transformation (A) gives

$$\text{(C)}\qquad x=\frac{Xnl-Ym}{\sqrt{l^2+m^2}},\qquad y=\frac{Xmn+Yl}{\sqrt{l^2+m^2}},\qquad z=-X\sqrt{l^2+m^2}.$$

The procedure from this point on is exactly that used in the previous case except that transformation (C) is used instead of (B). Hence we shall list hereafter simply the results obtained.

3.1. Representative of the point (α, β, γ).

$$\text{(12)}\quad X=\frac{p[\gamma(l^2+m^2)-n(\alpha l+\beta m)]}{(\alpha l+\beta m+\gamma n-p)\sqrt{l^2+m^2}},\qquad Y=\frac{p(\alpha m-\beta l)}{(\alpha l+\beta m+\gamma n-p)\sqrt{l^2+m^2}}.$$

3.2. Representatives of coördinate axes. Same results as in 2.2.

3.3. Representatives of curves in space. Same as 2.3.

3.4. Curves of visibility. The curve of visibility of $ax^2+by^2+cz^2=1$ is found to have the equation

$$\text{(13)}\quad \begin{aligned}&[(al^2+bm^2)(cp^2-n^2)-c(l^2+m^2)^2]X^2-2lmn(a-b)(cp^2-1)XY\\&+[abp^2(l^2+m^2)^2+(am^2+bl^2)(cp^2n^2-1)]Y^2+2plm(b-a)\sqrt{l^2+m^2}\,Y\\&+2pn[al^2+bm^2-c(l^2+m^2)]\sqrt{l^2+m^2}\,X\\&-p^2(al^2+bm^2+cn^2)(l^2+m^2)=0,\end{aligned}$$

and that of $ax^2+by^2+z=0$ is

$$\text{(14)}\quad \begin{aligned}&[4pn(al^2+bm^2)-(l^2+m^2)^2]X^2+4plm(b-a)(1+n^2)XY\\&+4[abp^2(l^2+m^2)^2+pn(am^2+bl^2)]Y^2+4p^2lmn(a-b)\sqrt{l^2+m^2}\,Y\\&-[4p^2(al^2+bm^2)+2pn(l^2+m^2)]\sqrt{l^2+m^2}\,X-p^2n^2(l^2+m^2)=0.\end{aligned}$$

3.5. Vanishing points. Representatives of lines parallel to the x-axis meet at $(-pn/\sqrt{l^2+m^2},\ pm/l\sqrt{l^2+m^2})$, those of lines parallel to the y-axis meet at $(-pn/\sqrt{l^2+m^2},\ -pl/m\sqrt{l^2+m^2})$, and those of lines parallel to the z-axis at $(p\sqrt{l^2+m^2}/n,\ 0)$. Note that in this case the representatives of lines parallel to the z-axis meet in a point instead of being parallel. It is interesting to note that the distances of the vanishing points from the origin are, respectively, $p\sqrt{m^2+n^2}/l$, $p\sqrt{l^2+n^2}/m$, $p\sqrt{l^2+m^2}/n$.

4. The plates. To afford comparison and contrast, three drawings of the same surface are shown in the plates, one vertical perspective, one normal perspective, and the third isometric. The equation of the surface is $x^2/9-y^2/4=z$. In all the drawings $l=m=n=1/\sqrt{3}$, and in the first two $p=40\sqrt{3}$. The traces shown are those in the coördinate planes and in the planes $z=-9$ and $x=\pm 12$. Curves of visibility are drawn in all cases.

5. Stereoscopic drawings. The author has been experimenting with pairs of drawings to be used in a stereoscope to produce an impression of depth. One drawing with $l=m=n=1/\sqrt{3}$ and another with $l=0.608$, $m=0.545$, and $n=0.577$ have been used together with fair success, but the experiment is far from completion.

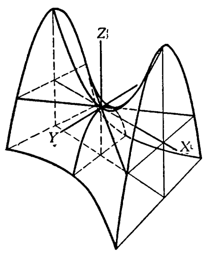

PLATE I. Vertical perspective.

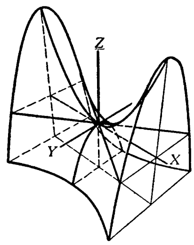

PLATE II. Normal perspective.

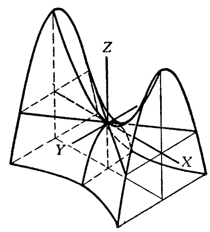

PLATE III. Isometric.

REMARKS ON SOME GENERALIZATIONS OF CAUCHY'S CONDENSATION AND INTEGRAL TESTS

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1. This paper is a collection of theorems which may be placed in the same context as the two in Hukam Chand's article, *On some generalizations of Cauchy's condensation and integral tests*, this MONTHLY, vol. 46, 1939, pp. 338–341. The two theorems in question are contained in de la Vallée-Poussin's *Cours d'Analyse* [8].

THEOREM 1. *If* $d_n>0$, $(n=1, 2, 3, \cdots)$, $D_n=\sum_{\nu=1}^{n} d_\nu \to \infty$ *with* n *and if* $F(x)$ *is positive monotone decreasing, then* $\sum^\infty d_n F(D_n)$ *converges with* $\int^\infty F(x)dx$ *and* $\sum^\infty d_n F(D_{n-1})$ *diverges with* $\int^\infty F(x)dx$.

It is easy to deduce from this result the following:

THEOREM 1a. *If, in Theorem* 1, *either* d_n [5], *or* d_{n+1}/d_n, *or* D_{n+1}/D_n *is bounded, then* $\sum^\infty d_n F(D_n)$ *converges or diverges with* $\int^\infty F(x)dx$.

2. I have obtained elsewhere [6] certain theorems which may be regarded as companions to the above. One of them reads, after a slight modification:

THEOREM 2. *If* d_n *(defined in Theorem* 1) *is bounded and if* $f(x)$ *has a continuous derivative* $f'(x)$ *such that* $\int^\infty |f'(x)|dx$ *is convergent, then*

$$\sum^\infty d_{n+1} \exp\left(\sum^{\nu=n} d_\nu f(D_\nu)\right) \textit{ converges or diverges with } \int^\infty \exp\left(\int^x f(t)dt\right)dx.$$

When $d_n=1$, this theorem leads to certain integral tests of R. W. Brink [1] for the convergence of series of positive terms. In conjunction with Theorem 1a, it also suggests forms sufficiently general to include all the familiar convergence tests for such series [6]. These forms are merely indicated here.

Theorem 1a shows that $\sum a_n$, $(a_n>0)$, is convergent [or divergent] provided

$$\frac{a_n}{d_n} \leqq F(D_n), \quad [\text{or}, \ \geqq F(D_n)], \tag{1}$$
$$\int^{\infty} F(x)dx \text{ is convergent [or divergent].}$$

Theorem 2 establishes the convergence [or divergence] of $\sum a_n$ when

$$\frac{a_{n+1}}{a_n} \leqq \frac{d_{n+1}}{d_n} \exp\,(d_n f(D_n)) \quad \left[\text{or}, \ \geqq \frac{d_{n+1}}{d_n} \exp\,(d_n f(D_n))\right], \tag{2}$$
$$\int^{\infty} \exp\left(\int^{x} f(t)dt\right)dx \text{ is convergent [or divergent].}$$

An alternative form of (2) is

$$\frac{a_{n+1}/a_n}{d_{n+1}/d_n} - 1 \leqq d_n f(D_n) \quad \left[\text{or}, \ 1 - \frac{d_{n+1}/d_n}{a_{n+1}/a_n} \geqq d_n f(D_n)\right], \tag{2a}$$
$$\int^{\infty} \exp\left(\int^{x} f(t)dt\right)dx \text{ is convergent [or divergent].}$$

3. The following [**7**] is a generalization of one part of Theorem 1a.

THEOREM 3. *If d_n (in Theorem 1) is bounded and if $F(x)$ is any function with a continuous derivative $F'(x)$ such that $\int^{\infty}|F'(x)|dx$ is convergent, then*

$$\lim_{n\to\infty}\left[\sum_{\nu=1}^{n} d_\nu F(D_\nu) - \int_{D_1}^{D_n} F(x)dx\right]$$

is finite.

Further, if $\lim_{x\to\infty} F(x)=0$, the convergence of $\int^{\infty}F(x)dx$ is necessary and sufficient for that of $\sum^{\infty} d_n F(D_n)$.

The second half of the theorem reduces to Hardy's generalization [**3**] of the Maclaurin-Cauchy integral test, when $d_n=1$. The first half gives, when $F(x)=x^{-\mu}$, the following:

COROLLARY. *The series $\sum d_n/D_n^{\mu}$ converges when $\Re\mu>1$; diverges (in the sense that the sequence of moduli of the partial sums diverges) when $0<\Re\mu<1$, the sum to n terms being $\sim D_n^{1-\mu}/(1-\mu)$; oscillates finitely when $\Re\mu=1$.*

4. We can restate (1) and (2) or (2a) as sufficient conditions for the absolute convergence of a complex series $\sum a_n$. For instance, Kummer's test for the absolute convergence of $\sum a_n$ may be given the form [**2**]

$$\left|\frac{a_n}{a_{n+1}}\right| - \frac{d_n}{d_{n+1}} \geqq \rho d_n > 0. \tag{2'a}$$

A similar ratio-test for the *non-absolute* convergence of a complex series $\sum a_n$ is embodied in the theorem which follows. When $d_n=1$, this theorem reduces to Weierstrass's generalization [**2, 4**] of Gauss's test for the convergence of positive series.

THEOREM 4. *If* $\sum d_n$ *is a divergent series of positive terms with* $d_n = O(1)$ *and if* $\sum a_n$ *is any series of complex terms satisfying*

$$\frac{a_n}{a_{n+1}} - \frac{d_n}{d_{n+1}} = \mu \frac{d_n}{D_n} + O\left(\frac{d_n}{D_n^\lambda}\right), \qquad (\lambda > 1), \tag{2'b}$$

then

(i) *when* $\Re\mu > 1$, $\sum a_n$ *is absolutely convergent*;
(ii) *when* $0 < \Re\mu < 1$, $\sum a_n$ *is divergent in the sense already explained*;
(iii) *when* $\Re\mu = 1$, $\sum a_n$ *is finitely oscillating.*

The proof of the theorem makes use of two well known results quoted below as lemmas.

LEMMA 1. *If* (ϵ_n) *is any null sequence and* (α_n) *is such that*

$$|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \leqq K|\alpha_1 + \alpha_2 + \cdots + \alpha_n| \to \infty \quad as \ n \to \infty,$$

then

$$\frac{\alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \cdots + \alpha_n\epsilon_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \to 0 \quad as \ n \to \infty.$$

This result, due to Jensen, comes out as a particular case of a limit theorem of Toeplitz [**4**].

LEMMA 2. *If* $\sum b_n$ *is a finitely oscillating series and if* (v_n) *is a sequence convergent to a finite non-zero limit and such that* $\sum(v_n - v_{n+1})$ *is absolutely convergent, then* $\sum b_n v_n$ *is also finitely oscillating.*

This is an immediate consequence of Abel's partial summation lemma for complex series [**2**].

Proof of Theorem 4. Condition (2'b) gives

$$\frac{a_n/d_n}{a_{n+1}/d_{n+1}} = 1 + \mu \frac{d_{n+1}}{D_n} + O\left(\frac{d_{n+1}}{D_{n+1}^\lambda}\right).$$

Now

$$\frac{D_n^\mu}{D_{n+1}^\mu} = \left(1 + \frac{d_{n+1}}{D_n}\right)^{-\mu} = 1 - \mu \frac{d_{n+1}}{D_n} + O\left(\frac{d_{n+1}}{D_{n+1}^2}\right).$$

Hence

$$\frac{\dfrac{a_n}{d_n} D_n^\mu}{\dfrac{a_{n+1}}{d_{n+1}} D_{n+1}^\mu} = 1 + O\left(\frac{d_{n+1}}{D_{n+1}^k}\right), \text{ where } k = \min(\lambda, 2),$$

which is the typical factor of an absolutely convergent product.

Thus $v_n \equiv (a_n/d_n)D_n^\mu$ tends to a finite non-zero limit as $n \to \infty$; and $\sum(v_n - v_{n+1})$ is absolutely convergent.

Case (i). Since $(a_n/d_n)D_n^\mu \to A$ as $n \to \infty$, it follows that

$$| a_n | = O\left(\frac{d_n}{D_n^{R\mu}}\right)$$

and $\sum a_n$ is absolutely convergent when $\Re\mu > 1$.*

Case (ii).† Here $(a_n/d_n)D_n^\mu = A + \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$. Hence

$$a_n = A \frac{d_n}{D_n^\mu} + \epsilon_n \frac{d_n}{D_n^\mu}$$

and

$$\sum_{\nu=1}^{n} a_\nu = A \sum_{\nu=1}^{n} \frac{d_\nu}{D_\nu^\mu} \left\{ 1 + \frac{\displaystyle\sum_{\nu=1}^{n} \epsilon_\nu \frac{d_\nu}{D_\nu^\mu}}{\displaystyle A \sum_{\nu=1}^{n} \frac{d_\nu}{D_\nu^\mu}} \right\}.$$

Setting $d_n/D_n^\mu = \alpha_n$, $(0 < \Re\mu < 1)$, we find that $|\sum^{\nu=n} \alpha_\nu| \to \infty$ with n in consequence of the corollary to Theorem 3. The corollary further establishes that

$$\sum^{\nu=n} \alpha_\nu \sim \frac{D_n^{1-\mu}}{1-\mu}, \qquad \sum^{\nu=n} | \alpha_\nu | \sim \frac{D_n^{1-R\mu}}{1-\Re\mu},$$

whence it follows that

$$\frac{\displaystyle\sum^{\nu=n} | \alpha_\nu |}{\displaystyle\left| \sum^{\nu=n} \alpha_\nu \right|} \to \frac{| 1-\mu |}{1-\Re\mu} \quad \text{as} \quad n \to \infty.$$

Lemma 1 now shows that

$$\frac{\displaystyle\sum^{\nu=n} \alpha_\nu \epsilon_\nu}{\displaystyle\sum^{\nu=n} \alpha_\nu} \to 0 \quad \text{as} \quad n \to \infty$$

which, in conjunction with the fact that

$$\left| \sum^{\nu=n} \frac{d_\nu}{D_\nu^\mu} \right| \to \infty \quad \text{as} \quad n \to \infty,$$

implies that

$$\left| \sum^{\nu=n} a_\nu \right| \to \infty.$$

---

* In the exponent, R is used for $\Re$.

† Knopp [2] refers to a discussion by J. A. Gmeiner, restricted to $d_n = 1$, of the cases (ii) and (iii). This discussion (Monatshefte für Math. und Phys., vol. 19, 1908, pp. 149–163) is not accessible to me.

Case (iii). In Lemma 2, taking $b_n = d_n/D_n^\mu$ and $v_n = (a_n/d_n)D_n^\mu$, we infer the finite oscillation of $\sum a_n$, $(\Re\mu = 1)$, from that of $\sum d_n/D_n^\mu$, $(\Re\mu = 1)$.

THEOREM 4a. *Under the conditions of Theorem* 4,

$$\sum \left| \frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right| \qquad \textit{and} \qquad \sum (-1)^{n-1} \frac{a_n}{d_n}$$

are convergent when $\Re\mu > 0$.

Proof. We have

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} = a_{n+1}\left\{\frac{\mu}{D_n} + O\left(\frac{1}{D_n^\lambda}\right)\right\} = O\left(\frac{d_{n+1}}{D_{n+1}^{1+R\mu}}\right).$$

This proves the first conclusion of the theorem.

To prove the second, we note that $\sum (a_n/d_n - a_{n+1}/d_{n+1})$ is a convergent series from which the brackets can be omitted since the resulting series is convergent by virtue of the restriction $\lim_{n\to\infty} a_n/d_n = 0$ implied in (2′b) when $\Re\mu > 0$.

Example of Theorem 4. We can consider, in the light of the last theorem,

$$\sum^\infty a_{n+1} \equiv \sum^\infty d_{n+1} \frac{\prod_{\nu=0}^{n} (\alpha d_{\nu+1} + D_\nu)(\beta d_{\nu+1} + D_\nu)}{\prod_{\nu=0}^{n} (\gamma d_{\nu+1} + D_\nu)(\delta d_{\nu+1} + D_\nu)}, \qquad (D_0 = 0).$$

It is obvious that the series reduces to the hypergeometric type when $d_n = \delta = 1$. For this series,

$$\frac{a_n/d_n}{a_{n+1}/d_{n+1}} = \frac{\left(1 + \gamma \dfrac{d_{n+1}}{D_n}\right)\left(1 + \delta \dfrac{d_{n+1}}{D_n}\right)}{\left(1 + \alpha \dfrac{d_{n+1}}{D_n}\right)\left(1 + \beta \dfrac{d_{n+1}}{D_n}\right)}$$

$$= 1 + (\gamma + \delta - \alpha - \beta)\frac{d_{n+1}}{D_n} + O\left(\frac{d_{n+1}}{D_n^2}\right).$$

Theorem 4 now settles the behaviour of $\sum a_n$ in the three cases: (i) $\Re(\gamma+\delta-\alpha-\beta) > 1$, (ii) $0 < \Re(\gamma+\delta-\alpha-\beta) < 1$, (iii) $\Re(\gamma+\delta-\alpha-\beta) = 1$.

Also, Theorem 4a establishes the convergence of

$$\sum (-1)^n \frac{\prod_{\nu=0}^{n} (\alpha d_{\nu+1} + D_\nu)(\beta d_{\nu+1} + D_\nu)}{\prod_{\nu=0}^{n} (\gamma d_{\nu+1} + D_\nu)(\delta d_{\nu+1} + D_\nu)}$$

when $\Re(\gamma+\delta) > \Re(\alpha+\beta)$.

5. To conclude, Theorem 1 may be made the starting-point of a discussion much wider in scope than Hukam Chand's. This theorem and the others following it reveal some of the possibilities in the idea behind the Maclaurin-Cauchy integral test. They also contain a clue to the paradox of convergence tests for $\sum a_n$ which turn on a comparison of $\sum a_n$ with a *divergent* series of positive terms.

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A PECULIAR PROPERTY OF THE PRIMITIVE ROOTS OF 13

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In logarithms, $\log_b a = 1/\log_a b$. If we seek a solution of $\log_b a = \log_a b$ we have $b^x = a$ and $a^x = b$, whence $b^{x^2} = a^x = b$ and $x = \pm 1$. Hence, either $b = a$ or $b = a^{-1}$.

In number theory, any primitive root g of a prime p may serve as a base whose powers form a complete residue system modulo p, *i.e.*, the terms of the series $g, g^2, g^3, \cdots, g^{p-1}$ modulo p are the integers $1, 2, 3, \cdots, p-1$ in some order. The exponents are termed indices and if $g^n \equiv N \pmod p$, we can write $\text{ind}_g N = n$. In fact, since g^{p-1} is always congruent to unity modulo p, $\text{ind}_g N \equiv n \pmod{p-1}$, so that adding multiples of $p-1$ to the exponent is the same as multiplying g^n by unity, thus leaving it unchanged.

The question arises: Under what conditions is

$$\text{ind}_g h \equiv \text{ind}_h g \pmod{p - 1};$$

that is, for what values of a prime p and a pair of its primitive roots g and h can we have $g^x \equiv h$ and $h^x \equiv g \pmod p$? If the congruence is possible, then $g^{x^2} \equiv h^x \equiv g \pmod p$ and therefore $x^2 \equiv 1 \pmod{p-1}$. Then $x \equiv \pm 1 \pmod{p-1}$ are solutions. The case $x \equiv +1 \pmod{p-1}$ is trivial since $g^{1^2} \equiv h^1$ makes g and h identical. But for $x \equiv -1 \pmod{p-1}$, we have $g^{(-1)^2} \equiv h^{-1} \pmod p$ or $g \equiv h^{p-2} \pmod p$. From the properties of primitive roots we know that in order for a power of a primitive root, h, to be congruent modulo p to another primitive root, g, its exponent must be one of the $\phi(p-1)$ integers prime to $p-1$. The exponent $p-2$ is always such an integer; hence all the primitive roots of every

prime can be paired so that for every pair, g and h, $\operatorname{ind}_g h \equiv \operatorname{ind}_h g \pmod{p-1}$, and if h is one such root, then h^{p-2} is the other one.

Example. The prime 17 has the $\phi(17-1)=8$ primitive roots 3, 5, 6, 7, 10, 11, 12, and 14 which can be paired modulo 17 as follows:

$$\begin{aligned} 3^{15} &\equiv 6, & 6^{15} &\equiv 3; \\ 5^{15} &\equiv 7, & 7^{15} &\equiv 5; \\ 10^{15} &\equiv 12, & 12^{15} &\equiv 10; \\ 11^{15} &\equiv 14, & 14^{15} &\equiv 11. \end{aligned}$$

We now ask: Do there exist primes for which this relation holds not only for *particular* pairs of primitive roots, but for each root paired with *every* other root? This requires that the congruence $x^2 \equiv 1 \pmod{p-1}$ hold for *every* value of x prime to $p-1$, not just for the particular values $x \equiv \pm 1$.

To answer this, we note that $p-1$ is necessarily even; hence every integer prime to it must be odd, and therefore x^2 must be of the form $8y+1$, and $8y \equiv 0 \pmod{p-1}$. This will hold for all values of y if $p-1$ is 1, 2, 4, or 8, and then $p=2$, 3, or 5 only, since $p=9$ is not a prime. However, 2 and 3 have only 1 primitive root each and do not therefore apply here; and for $p=5$ we have the single pair $2^3 \equiv 3$, $3^3 \equiv 2 \pmod 5$ which would have been obtained from the first case, $2^{p-2} \equiv 3$, $3^{p-2} \equiv 2 \pmod 5$.

If now we assume $p-1$ is divisible by 3, we can further restrict x so as to be prime to 3; that is, x^2 must be of the form $3w+1$. It is also of the form $8y+1$; hence it is of the form $24z+1$, and $24z \equiv 0 \pmod{p-1}$. This holds for all values of z if $p=2$, 3, 5, 7, or 13. The prime 7 has only 1 pair of primitive roots, 3 and 5, so that $3^5 \equiv 5$, $5^5 \equiv 3 \pmod 7$ as would be obtained from the first case. But for the 4 primitive roots 2, 6, 7, and 11 of 13, we now have not only the pairs $2^{11} \equiv 7$, $7^{11} \equiv 2$; $6^{11} \equiv 11$, $11^{11} \equiv 6 \pmod{13}$, but also:

$$\left.\begin{aligned} 7^5 &\equiv 11, & 11^5 &\equiv 7; \\ 2^7 &\equiv 11, & 11^7 &\equiv 2; \\ 6^5 &\equiv 2, & 2^5 &\equiv 6; \\ 6^7 &\equiv 7, & 7^7 &\equiv 6; \end{aligned}\right\} \quad \text{all modulo 13,}$$

so that *each* primitive root has been paired with every other one to exhibit the specified property.

Is it possible to obtain other solutions? Two necessary though insufficient conditions must always be fulfilled:

(1) x^2-1 must be a multiple of $p-1$.

(2) Since x^2-1 must be divisible by $p-1$ for *each* of the $\phi(p-1)$ values of x prime to $p-1$, the minimum value of x must be at least large enough so that $x^2-1>p-1$.

To prevent this minimum value of x being 3, since that would limit $p-1$

to 8, $p-1$ must be a multiple of 3; to prevent it being 5, $p-1$ must also be a multiple of 5, *etc.* This condition causes $p-1$ to increase so rapidly compared to x^2-1 that it is easy to show that there can be no other solutions. Table 1 below illustrates the possibilities:

TABLE 1

Minimum value of x	x^2-1	Possible values of $p-1$ to satisfy condition: (1)	(2)	Resultant values of p, a prime	Number of primitive roots
3	8	2	2	3	1
3	8	4	4	5	2
5	24	6	6	7	2
5	24	12	12	13	4
7	48	—	$2\cdot 3\cdot 5$	—	—
11	120	—	$2\cdot 3\cdot 5\cdot 7$	—	—
13	168	—	$2\cdot 3\cdot 5\cdot 7\cdot 11$	—	—

It can be seen from this table that even if all other conditions are set aside, the value of $p-1$ beyond $2\cdot 3\cdot 5$ exceeds the value of x^2-1. In general, the product of the first n primes is always larger than the square of the $(n+1)$th prime for $n>3$. This follows as a consequence of Bertrand's Postulate which states that there is always a prime between u and $2u-2$ when $u>3$. Hence x_{n+1}^2-1 can never be greater than $4(x_n^2-1)$, whereas the successive values of $p-1$ are multiplied by x_n which is always greater than 4 when $n>3$. Symbolically, if $\prod(x_n)$ is the product of the first n successive primes starting from 2, then $\prod(x_n) > x_{n+1}^2-1$ if $n>3$.

Hence we have the two theorems:

1. *If g is a primitive root of a prime p, then $h=g^{p-2}$ is another primitive root such that* $\operatorname{ind}_g h \equiv \operatorname{ind}_h g \pmod{p-1}$.

2. *For primes having more than two primitive roots, the relation* $\operatorname{ind}_g h \equiv \operatorname{ind}_h g$ $\pmod{p-1}$ *exists for every pair of primitive roots g and h only for the prime* 13.

THE MIL AS AN ANGULAR UNIT AND ITS IMPORTANCE TO THE ARMY*

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In the Army of the United States, two systems for measuring angles are in wide use; the *mil* system and the familiar *sexagesimal* system. In the mil system, the fundamental unit is the *mil*, where by definition 1600 mils equal one right angle. In contrast, in the sexagesimal system, the fundamental unit is the degree, where 90 degrees equal one right angle. Radian measure is also used to some extent by the Army. Most American mobile artillery units as well as many heavy railway mounts have the scales on their sights, azimuth circles, and quadrants graduated in mils, and some of these units have their scales graduated in both mils and degrees. The mil is also employed to a large extent by the Infantry.

The *mil* gains its name from the fact that one mil is approximately the angle subtended by one yard at a distance of 1000 yards. This simple approximate relation makes the mil well adapted to certain types of practical rapid calculations. It is principally for this reason that the mil system is used extensively in several branches of the Army.

In view of the preceding facts, it is obvious that many of the students now studying secondary or college mathematics will shortly find knowledge of the mil system highly desirable. However, the mil is mentioned in practically none of the current geometry or trigonometry texts and is seldom included in trigonometry courses at present. Hence, the Sub-Committee on Education for Service of the War Preparedness Committee recommends that teachers of trigonometry add to their courses work involving the mil system along with the usual material on radian measure. The following types of exercises should be included (the reader will no doubt want to add many more types).

1. Convert the angle $36° 10' 20''$ into mils.
2. Convert the angle 22 mils into degrees, minutes, and seconds.
3. Convert the angle 0.7 radians into mils.
4. Convert the angle 19 mils into radians.
5. Draw an angle, estimate its value in degrees; radians; mils.
6. How many mils are there in the central angle intercepting an arc of 20 inches on a circle of 25 inches radius?
7. What length of arc at 2000 yards will 3 mils intercept?
8. A circular target at 5000 yards subtends an angle of 2 mils at the edge. What is the diameter of the target?
9. From the position of an observer, an automobile 20 feet in length at right angles to the line of sight subtends an angle of 2 mils. What is its distance from the observer?

Problems involving the solution of triangles where the angles are measured in mils, should also be included.

The student should be given some field practice in estimating angles in mils;

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* Prepared at the request of the Sub-Committee on Education for Service of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America.

and he should be encouraged to work many problems involving the principles encountered in Exs. 6, 7, 8, and 9, approximately and without the use of pencil and paper.

Teachers at the secondary level should note that problems like the preceding Examples 1 through 9 could be included in the course in plane geometry if radian measure and related principles are discussed.

For the convenience of the reader, a few conversion factors are listed below.

90 degrees = 1600 mils	1 radian = 57° 17′ 45″
1 degree = 17.77778 mils	1 radian = 1018.6 mils
1 minute = 0.296296 mils	1 degree = 0.0174533 radians
1 mil = 0.05625 degrees	1 minute = 0.0002909 radians
1 mil = 3.37500 minutes	1 mil = 0.0009817 radians

MATHEMATICIAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

MATHEMATICS INSTRUCTION FOR PURPOSES OF GENERAL EDUCATION*

This report represents one phase of a study undertaken by a special committee of the American Association for the Advancement of Science on the improvement of science teaching.† The attention of this special committee is to be mainly devoted to the problem of improving science instruction in colleges and universities for the purposes of general education. The committee has assumed the following two responsibilities as an initial attack on the problem:

1. To make a study of the current instructional practices in those courses which are designed primarily for purposes of general education, *i.e.*, of those courses in science for students who take but few courses in science and for whom such courses are terminal.
2. To determine those experimental studies which seem to be the most important and urgent to be carried on with the aim of improving science instruction for non-science students.

The committee, therefore, in attacking the problem decided that some effort should be made to determine the point of view of science and mathematics teachers with respect to certain issues involved in this problem and also to discover the present practices in those courses designed primarily for purposes of general education. The committee decided that a first step might be that of obtaining certain of this information by means of a questionnaire.

* A preliminary report of the special committee of the American Association for the Advancement of Science on the improvement of science teaching in colleges and universities. This committee is made up of representatives from each of the following science fields: physics, mathematics, chemistry, botany, zoology, and the earth science (geology and geography). Representatives of mathematics are J. S. Georges and E. R. Hedrick.

† *Science*, vol. 87, 1939, page 454.

The purposes of the questionnaire. The purpose of the questionnaire was twofold:

1. To obtain the reactions of science teachers concerning some of the issues involved in the problem of instruction for purposes of general education.
2. To locate those science departments which have given considerable thought to and which have had considerable experience with courses designed for the non-science student.

Those departments which seem to have had considerable experience with this type of course were then to be contacted further by correspondence and personal visitation. In this way, more detailed information was obtained concerning course procedures, the content of the course, methods of testing, *etc.* The results tabulated in this report have to do only with the first aspects of this preliminary study, namely the questionnaire.

The questionnaire concerning the instruction of mathematics for purposes of general education was sent to approximately 500 colleges and universities. To obtain the opinions of a representative cross-section of mathematics teachers concerning the problem, the questionnaire was sent to the 500 colleges and universities irrespective of whether the college or university was known in advance to be favorable or unfavorable to certain of the issues involved.

The returned questionnaires were distributed among the various types of colleges and universities in the following ways: 158 colleges and universities, 50 teachers colleges, and 5 professional colleges.

Over half (56 per cent) of the questionnaires were answered by the head or chairman of the department in which mathematics was found (mathematics department, physical science department, science department, *etc.*). Thirty-one per cent were answered by individuals with the rank of full professor in mathematics, while twelve per cent were answered by persons holding the rank of associate professor or lower. Two of the questionnaires were answered by committees of teachers within the department of mathematics.

Four major themes or questions concerning mathematics instruction were incorporated into the questionnaire:

A. What are the opinions of mathematics teachers concerning some of the important issues in the problem of mathematics for the purposes of general education?

B. What do mathematics teachers believe should be accomplished in a mathematics course for the non-science or non-mathematics student?

C. What has been done by various mathematics departments in an attempt to meet this problem?

D. What are some of the major problems to be solved if mathematics instruction for the non-science or non-mathematics student is to be improved?

In order to shorten the questionnaire and thereby obtain a large yield, only

those issues and problems were included in the questionnaire which have been most frequently mentioned and discussed. Additional comments were invited and many teachers expanded their answers to the questions with several pages of written comments.

The questionnaire and the results. In preparing the questionnaire, specific questions were formulated to obtain information concerning the more general questions or items stated above. The introduction to the questionnaire was as follows:

A relatively small number of the students who enter college mathematics continue their study in more advanced courses. A much larger group take the course in order to meet certain requirements or solely for its contribution to their general education. The assumption has usually been made that essentially the same type of course meets the needs of these different groups of students. One of the questions of general concern which this committee believes should be studied may be phrased as follows: Does the teaching of mathematics through its content and method of instruction now adequately meet the needs of those students who do not continue the study of the subject?

The Committee would appreciate your coöperation in answering the following questions related to the problem above. The Committee makes no pretense that this questionnaire is complete and you may wish to add comments on the reverse side of these sheets related to the questions or to add other questions which are not presented in the questionnaire.

TABLE I

Part A of the Questionnaire

	Yes	*No*	*Uncertain*
Do you consider that the conventional introductory college course in mathematics as represented by a majority of current text-books:			
1. Is in general satisfactory for the non-specializing student?	51	130	24
2. Is more appropriate for students who later specialize in mathematics than for those who do not?	149	31	17
3. Could be significantly improved for the non-specializing student?	136	31	32
4. If modified for the non-specializing student would be in danger of becoming superficial?	98	62	41
5. Should be replaced for the non-specializing student by a mathematical survey course or a course which attempts to unify the elementary parts of algebra, trigonometry, analytic geometry, and calculus?	100	53	42
Do you consider that:			
6. The time and cost involved are essential factors which have tended to retard the introduction of courses designed to meet the needs of the non-specializing student?	70	91	33
7. The emphasis placed on "pure research" as a basis for advancement of the instructional staff has retarded the development of a real concern about and research upon teaching problems related to general introductory courses in mathematics?	110	50	41

The responses to Part A of the questionnaire indicate that the majority of mathematics teachers answering these questions believe the introductory course in mathematics to be unsatisfactory for the non-specializing students, and more appropriate for the students who later specialize in mathematics. The majority believe this course could be significantly improved for the non-specializing student. About half of the instructors believe that modification of the introductory

course for the non-specializing student might lead to superficiality. Half the instructors believe the introductory course should be replaced for the non-specializing student by a mathematical survey or a course which attempts to unify the elementary parts of algebra, trigonometry, analytic geometry, and calculus.

That too great emphasis is placed on "pure research" was given by about half of the group as a reason for a retarded development of a real concern about and research upon teaching problems related to general introductory courses in mathematics. A somewhat smaller group seemed to be of the opinion that time and cost involved are essential factors which have tended to retard the introduction of courses designed to meet the needs of the non-specializing student.

The comments on this part of the questionnaire suggest that many of the changes which might be made for the non-specializing student should also be made for the specializing student. A number of the instructors are of the opinion that there is danger of superficiality, but that this is not necessarily the case and can be avoided.

Question A-5 aroused the most comment. Although the majority of those answering the questionnaire were in favor of the proposal, most of the comments were made by those instructors who are opposed to the survey course. The following comments illustrate the character of this opposition:

> Such a course would give no power to use, and hence, the knowledge would be largely useless—a smaller content thoroughly learned is more useful.
>
> The elements and techniques may be taught separately, but for heaven's sake, give the student free rein to use everything he knows as he sees fit to attempt it. He will unify his mathematics knowledge.
>
> The problem of mathematics in general education cannot be solved by a mere unification of the elementary parts of algebra, geometry, and the calculus. What is needed is a knowledge of where mathematics touches the life of the individual in a cultural and practical way. Furthermore, why exempt the specializing student from contact with this type of course? In the not far distant future, he will be the teacher and will have a voice in determination of policies. Why delay introducing the specialist at an early date, to one or two courses that might orient him in a point of view as to the primary purpose of mathematics?

Others would be in favor of the survey course with certain restrictions or under certain specified conditions. The temper of this group is illustrated by the following quotations:

> A good unified course could be given for all students.
>
> The course for the non-specializing student while unifying the conventional courses, as suggested, should put more emphasis on the history of the development of mathematics and on its influence on, and relation to, the culture and civilization of the past and present.
>
> It is my belief that mathematics has a great deal to contribute to the educational development of the college students quite independently of whether or not they are to make technical use of the subject. This aim cannot be accomplished, however, by giving them merely elementary parts of the traditional courses. Any departure from the traditional courses should aim at insight into the significance of the subject for an intelligent understanding of our environment.

The comments on Question A-6 bring out a number of other factors which have retarded the introduction of these new courses. These are:

Inertia and conservatism of the instructors.
Schedule difficulties in small colleges and universities.
Conservative attitude of text-book publishers.
Uncertainty of science teachers as to what constitutes the modern trend.
Lack of educational vision.
Lack of adequate material.
Difficulties in the classification of transfer students who have taken the courses.

In a complex organization such as a department in a college or university, modifications in curriculum result from the opinions and attitudes of the men within the department, and are conditioned to a large extent by the curriculum which already exists within that department. With this in mind, an attempt was made to determine the relationship between the attitudes of the persons who responded to this questionnaire and the modifications which they report to have taken place in their departments (the department of mathematics) within the last five years. (Part C of the questionnaire.)

Little relationship was observed between the stated modifications of the introductory course in mathematics and the belief that if the course is modified for the non-specializing student, there is danger of its becoming superficial.

Likewise, little relationship was found between modifications of the introductory course and opinion as to whether the course as it now exists is satisfactory or not for the non-specializing students.

A large proportion of those instructors who are favorable toward the introduction of the survey course report that their departments have already replaced an old course with either a survey or a unified course, while the majority of those instructors who are opposed to this change report that their departments have not made this type of change. Thus, there seems to be a positive relationship between a favorable attitude toward the survey course and the introduction of such courses.

TABLE II

Part B of the Questionnaire

What do you believe are the most significant contributions which a study of mathematics should make for those students who are not to specialize in mathematics? (Write the number 1 in the parentheses below if you believe the contribution to be very important; the number 2 if you believe the contribution to be of some importance; and the number 3 for those contributions which a mathematics course should not attempt to make.)

The course should:	*1*	*2*	*3*
1. Develop the ability to do logical thinking	168	27	3
2. Show how the discoveries of mathematics have contributed to the "world view" characteristics of the present scientific era	88	88	14
3. Show the relationship which exists between the development of certain mathematical concepts and their use in different fields of science	109	83	3
4. Develop the ability to interpret mathematical data and to apply the mathematical principles related to one situation in other similar problems	148	43	5
5. Develop certain manipulative skills involved in solving mathematical problems	66	102	31

6. Expand the interest of students by encouraging certain hobbies which are naturally related to mathematics	5	105	80
7. Make students familiar with the facts, principles, and concepts of mathematics	133	54	9

The purpose of Part B of the questionnaire was to obtain the beliefs of mathematics teachers concerning the contributions or purposes of mathematics instruction for the student who is not specializing in mathematics. The ability to do logical thinking seemed to be the most important contribution which a study of mathematics should make for those students who are not to specialize in mathematics, according to the group answering the questionnaire. Next in order of importance were B-4 and B-7. These instructors also regard as important the demonstration of the relationship which exists between the development of certain mathematical concepts and their use in different fields of science. The ranking of the above purposes or objectives for the desirable course for non-mathematicians seems to indicate the direction of certain changes which could be made in the existing courses.

TABLE III

Part C of the Questionnaire

If your department has made some change in the introductory course in the last five years, what direction has it taken?

	Yes	*No*	*Uncertain*
1. Has the content and program of instruction been considerably modified within the framework of the old course?	63	82	3
2. Has the department replaced an old course with either a survey course or one in which the content has been more unified or integrated than that usually found in conventional mathematics courses?	73	89	1
3. Has the department introduced new courses (such as the survey courses) in addition to the regular introductory courses?	60	97	1
4. Do you rely upon a single mathematics text-book in the new or revised course?	68	66	1
5. Are you using an outline or syllabus which you have prepared especially for the new course?	68	66	1
6. Do you attempt to cover most of the traditional content such as determinants, the solution of triangles, conics, *etc.*, in the revised course?	58	44	9
7. Do you expect more outside reading in a survey course than you do in the course designed primarily for further work in mathematics?	58	41	9
8. Is the work in the new or revised course designed to show the applications of mathematics in other science fields?	94	10	10
9. Has your department prepared a bibliography of reading material related to the interests of those people who may not specialize in mathematics?	42	100	5
10. Has your department prepared a list of problems requiring investigations which can be carried on by the student outside of the classroom?	16	121	2
11. Has your department prepared special tests or other means of evaluating student achievement of the distinctive aims for the new or revised course?	18	110	1

The questions in Part C were designed to obtain evidence concerning present practices utilized in courses primarily for the non-specializing student. Forty-three per cent of the mathematics departments represented by the persons answering the questionnaire have modified the content and program of instruction of the introductory course. Forty-five per cent have replaced an old course with a survey or unified course, while thirty-eight per cent have introduced new courses (such as survey courses) in addition to the regular introductory courses.

The data indicate that of those institutions which have made changes in the introductory courses, sixty-four per cent have modified the regular course while thirty-six per cent have made more drastic changes consisting of replacing the regular introductory course with a completely revised course or adding a survey course. However, either of these changes may have represented a major change for the department concerned. This evidence seems to suggest that very few of the departments made sweeping and drastic changes in a short period of time.

Two procedures in the use of a bibliography of reading material are illustrated by the following comments on item C-9:

We have a list of books as well as references but we use them only for the better students.

Our bibliography is a card file plus a department list of another poster file hung in the library. In each, the books available in our library are listed with authors, titles, publishers, *etc* , and classified as 'art and mathematics,' 'recreation,' 'business,' *etc.* Informally, as a rule, students in any class are referred to selected groups of books and sent to these lists to find others on the same or allied topics. Our department office has a large number of books, also listed in other files and available for student use (in open book shelves).

The comments on Part C of the questionnaire are illustrative of the reasons for lack of changes in mathematics departments. These comments represent a minority of the respondents to this questionnaire. But because they represent the opinions of persons who for the most part are not enthusiastic about the survey or unified course, they are here presented:

Essentially, no change in introductory course and staff not large enough to add survey course.

We have tried a survey course and dropped it. Nearly all of our students take mathematics for use as a tool. Present survey courses as such are dilute, or take in too much territory. Concepts must be introduced more slowly.

Some years ago, our department offered unified courses, but discontinued them on account of difficulties in the distribution of 'credits' required by the prevailing 'pigeon hole' method of instruction.

TABLE IV

Part D of the Questionnaire

What are some of the things which you would like to have developed in a further consideration of the problems stated above? (Indicate by the number 1 those things which you believe to be very important; by the number 2 those which you believe to have some importance; and by the number 3 those which you believe are not at all important for this problem.)

	1	*2*	*3*
1. The clarification of a point of view for teachers of mathematics with regard to the place of mathematics in general education at the college level	170	25	4

2. The preparation of a list of problems suitable for purposes of general education which require:			
(a) Investigations making use of library materials	61	106	24
(b) Investigations requiring the solution of a somewhat original problem	58	103	24
3. The preparation of reading material designed:			
(a) To develop the ability to see the implications of discoveries in mathematics in everyday living	112	71	13
(b) To encourage thinking as to how mathematical discoveries may be applied to improve everyday living	100	76	15
4. The preparation of a bibliography of readings designed for purposes of general education suitable for use in mathematics courses	84	91	19
5. The development of methods for discovering the particular needs and interests of students and for selecting content and teaching procedures to meet those needs and interests	103	61	27
6. The preparation of tests designed to measure the achievement of students with respect to certain aims generally not now specifically tested such as the understanding and use of the structure of mathematical science and the ability to do logical thinking	100	68	22
7. The development of techniques for interpreting and using test results for the purpose of improving the achievement of students	82	73	30
8. The training of those people already engaged in the teaching of mathematics to meet the problem of mathematics in general education	136	42	10
9. The institution of upper level or graduate courses in history and philosophy of mathematics taught by mathematicians	84	83	20

In Part D of the questionnaire, an attempt was made to determine the opinions of mathematics teachers concerning the importance of certain projects which, if adequately developed, may increase the effectiveness of mathematics instruction for the student not specializing in mathematics. These mathematics instructors are of the opinion that the most important problem which should receive consideration is the clarification of a point of view for teachers of mathematics with regard to the place of mathematics in general education at the college level. When one considers the general dissatisfaction with the introductory course in mathematics and the strong belief that this course could be significantly improved for the non-specializing student, as reported in Part A of this questionnaire, the importance of this problem becomes readily apparent. The interest of teachers in this project seems to indicate that a general uncertainty exists concerning what mathematics instruction should do for the non-specializing student.

The large proportion of the instructors, who regard "logical thinking," interpretation of mathematical data, application of mathematical principles, and familiarity with the facts, principles, and concepts of mathematics as the important outcomes for the non-specializing students, lends support to the need for a further analysis of these aims. Such analysis might include suggestions on such questions as:

What is meant by logical thinking?
Can the ability to do logical thinking be developed?

How similar must the situations be if the student is to be able to apply the mathematical principles related to one situation in other similar problems?

What types of mathematical data will these non-specializing students need to interpret?

What are the facts, principles, and concepts of mathematics which might be taught to students taking an introductory course in mathematics?

In what situations might the student in general education use the above facts, principles, and concepts?

The above questions are not to be interpreted as a desire for standardizing the introductory course in mathematics. But, if well done, the project D-1 would yield a body of data which would be of considerable aid to mathematics teachers in clarifying their view-point on the subject of mathematics in general education.

The project D-5 ranked fourth in importance according to the judgment of these teachers. Many of the problems involved in this project are inextricably related to the problem of formulating a point of view concerning mathematics in general education, while others involve experimental studies, such as determining the value of a "mathematics laboratory" in which students can solve their assignments with such aid as they need from an instructor.

At least two possible interpretations can be made of the responses to question D-7. First, that instructors have fairly adequate techniques for interpreting test results; that is, in the field of mathematics, instructors feel fairly confident of the advice concerning improvement which they give their students; or second, that the problem of devising test interpretation techniques is an important one but certain other problems must receive prior consideration.

There were few comments on Part D of the questionnaire. However, one instructor did insist that the clarification of a point of view for teachers of mathematics with regard to the place of mathematics in general education at the college level be done by mathematicians and not by "education" people. Another believes that this clarification of view-point represents a hopeless task. With regard to item D-2, some of the instructors point out that this is "not for freshmen" and that the students need drill in simple arithmetic and high school algebra. In regard to the use of a bibliography of readings, the comments indicate that the instructors believe this is desirable if possible. Others feel that it is impossible with first-year students, and that the general student will never get far enough to make this have the slightest significance—"Can't do in one course, need several years training in mathematics."

The above report represents an attempt to summarize the returns of the questionnaire to mathematics teachers. The report may serve to provide some evidence concerning the feelings and opinions of mathematics teachers on the problem of mathematics for the purposes of general education. The committee will present a formal report to the executive committee of the American Association for Advancement of Science in which it will attempt to summarize its opinions concerning possible steps to take in the attack on certain of the problems pertaining to improvement of science instruction for purposes of general education.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A RULE FOR COMPUTING THE INVERSE OF A MATRIX

A. A. ALBERT, University of Chicago

If one uses the usual formula to compute the inverse of an n-rowed nonsingular matrix A, one must compute the determinant of A as well as all n^2 of its $(n-1)$-rowed minors. This is actually the way inverses of numerical matrices are customarily computed, and it is regrettable that a really simple method for this computation seems to have been overlooked* in the literature. This latter method may not be new but it is certainly known to very few mathematicians and, since it involves very little more than the computation of the determinant of A alone, it deserves some publicity.

The method for constructing A^{-1} depends upon the fact that if A is nonsingular it is possible to carry A into the identity matrix I by elementary *row* transformations alone. We may now state the method as the following rule:

Apply elementary row transformations to A which carry it into I, noting each transformation as used. Apply the same transformations to I and thereby obtain A^{-1}.

The rule is a consequence of the following rather evident

LEMMA. *Let $C=BA$ and apply an elementary row transformation to B resulting in what we shall designate by B_0, and then the same transformation to C resulting in C_0. Then $C_0=B_0A$.*

The lemma evidently generalizes to the case where C_0 and B_0 are obtained respectively from C and B by a finite sequence of elementary row transformations. We now take $B=I$, $C=A=IA$, so that $A_0=I=I_0A$. Thus $I_0=A^{-1}$ and we have our rule. To illustrate the rule we may, for example, take the simple case

$$A=\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{pmatrix}.$$

We add the second row to the fourth row of A, subtract twice the first row from the third, add twice the second row to the first, and interchange the first and second rows obtaining as the result

* The rule was unknown to me until I observed it recently, while engaged in constructing numerical exercises for my Introduction to Algebraic Theories. I give the rule in that text only in exercises, and was persuaded by W. D. Cairns to present it to the readers of this MONTHLY.

$$A_1 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We next add the fourth row of A_1 to each of its other rows, add the second row of the result to this third row to obtain

$$A_2 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and then subtract twice the third row from the first, four times the third row from the second to obtain I. Applying the same transformations we obtain

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

and finally

$$A^{-1} = \begin{pmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

ON THE GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP OF ORDER n^m AND TYPE $(1, 1, \cdots, 1)$

F. A. Lewis, University of Alabama

1. Introduction. Since the group of isomorphisms of an Abelian group is the direct product of the groups of isomorphisms of its Sylow sub-groups, it is natural that the involved study of the group of isomorphisms has centered on prime power Abelian groups and in particular on Abelian groups of order p^m and type $(1, 1, \cdots, 1)$. In the latter case the group of isomorphisms is simply isomorphic with the linear homogeneous group modulo p of order $(p^m-1)(p^m-p)(p^m-p^2) \cdots (p^m-p^{m-1})$. Only slight modifications of the usual proof of this theorem* are necessary in order to establish the corresponding property of an Abelian group of order n^m and type $(1, 1, \cdots, 1)$.

* See, *e.g.*, R. D. Carmichael, Introduction to the Theory of Groups of Finite Order, p. 111.

2. The order of the group of isomorphisms of G. The order of the group of isomorphisms equals the number of ways in which a set of independent generators may be selected. Let $S_1, S_2, \cdots, S_m$ denote a fixed set of generators of an Abelian group G of order n^m and type $(1, 1, \cdots, 1)$. Every element of G may be represented in the form

$$S_1^{x_1}S_2^{x_2}\cdots S_m^{x_m},$$

where the x's range independently from 0 to $n-1$. Any element of G is of period n if and only if the greatest common divisor of $x_1, x_2, \cdots, x_m$ is relatively prime to n. Hence, if $\phi_m(n)$ denotes the number of ordered sets $x_1, x_2, \cdots, x_n$ which satisfy these conditions, the first generator can be selected in $\phi_m(n)$ ways. It will be convenient to assume that S_1 is selected as the first generator. In selecting the second generator we may choose $x_1, x_2, \cdots, x_m$ in any way whatever, provided $S_1^{x_1}S_2^{x_2}\cdots S_m^{x_m}$ is of period n and no one of its powers less than the nth equals a power of S_1. A necessary and sufficient condition for this is that the greatest common divisor of $x_2, \cdots, x_m$ be prime to n. Hence the second generator can be selected in $n\phi_{m-1}(n)$ ways. Similarly, the third generator can be selected in $n^2\phi_{m-2}(n)$ ways, and so on. We therefore obtain the following:

THEOREM 1. *The order of the group of isomorphisms of an Abelian group of order n^m and type* $(1, 1, \cdots, 1)$ *is*

$$\theta_m(n) = \phi_m(n)\cdot n\phi_{m-1}(n)\cdot n^2\phi_{m-2}(n)\cdots n^{m-1}\phi_1(n),$$

where $\phi_i(n)$ represents the number of ordered sets of i non-negative integers such that each integer is less than n and their greatest common divisor is relatively prime to n.

If $p_1, p_2, \cdots, p_r$ are the distinct prime factors of n, then*

$$\phi_i(n) = n^i\prod_{j=1}^{r}(1 - p_j^{-i}).$$

Hence we obtain

$$\theta_m(n) = n^{m^2}\prod_{j=1}^{r}\prod_{i=1}^{m}(1 - p_j^{-i}).$$

3. The general linear congruence group. Let $S_1, S_2, \cdots, S_m$ denote a fixed set of generators of G. An isomorphism of G with itself may be represented by means of the symbol

$$\begin{pmatrix} S_1, & S_2, & \cdots, & S_m \\ T_1, & T_2, & \cdots, & T_m \end{pmatrix}, \tag{3.1}$$

where $T_1, T_2, \cdots, T_m$ is any ordered set of generators of G, and T_i corresponds to S_i for $i=1, \cdots, m$. Since the T's in the second line of (3.1) are products of

* L. W. Reid, The Elements of the Theory of Algebraic Numbers, p. 54.

powers of the S's, we may write the m components of (3.1) as

$$\begin{pmatrix} S_i \\ S_1^{a_{1i}} S_2^{a_{2i}} \cdots S_m^{a_{mi}} \end{pmatrix}, \qquad (i = 1, 2, \cdots, m). \tag{3.2}$$

Now (3.2) replaces $S_1^{x_1} S_2^{x_2} \cdots S_m^{x_m}$ by $S_1^{y_1} S_2^{y_2} \cdots S_m^{y_m}$, where

$$y_i \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m \pmod{n}, \qquad (i = 1, 2, \cdots, m). \tag{3.3}$$

An isomorphism of G with itself is defined by (3.3) if and only if this system of m simultaneous congruences admits a unique solution for the x's when the y's are given. A necessary and sufficient condition for this is that the determinant $|a_{ij}|$ of the system be relatively prime to n. Therefore we have the following:

THEOREM 2. *The group of isomorphisms of an Abelian group of order n^m and type* $(1, 1, \cdots, 1)$ *is simply isomorphic with the general linear homogeneous group* $\sum_{j=1}^{m} a_{ij}x_j \equiv y_i \pmod{n}$, $(i=1, \cdots, m)$, *where the coefficients a_{ij} take on all values such that the determinant $|a_{ij}|$ is relatively prime to n.*

ON CHARACTERISTIC ROOTS OF MATRIX PRODUCTS

W. M. SCOTT, University of Alabama

1. Introduction. It is well known that two square matrices C_1 and C_2 are such that C_1C_2 and C_2C_1 have the same determinant and that the trace of C_1C_2 is equal to the trace of C_2C_1. H. S. Thurston* proved that C_1C_2 and C_2C_1 have the same characteristic roots. His proof depended upon a continuity concept. MacDuffee† gives a proof and lists references to the theorem.

The proof given here is a purely matrix one.

2. We prove the following:

THEOREM 1. *If C_1 and C_2 are two square matrices, then C_1C_2 and C_2C_1 have the same characteristic roots.*

Proof. It is well known that similar matrices have the same characteristic roots.

We have the two following cases to consider:

(i). If either C_1 or C_2 is non-singular, say $|C_1| \neq 0$, then

$$C_1C_2 \sim C_2C_1$$

for

$$C_1^{-1}C_1C_2C_1 = C_2C_1.$$

(ii). If both C_1 and C_2 are singular, let $C_1C_2 = X$. We know that there exist

* H. S. Thurston, On the characteristic equations of products of square matrices, this MONTHLY, vol. 38, 1931, pp. 322–324. See also, J. H. M. Wedderburn, Lectures on Matrices, American Mathematical Society Publications, 1934, p. 25.

† C. C. MacDuffee, Theory of matrices, Ergebnisse der Mathematik, Springer, Berlin, 1933.

two non-singular matrices, M and N, such that

$$MC_1N = \begin{pmatrix} E_r & O_2 \\ O_3 & O_4 \end{pmatrix},$$

where E_r is a unit matrix of rank r, the rank of C_1, and the other three, O_2, O_3, O_4, are three zero-matrices of r rows and $n-r$ columns, $n-r$ rows and r columns, and $n-r$ rows and $n-r$ columns, respectively.

We also know that since $|M| \neq 0$ and $C_1C_2 = X$,

$$MC_1C_2M^{-1} \sim X$$

or

$$MC_1\,NN^{-1}C_2M^{-1} \sim X.$$

Now we split $N^{-1}C_2M^{-1}$ into sub-matrices as follows:

$$N^{-1}C_2M^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where C_{11} is r by r, C_{12} is r by $n-r$, C_{21} is $n-r$ by r, and C_{22} is $n-r$ by $n-r$.

We now have that

$$MC_1\,NN^{-1}C_2M^{-1} = \begin{pmatrix} E_r & O_2 \\ O_3 & O_4 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ O & O \end{pmatrix} = P \sim X.$$

But the characteristic equation of P depends only upon C_{11}, that is, the characteristic roots of P are entirely determined from the sub-matrix C_{11}.

Similarly, we put $C_2C_1 = X'$, and write

$$N^{-1}C_2C_1N \sim X'$$

or

$$N^{-1}C_2M^{-1}MC_1N \sim X'.$$

But we have already written

$$N^{-1}C_2M^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

and

$$MC_1N = \begin{pmatrix} E_r & O_2 \\ O_3 & O_4 \end{pmatrix},$$

so that we have

$$N^{-1}C_2M^{-1}MC_1N = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} E_r & O_2 \\ O_3 & O_4 \end{pmatrix} = \begin{pmatrix} C_{11} & O \\ C_{21} & O \end{pmatrix} = Q \sim X'.$$

But this matrix, Q, depends only upon the sub-matrix C_{11} for its characteristic roots. That is, X' and X have the same characteristic roots, or, that is, C_2C_1 and C_1C_2 have the same characteristic roots.

As was shown by Thurston* this can immediately be extended to any cyclic permutation of n matrices and we have the following:

THEOREM 2. *Any cyclic permutation of square matrices* $C_1, C_2, C_3, \cdots, C_n$ *will have the same characteristic roots, when the product is taken, as the matrix formed from the product* $C_1 \cdot C_2 \cdot C_3 \cdots C_n$.

IS A MANTISSA NECESSARILY POSITIVE?

C. B. READ, University of Wichita

Texts on the high school and freshman college level have various ways of defining the characteristic and the mantissa of a logarithm. There is disagreement as to the meaning of the terms. Many texts use a definition equivalent to: The integral portion of the logarithm of a number is called the *characteristic*, the decimal portion of the logarithm is called the *mantissa*. Other texts only define the mantissa for the situation where the logarithm is expressed as the sum of an integer and a decimal fraction, which is positive or zero, and less than one. Under these conditions, the *positive* fraction is called the mantissa, and the integer the characteristic. That there is a distinct difference is emphasized by such statements as: "This agreement always to write a logarithm in such a way as to have its mantissa positive is based on grounds of convenience in the use of the tables," "To avoid a negative mantissa . . ."; and, on the other side: "Warning: in the logarithm -1.6029 the decimal part is not the mantissa since it is not positive."

As a matter of curiosity, fifty recent books were examined. Twenty restrict the mantissa by defining it as a positive number, thirty use some form of the other definition. Of the thirty, twelve then point out the convenience of a positive mantissa in using tables, the remaining eighteen make the statement that the mantissa is independent of the position of the decimal point, without any statement that this is only true if the mantissa is positive, four of these eighteen fail to point out that even then the statement is only true for common logarithms. On the other hand, at least two books using the other definition make the definite statement that a logarithm *must* consist of an integer, positive or negative, and a decimal fraction, positive or zero; in other words, $-2\frac{1}{2}$ could not be a logarithm.

Without expressing a preference, it would seem that whatever definition is used, it should not be followed, even in the interest of brevity, by statements which can not be rigorously defended. The small sample of books examined seems to indicate less possibility of error if the term mantissa is restricted to a positive fraction (the definition of such standard references as Chrystal and

* H. S. Thurston, *op. cit.*

Fine is given with this restriction). With such a definition, do the terms characteristic and mantissa have any meaning when we consider, for example, log $.0275 = -1.5607$?

Granted that occasionally different definitions may be advisable depending upon the conditions under which the term defined is to be used, it seems rather doubtful that this is one of these situations. It seems unfortunate that there should be such a difference of opinion at a place which may confuse the student who is interested enough to consult a book other than his own text.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Dozen System. An Easier Method of Arithmetic. By G. S. Terry. London, New York, and Toronto, Longmans Green and Co., 1941. 53 pages. $0.50.

Methods of Apportionment in Congress—A Survey of Methods of Apportionment in Congress. (Senate Document, No. 304.) Washington, D. C., Government Printing Office, 1940. 41 pages. $0.10.

Symmetric Functions in the Theory of Integral Numbers. By H. Gupta. (Lucknow University Studies, No. 14.) Lucknow University, 1940. 7+106 pages.

A Bibliography on Orthogonal Polynomials. By J. A. Shohat, Einar Hille, and J. L. Walsh. (Bulletin of the National Research Council, No. 103.) Washington, D. C., National Research Council, 1940. 9+204 pages. $3.00.

Report of the Sixth Annual Research Conference on Economics and Statistics at Colorado Springs, July 1 to 26, 1940. Chicago, Cowles Commission for Research in Economics, 1940. 99 pages.

Elements of Calculus. By Abraham Cohen. Boston, D. C. Heath and Co., 1940. 5+583 pages. $3.50.

Matrix and Tensor Algebra for Engineers and Chemists. By C. E. Rose. New York, Chemical Publishing Co., 1940. 8+143 pages.

Exterior Ballistics. A reprint of Chapter X, Exterior Ballistics, and Chapter XII, Bombing from Airplanes, from *Elements of Ordnance.* Prepared under the direction of Lt. Col. Thomas J. Hayes. New York, John Wiley and Sons, 1938. 98 pages. $1.00.

REVIEWS

Advanced Mathematics for Engineers. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc., 1938. 10+473 pages.

The title of this book indicates the type of student to whom it is addressed and the scope of the book is evident from the table of contents which lists Ordinary Differential Equations, Hyperbolic Functions, Elliptic Integrals, Infinite

Series, Fourier Series, Gamma and Bessel Functions, Partial Derivatives and Partial Differential Equations, Vector Analysis, Probability, Functions of a Complex Variable, and Operational Calculus.

It seems to this reviewer that the authors have made a judicious choice of material for inclusion in this text and that this choice of material includes considerable advanced mathematics which should be useful to engineers and which can be presented in a course to follow the standard courses in the calculus. It is gratifying to find in this text, material on operational calculus, because this subject has a vast array of applications in the field of engineering and, for the most part, second courses in calculus for engineers usually neglect this subject.

Students of engineering are interested in knowing how mathematics can be applied to their fields. The authors of this book have attempted to show this by including a wealth of problems in which the mathematical theory is applied to the fields of Civil, Electrical, Mechanical, and Chemical Engineering. It would seem that these problems should furnish the engineering student with a motive for making his mathematics a valuable tool in his work.

This book should be of interest to all who are engaged in teaching advanced calculus to engineering students and also to engineers who wish to learn more mathematics for use in their work. The reviewer believes that for students in engineering and science the book will meet a real need.

E. B. ALLEN

Introductory Mathematical Analysis. By J. S. Georges and J. M. Kinney. New York, The Macmillan Co., 1938. 15+605 pages. $3.00.

This text is the result of experimentation to determine "the aims of mathematical instruction for college students, the selection and organization of instructional materials for the attainment of those aims, and the methods and modes of instruction and of evaluation of instruction."

Unfortunately the authors do not tell us in the preface what the aims of mathematical instruction are. However, they have included in the six hundred pages an abundance of material which they estimate as sufficient for a five-hour course for two semesters.

Roughly, one-fourth of the text is algebra, one-third is analytic geometry (plane and solid), one-sixth is trigonometry, and about one-eighth is calculus. These are joined together by the function concept into a unified whole which ought to make a rather thorough foundation in college mathematics for those majoring in science. The book is not a "survey" course for the ordinary college student, but each of the subjects mentioned above is pretty thoroughly covered. We find all the topics of the ordinary college algebra, such as complex numbers, theory of equations, determinants, logarithms, and even interest and annuities. In analytic geometry, there are full treatments of the straight line, the circle, the conics, transformation of coördinates, parametric equations, and solid geometry.

Though the treatment of plane trigonometry is brief, it is adequate for gen-

eral uses in science. There are abundant examples in trigonometric equations and formulas.

The treatment of the calculus includes both differential and integral and these applied in both the algebra and the analytics to good advantage to all.

There are tables of squares and square roots, of natural trigonometric functions, of common logarithms, of trigonometric functions, and of natural logarithms.

Each topic is accompanied by excellent exercises which furnish not only adequate drill but typical applications to practical problems in physics, chemistry, and business.

The authors suggest that the first six chapters, 275 pages, may be used as a semester course of five hours per week. Such a course would embrace a good elementary course in analytic geometry, and college algebra together with a short introduction to the calculus.

Those who like unified courses will find this an excellent text where the unifying principle has been used in such a skillful way as to hide the old lines of separation. The publishers have done their work well and have produced an excellent volume. There are few errors and the teacher should find it most satisfactory.

R. P. STEPHENS

Vierstellige Tafeln und Gegentafeln. By H. Schubert. New edition by R. Haussner. (Sammlung Göschen, Band 81.) Berlin, Walter de Gruyter and Co., 1940. 181 pages. RM 1.62.

The salient feature of this new edition is that the tables are printed in two colors, entries in blue and values in brown. For the inverse tables, the colors are interchanged. The type is large and the page open, making the tables easy to use. There are twelve tables, including logarithms of numbers and of trigonometric functions, addition and subtraction tables, natural logarithms, squares and cubes, square and cube roots, and an appendix containing various useful tables and constants.

VIRGIL SNYDER

Correction. In my review of the *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments* (vol. 47, 1940, page 652), I stated that the actual work of computation was undertaken by a staff of six computors. In fact, the six mentioned were the supervisors of the actual computors, who included more than two hundred and fifty persons.

VIRGIL SNYDER

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

CONTESTS

The *Pi Mu Epsilon* Metropolitan New York Intercollegiate Mathematics Contest, sponsored annually by the chapter at *Brooklyn College*, was held on April 26, 1940, with the following scores in points: Brooklyn College, Team I, 46; Cooper Union, Team I, 41; Cooper Union, Team II, 30; New York University, 18; Brooklyn College, Team II, 18; Queens College, 12; and Manhattan College, 7. The highest individual scores were: Benjamin Lax, Cooper Union, 13; Richmond Albert, Brooklyn College, 11; and Peter Chiarulli, Brooklyn College, 11.

The Seventh Annual *Pi Mu Epsilon* Interscholastic Mathematics Contest under the direction of the *Delta* chapter at *New York University* was held on April 20, 1940. The highest team score was made by Boys High School of Brooklyn, N. Y. which was awarded the Interscholastic Mathematics Contest Cup. Four sectional cups were awarded, one to Weequahic High School of Newark, N. J., one to Garden City High School of Garden City, N. Y., one to White Plains High School of White Plains, N. Y., and one to Stuyvesant High School of New York City for the highest team score in each section. Three medals, one gold, one silver, and one bronze, were awarded to the contestants making the three highest individual scores: Robert Gluckstern and Abraham Mark, both of Boys High School of Brooklyn, N. Y., and George Arfken, Jr. of Montclair High School, Montclair, N. J.

The second annual *Pi Mu Epsilon* Interscholastic Mathematics Contest of the New York *Epsilon* chapter at *St. Lawrence University* was held on May 11, 1940, with the help of the members of *Alpha Mu Gamma*. Six chosen high schools entered with a total of eighteen contestants. The Interscholastic Mathematics Cup was won by the Massena High School of Massena, N. Y.

CLUB REPORTS, 1939-1940

Mathematics Club, Wellesley College

This club which has for members sophomores, juniors, and seniors interested in mathematics, met five times during the year. Famous women in mathematics discussed at one meeting included Sonia Kovalevsky, by Norma Gould; Maria Agnesi, by Ellen Holt; and Emmy Noether, by Mary Gaylord. Another meeting included a discussion of Various methods of trisecting an angle: McLaurin's theory, by Elizabeth Paul; The method of conchoid and quadratrix, by Mildred Boyden; The hyperbolic method, by Selma Gottlieb; and the Impossibility of trisecting by compass and ungraduated ruler, by Professor Marian Stark. Mathematics and related fields, another theme, involved Mathematics and physics, by Mildred Boyden; Mathematics and astronomy, by Martha Stahr; Certain aspects of the Fourier series, by Ann Gray; Mathematics and economics, by Betty Colby; Mathematics and art, by Sue Norton; Curves of light, by Elizabeth Powers; Mathematics and literature, by Sally Sells; and Mathematics and poetry, by Mary Wells. Officers were: President, Mary Gaylord; Vice-President, Ellen Holt; Junior Executive, Doris Mosher; Secretary, Norma Gould; Treasurer, Jeanne Pope; Faculty Adviser, Professor Mabel Young.

Pi Mu Epsilon, University of Kentucky

The following books were presented to the Mathematics Library by the *Kentucky Alpha* chapter: *Opere Mathematiche* by Francesco Brioschi, 1901, in 5 volumes; *Sophus Lie, Gesammelte Abhandlungen* by Frederick Engel, in 8 volumes; and *Gesammelte Werke* by Jacob Steiner, 1881, in 2 volumes. Papers given during the year included: Topology, by Dr. L. W. Cohen; Infinite products, by Eugene Corum; Fermat's last theorem, by Glenn Clark; The poloconic, by W. F. Atchison; Multiple integrals, by F. M. McGee; and Ordinary annuities, by W. H. Clatworthy.

Mathematics Club, Smith College

Monthly meetings were held at which Dorothy Bray spoke on Topology, Katherine Fisk on Non-euclidean geometry, Margaret Stephens on Magic squares, Marlies Schaeffer on Algebras and non-Archimedean numbers, and Polly Puryear on Postulates of serial order.

Pi Mu Epsilon, Ohio State University

Twenty-two portraits of famous mathematicians were purchased by this chapter in December 1939 and hung in the offices of the department. This led to the addition of other pictures in the course of the year. A number of models of quadric surfaces were also constructed with the coöperation of the department of Ceramic Engineering and placed in the university classrooms. The members of the chapter meet every two weeks with the *Graduate Mathematics Club* to hear papers presented by faculty members and graduate students in mathematics. At the annual initiation the guest speaker was Professor L. M. Graves of the University of Chicago, who spoke on Generalized curves and existence theorems in the calculus of variations. Officers were: Director, Robert Westhafer; Vice-Director, William Scott; Secretary, Paul Young; Treasurer, John Ault.

Delta Rho, Southern Illinois Normal University

Interesting magazine articles from current issues of the various mathematical magazines are reported on at every meeting of the club. In addition, members prepared papers which could be presented before high school mathematics clubs or classrooms. In February every member coöperated in a Mathematics Field Day for high school students at which books for the high school library were exhibited, a contest was held for students in algebra and geometry, and student-made models were exhibited and demonstrated. In April high school teachers and students were invited to attend a special lecture sponsored by this club at which Dr. H. C. Christofferson of Miami University spoke on Mathematics as a pattern of thinking. Among the books reviewed at meetings were: Coolidge's *The Origin of Analytic Geometry*, by Russell Stevens; Bell's *The Search for Truth*, by Mabel Wallace and Lorelli Baker. Meetings were also held during the summer term at one of which Mr. James Slechticky, an alumnus, spoke on An application of the theory of functions to the two-dimensional flow around the wings of an airplane. Officers were: President, Fred Banes; Vice-President, Fred Basolo; Secretary-Treasurer, Beulah Freeman; Program Chairman, Ted Rodd.

Pi Mu Epsilon, Hunter College

This chapter celebrated the fifteenth anniversary of its founding at a dinner held in May at which Dr. Harriet Griffin of Brooklyn College, a charter member of the chapter and its first vice-director, was toastmaster. Guests were Professor Tomlinson Fort of Lehigh University, who was head of the Department of Mathematics at Hunter College when the chapter was founded, Professors Fite and Kasner of Columbia University, honorary members of the chapter, and Dr. George N. Shuster, President of Hunter College. Announcement was made of three new scholarships. The chapter together with members of the staff, students, and alumnae is raising funds for a Lao G. Simons Scholarship Fund. Professor Simons, the chairman of the department and a teacher of mathematics at Hunter College for forty-five years, retired on September 1, 1940. The chapter also sponsored the Lillian Marek Scholarship awarded to a high ranking student in the department; the nucleus of this scholarship was the Joseph A. Gillet prize of $40 for excellence in mathematics which was awarded posthumously at the commencement exercises the previous year to Lillian Marek. Professor Simons announced that she was founding the Pi Mu Epsilon Scholarship Fund, the first award to be made at the June commencement, for excellence in mathematics. Meetings held during the year dealt with a study of the topic The calculus of variations. Bliss's *The Calculus of Variations*, a Carus monograph, was awarded as a prize each semester for the best paper presented, and was given to Shirley Orlinoff and Emma Panaro. Officers were: Director, Dr. Mary K. Landers; Vice-Director, Marjorie Schaffner; Treasurer, Bertha Link; Recording Secretary, Alice Sternberg; Corresponding Secretary, Bernice Scherl.

Pi Mu Epsilon, Bucknell University

Subjects for meetings were: The use of the partial derivative in chemistry, by Thomas Fagley; Wine glass problem, by Betty Eyler; Symmetric products, by Professor Richardson; Irrational numbers, by Roger Keeney; Asymptotes in rectangular coördinates, by Carl Bennett; Areas without integration, by Robert Lewis; The method of least squares, by William James; Summation of simple series, by Peggy Reiff; The applications of mathematics in astronomy, by Louis McKee; and Tracing conics by differentiation, by Irene Harnish. Officers were: Director, Paul Benson; Vice-Director, Robert Stanton; Secretary, Margaret Bortz; Treasurer, Mary Eyler.

Mathematical Club, Case School of Applied Science

The speakers for the year included Dr. C. C. Torrance who spoke on The integral concept, Gail Adams on A short history of mathematics, L. L. Foldy on Dimensional analysis, R. C. Dorris on Probability, J. C. Roth on Nomography, J. D. Lubahn on Solutions of special trigonometric problems involving angles and also their functions, and J. I. Ehrhart on The theory of numbers. Officers were: President, J. W. Fitzwilliam; Vice-President, E. C. Gregg; Secretary, P. H. Houser.

Hall Mathematics Club, Lafayette College

Twenty members of the sophomore class acted as the "Board of Experts" at a mathematical "Information Please" contest quizzed by Professor Smith. Prizes of \$25, \$15, and \$10 were awarded to the winners. Similar awards were also made to sophomores earlier in the year in an essay contest on the topic The history and significance of pi. Professor Yates of Louisiana State University was guest speaker during the year; he spoke on Parallelograms, and presented the club with a collection of model linkages used in illustrating his talk. At other meetings, talks were given by student and faculty members on such subjects as: The general principles and fundamental theorems of dimensional analysis, Proof of the fundamental theorem, Solution of problems by means of dimensional analysis, The binary scale of notation, A Russian peasant method of multiplication, and A mathematical treatment of the game of Cardan's rings, Nim_1 and Nim_{k+1}. Officers were: President, Samuel Labate; Secretary, Richard May.

Pi Mu Epsilon, University of California

Pledges of the fall semester were presented at a regular meeting at which they were required to give short debates and speeches. In the spring, the pledges were blindfolded and put through a cross examination before the members present. Guest of honor and speaker of the evening at the spring banquet was Miss Harriet Glazier who retired from the faculty at the end of the year. The annual calculus contest sponsored by this organization was won by Alfred Landau. Meetings were held every month at which the following papers were presented: Vectors, quaternions, and Cayley numbers, by Dr. Max Zorn; Topological space, by Dr. T. Puckett, Jr.; Angling, by Dr. Frederick Valentine; Probability determination of pi, by Dr. Paul Hoel; Foundation of probability, by Dr. Reichenbach; and Possible implications of mathematics in the social sciences, by Dr. Paul H. Daus. Several outings were held including an ice skating party at Pan Pacific Auditorium, a mountain party at Camp Baldy, a motor trip and picnic in Palm Canyon, a picnic at Santa Anita Park, and a beach party at Manhattan Beach. Officers were: Director, Roy Luke; Vice-Director, Mrs. Margaret Lehman; Secretary, Annette Leimer; Treasurer, Wendell E. Mason.

Shuttlesworth Mathematical Society, University of Saskatchewan

The proceedings of this group during this year reflected the emphasis placed by the faculty on independent student investigation of mathematical principles and problems not ordinarily met with in the lecture room. Among the topics discussed were: Non-euclidean geometry, Elliptic integrals and functions, and Planetary motion. The social highlight of the year was the annual Mathematics-Physics banquet.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 461. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the difference equation $\Delta^k u_1 = u_k$, $u_1 = 1$, defines the sequence

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975,\ \cdots,$$

whose kth term is $f^{(k)}(0)$, where $f(x) = e^{e^x - 1}$. (*Cf.*, G. H. Hardy, *Pure Mathematics*, seventh edition, p. 424, Ex. 9.)

E 462. *Proposed by V. Thébault, San Sebastián, Spain.*

In what scale of notation can a square end with the digits 7777?

E 463. *Proposed by N. A. Court, University of Oklahoma.*

Determine the locus of the trilinear pole of a given line with respect to the triangle along which a variable plane through the line cuts a given trihedral angle.

E 464. *Proposed by Emma Lehmer, Berkeley, Calif.*

Prove that, for any prime $p > 3$,

$$\binom{kp^\alpha}{np^\beta} \equiv \binom{kp^{\alpha-\gamma}}{np^{\beta-\gamma}} \pmod{p^{\alpha-\gamma+3}}.$$

E 465. *Proposed by L. S. Johnston, University of Detroit.*

Without explicit use of the integral calculus, find the area enclosed by the curve $b^2y^2 = (b+x)^2(a^2 - x^2)$, where $b \geqq a > 0$.

SOLUTIONS

E 426 [1940, 395]. *Proposed by V. Thébault, Le Mans, France.*

Find the locus of a point whose polar planes with respect to four given spheres are concurrent, and the locus of the point of concurrence.

Solution by R. A. Johnson, Brooklyn College.

It is known that the polar plane of a point P with respect to a sphere S passes through the inverse P' of P; further, that any sphere through P and P' is orthogonal to S. Therefore if Q lies in the polar plane of P, the sphere on PQ as diameter passes through P' and is orthogonal to S.

If then the polar planes of a point P with respect to four given spheres are concurrent at Q, the sphere on PQ as diameter is orthogonal to the given spheres. In general, four given spheres have a single common orthogonal sphere, with center at their radical center. This sphere, then, is the required locus for both P (whose polar planes are concurrent) and Q (the point of concurrence).

If the given spheres are orthogonal to more than one sphere, they are orthogonal to a pencil of spheres. Through every point there is one of the orthogonal spheres, and the polar planes of every point are concurrent. If the given spheres are orthogonal to three spheres not belonging to a pencil, they are themselves members of a pencil, and the polar planes of any point have a common line.

Also solved by N. A. Court and the proposer. Court remarks that this proposition (as well as the corresponding proposition in the plane) is due to J. B. Durrande (1797–1825). They appeared in *Gergonne's Annales de Math.*, vol. 16, 1825, p. 112. See also, *Nouvelles Annales de Math.*, vol. 17, 1858, p. 239.

E 427 [1940, 395]. *Proposed by A. Gloden, Luxembourg.*

Find a palindromic pentagonal number greater than 22.

Solution by R. A. Johnson, Brooklyn College.

Pentagonal numbers are defined by the formula

$$f(n) = n(3n - 1)/2,$$

where n is a positive integer. Since the palindromic property is not inherent in a number, but depends on the use of the denary scale, palindromic numbers are most easily determined by actually computing and observing them. For the pentagonal numbers we have the relation

$$f(n + h) = f(n) + 3nh + f(h),$$

from which it follows at once that $f(n)$ and $f(n+20)$ have the same last digit, and $f(n)$ and $f(n+200)$ the same last two digits. One therefore prepares a table of the last two digits of $f(n)$ from $n=1$ to $n= 200$. Then, within a given range, it is sufficient to consider those values of n for which the last two digits of $f(n)$ are the same as the first two but in reverse order. Further, a palindrome with an even number of digits is divisible by 11; hence in this case n must be of one of the forms $11k$, $11k+4$. In this way the labor of determining the needed values of $f(n)$ is reduced appreciably; for example, in the range

$$5 \cdot 10^7 < f(n) < 10^8,$$

only some half-dozen values need be computed. Subject to possible correction by addition to the list, I find the only pentagonal palindromes less than $2 \cdot 10^8$ to be $f(1)=1$, $f(2)=5$, $f(4)=22$, $f(26)=1001$, $f(44)=2882$, $f(101)=15251$, $f(693)=720027$, $f(2173)=7081807$, $f(2229)=7451547$, $f(4228)=26811862$, $f(6010)=54177145$.

One greater than 10^9 is $f(26906)=1085885801$.

Also solved (not farther than 15251) by D. H. Browne, W. E. Buker, E. G. H. Comfort, Daniel Finkel, D. F. Johnson, E. P. Starke, C. W. Trigg, and the proposer.

E 429 [1940, 395]. *Proposed by Emma Lehmer, Berkeley, Calif.*

Prove that

$$\sum_{k=0}^{[n/3]} \binom{n}{3k} = \frac{2^n + r}{3},$$

where $r = (-1)^n 2$ or $(-1)^{[(n+1)/3]}$ according as n is or is not divisible by 3.

Solution by E. P. Starke, Rutgers University.

The stated formulas are proved in the proposer's solution of E 300 [1938, 320], where they are printed as

$S_0 = 2\{2^{n-1} + (-1)^n\}/3$ if n is a multiple of 3,

$S_0 = \{2^n - (-1)^n\}/3$ otherwise.

These expressions are seen to be identical with those now proposed if we can show that, in the latter case, $n - [(n+1)/3]$ is odd; but this is evident for $n = 3m + b$, $b = \pm 1$, since it becomes

$$3m + b - m - [(b + 1)/3] = 2m + b.$$

E 430 [1940, 487]. *Proposed by Louis Bauer, Hofstra College, Hempstead, N. Y.*

Somebody received a check, calling for a certain amount of money in dollars and cents. When he went to cash the check, the teller made a mistake and paid him the amount which was written as cents, in dollars, and vice versa. Later, after spending \$3.50, he suddenly realized that he had twice the amount of money the check called for. What was the amount on the check?

Solution by Michael Wilensky, Cincinnati, Ohio.

Let x and y be the numbers of dollars and cents called for in the check. Then the conditions of the problem are expressed by the equation

$$100y + x - 350 = 2(100x + y), \quad \text{or} \quad 199x - 98y + 350 = 0.$$

We deduce in turn,

$$\begin{aligned} 199x + 350 &\equiv 0 \pmod{98}, \\ 3x &\equiv 42 \pmod{98}, \\ x &= 14 + 98t, \qquad y = 32 + 199t, \end{aligned}$$

where t is an integer. Since $0 < y < 100$, we must have $t = 0$. Thus the amount of the check was \$14.32.

Also solved by F. A. Alfieri, R. K. Allen, W. E. Buker, M. L. Constable, C. H. Cunkle, William Douglas, C. D. Firestone, W. N. Huff, J. L. Hunter (with J. E. Grauel), Margaret Joseph, C. F. Kelley, III, Gertrude S. Ketchum,

William Lafferty, H. D. Larsen, H. R. Leifer, C. W. Moran, C. C. Oursler, P. W. A. Raine, W. C. Rufus, Hazel E. Schoonmaker, E. P. Starke, J. E. Trevor, Alan Wayne, B. C. Zimmerman, and the proposer.

E 431 [1940, 487]. *Proposed by V. Thébault, Le Mans, France.*

Find a number whose cube and fourth power together contain the ten digits, once each. (*Cf.*, E 116.)

Solution by J. L. Hunter, John Carroll University.

The desired number lies between 17 and 22, since there are, respectively, 4, 5, 5, 6 digits in 17^3, 17^4, 22^3, 22^4. The number cannot end in 0 or 1, as both the third and fourth powers would then end in 0 or 1; this eliminates 20 and 21. Finally, $19^4=130321$. Thus the unique solution is 18, as

$$18^3 = 5832, \qquad 18^4 = 104976.$$

Also solved by W. E. Buker, William Douglas, M. W. Fleck, P. W. Ketchum, C. W. Moran, E. P. Starke, Alan Wayne, and the proposer. By carrying through the same work in other scales of notation, Ketchum finds that ten is the only radix less than thirteen which admits a number whose cube and fourth power together contain all the available digits just once. The proposer shows that the only other case where a cube and a fourth power together contain the ten digits in the denary scale is

$$5^3 = 125, \qquad 44^4 = 3748096.$$

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3988. *Proposed by N. A. Court, University of Oklahoma.*

The symmetrics of a given straight line (plane) with respect to the sides (faces) of a given triangle (tetrahedron) form a second triangle (tetrahedron) perspective to the first, and the center of perspectivity is equidistant from the sides (faces) of the second triangle (tetrahedron).

3989. *Proposed by N. A. Court, University of Oklahoma.*

Three given spheres with non-collinear centers are touched by a (fourth) sphere in the points P, Q, R, and (p), (q), (r) are great circles, in parallel planes, on the three given spheres. Show that the three cones $P(p)$, $Q(q)$, $R(r)$ have a circle in common.

3990. *Proposed by V. Thébault, San Sebastián, Spain.*

Let A', B', C' be the centers of squares $BCA_1'A_2'$, $CAB_1'B_2'$, $ABC_1'C_2'$ constructed interiorly on the sides of triangle ABC with the centroid G and the angle V of Brocard. If $\cot V=7/4$, show that: (1) The centers A'', B'', C'' of the squares constructed interiorly on the sides of $A'B'C'$ lie on a straight line through G. (2) The angle V' of Brocard of $A'B'C'$ is such that $\cot V'=2$. (3) The straight lines joining A, B, C respectively to the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are parallel. (4) The distance of the circumcenter from the orthocenter of the orthic triangle is equal to one-fourth of the perimeter of the last triangle.

3991. *Proposed by V. Thébault, San Sebastián, Spain.*

Four straight lines Δ_i in a plane determine a complete quadrilateral (Q) forming four triangles $T_1\equiv(\Delta_2, \Delta_3, \Delta_4)$, $T_2\equiv(\Delta_1, \Delta_3, \Delta_4)$, *etc.*, with the orthocenters H_i. Show that the orthopoles of the straight line $\Delta\equiv(H_1, H_2, H_3, H_4)$, with respect to the four triangles, of the parallels to Δ_i through H_i are the orthogonal projections of the Miquel point on the sides of (Q).

3992. *Proposed by V. Thébault, San Sebastián, Spain.*

Show that the envelope of a variable sphere (S) which has its center on a quadric surface of revolution (Q) and which is orthogonal to a sphere (Σ) tangent to (Q) along a circle (C) is composed of two spheres passing through (C).

SOLUTIONS

3860 [1938, 122]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Given a tetrahedron, find the point such that the sum of its distances from the vertices is a minimum.

Editorial Note. A point P which lies outside the tetrahedron $T\equiv ABCD$ cannot give the minimum sum $\Sigma=PA+PB+PC+PD$. For, if P lies outside, there is at least one vertex, say D, such that P and D lie on different sides of the opposite face ABC; and the projection M of P on the plane of that face gives a smaller Σ. For, in the triangle DMP the angle DMP is greater than $\pi/2$ and $MD<PD$, and also $MA+MB+MC<PA+PB+PC$. Suppose now that P is a point within T which gives the minimum value of Σ, then P must be the point of tangency of two ellipsoids of revolution having respectively the foci A, B and C, D. For, if two such ellipsoids pass through P and are not tangent at P, they enclose a region such that at least a part lies within T, and any point Q within this part gives a smaller Σ. From this it follows that the common normal to the two ellipsoids at P is the intersection of the planes of angles APB, CPD and bisects each angle. Similar results hold for the two remaining ways of pairing the vertices. It will now be shown that angles APB and CPD are equal, and similarly for the remaining two pairs. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be unit vectors on PA, PB, PC, PD with origin at P giving $T'\equiv A'B'C'D'$ so that P is the circumcenter of T'. Then $\mathbf{a}+\mathbf{b}=k(\mathbf{c}+\mathbf{d})$ and $\mathbf{a}+\mathbf{c}=l(\mathbf{b}+\mathbf{d})$, and it follows that $(1+l)\mathbf{b}-(1+k)\mathbf{c}+(l-k)\mathbf{d}=0$. Since P is inside T, we must have $l=k=-1$; and thus $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$, and

P is also the centroid of T'. From $(\mathbf{a}+\mathbf{b})^2=(\mathbf{c}+\mathbf{d})^2$, we have $\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{d}$, which says that angles APB and CPD are equal, and similarly for the other pairs of opposite angles. It also follows that opposite edges of T' are equal. If the three faces through a vertex, say D', are folded into the plane of the opposite face, $A'B'C'$, we obtain the anticomplementary triangle of $A'B'C'$ whose vertices are the three positions of D'. The bimedians are axes of symmetry so that opposite dihedral angles are equal. Conversely, if T' is any tetrahedron whose opposite dihedral angles are equal, it is of the above type. For, the trihedral angles at the vertices are equal, having equal corresponding dihedral angles, and from this it follows that the faces of the tetrahedron are congruent triangles.

We now use the last result to prove that, if P is a point inside T such that $\angle APB=\angle CPD$, $\angle BPC=\angle DPA$, $\angle APC=\angle BPD$, then Σ_p for P is the minimum. Let $T_1\equiv A_1B_1C_1D_1$ be the antipedal tetrahedron of P with respect to T; then P lies within T_1, and T_1 has its opposite dihedral angles equal. Hence the faces of T_1 have equal areas σ; and the sum of the absolute normal coördinates of any point Q inside T_1 is equal to the quotient of three times the volume of T_1 divided by σ, and this quotient must be Σ_p. Thus Σ_q is greater than Σ_p, if Q is distinct from P.

A construction of a T with its minimizing P is as follows: Let $AB\overline{C}\overline{D}$ be a convex plane quadrilateral whose diagonals intersect in P. With APB fixed in position, rotate the triangle $\overline{C}P\overline{D}$ about the internal bisector of angle APB to the new position in space CPD, and we have the desired T and its P.

In order to complete the discussion, the cases where P lies on the boundary of T should be considered; also, there should be some consideration of the geometric construction of the minimizing point P. One method would be the determination of the locus of the points of tangency of the pair of ellipsoids of revolution with foci A, B and C, D; also for A, C and B, D. But it appears difficult to get sufficient information about the intersections of the two loci.

Several solutions of the similar problem for a triangle are given in the solution of 2742 [1920, 38] followed by *Remarks and Historical Notes* by R. C. Archibald which are of great interest. For n points in a plane, see the solution of 2980 [1923, 450] where there is also given a proof for the sufficiency of the condition for a minimum in the case of a triangle.

Michael Goldberg considered the minimizing point P within T, and assumed an infinitesimal displacement of P along the normal at P to the plane of APB. Then, except for infinitesimals of higher order, the change in CP is equal and opposite to that of DP; and therefore CP and DP lie on opposite sides of the plane of APB and make equal angles with the normal; that is, the plane APB cuts the plane CPD in the bisector of angle CPD. A similar result follows for any other pair of vertices; and this complete symmetry at P is satisfied when the six angles such as APB are equal. He also considered the mechanical device of a string attached at A and then passed successively through small rings at P, B, C, P, D. When the string is drawn tightly through D the sum Σ is minimized.

The proposer considered the special case where there exists within T a point

P such that the six angles such as APB are equal; and then gave a geometric proof, different from the above, that P gives the minimum Σ.

3918 [1939, 363]. *Proposed by B. M. Stewart, University of Wisconsin.*

Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

For $k=3$ this problem is called "The tower of Hanoï" in Ball's *Mathematical Recreations*, and the solution is given as 2^n-1.

I. *Solution by J. S. Frame, Brown University, Providence, R. I.*

Halfway through the process of moving the n washers, the largest washer lies alone on its original peg, and the chosen final peg is free to receive it. The other $n-1$ washers are distributed among the $h=k-2$ auxiliary pegs, and we may assume that the n_1 largest of these washers are on the first peg, the next n_2 on the next, *etc.* $\cdots$ and the n_h smallest ones on the last. In some cases the solution requiring the least number of moves is unique; in others it is not. We shall describe one of these "most economical" methods, understanding that others may be equally short but not shorter. In the trivial case $h>n-1$, only $n-1$ auxiliary pegs need be used, so the number of moves required is the same as for $h=n-1$. Otherwise, if the smallest washer is to cover n_h-1 others at this stage, it is a most economical method to have these be the smallest washers, so that these in turn do not block other pegs. Similarly in each of the h auxiliary piles the washers may be arranged in consecutive order according to size. It is also a most economical method to have the larger piles contain the smaller washers, since the latter have access to more pegs at the time of their transfer. Hence

$$n = 1 + n_1 + n_2 + \cdots + n_h, \qquad 1 \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_h. \tag{1}$$

To complete the transfer we move the largest washer to its destination, then move the n_1 next largest washers onto it using one auxiliary peg, then the next n_2 using two auxiliary pegs, *etc.* $\cdots$, and finally move the n_h smallest washers using h auxiliary pegs. The minimum number of moves, $m(h, n)$, required to move n washers using $h=k-2$ auxiliary pegs is thus given by

$$m(h, n) = 1 + 2[m(1, n_1) + m(2, n_2) + \cdots + m(h, n_h)], \tag{2}$$

if the best partition of n is chosen. A possible partition of $n-1$ is given by $1+n_1+\cdots+(n_r-1)+\cdots+n_h$, but since this is not necessarily the most economical partition, we have, instead of the equality (2), the inequality

$$m(h, n-1) \leqq 1 + 2[m(1, n_1) + \cdots + m(r, n_r - 1) + \cdots + m\ (h,n_h)]. \tag{3}$$

We define the cost of moving the nth washer to be

$$c(h, n) = m(h, n) - m(h, n-1), \tag{4}$$

and from (2), (3), and (4), we obtain the inequalities

$$c(h, n) \geqq 2c(r, n_r), \qquad (r = 1, 2, \cdots, h). \tag{5}$$

To minimize $c(h, n)$, we choose the partition of n so as to minimize the largest of the quantities $c(r, n_r)$. Then $c(h, n)$ can be taken to be twice this value. By induction we see at once that $c(h, n)$ is a power of 2, say 2^s. For fixed h, as n increases, s is non-decreasing, but may have constant stretches. We denote by $n_{h,s}$ the largest n for given h and s. Then

$$c(h, n) = 2^s, \qquad n_{h,s-1} < n \leqq n_{h,s}. \tag{6}$$

The maximum value of $c(r, n_r)$ must be 2^{s-1}. Without increasing this, we may choose our partition in a unique manner so that

$$n_r = n_{r,s-1}, \quad r < h; \qquad n_h = n - (1 + n_1 + n_2 + \cdots + n_{h-1}). \tag{7}$$

The largest n satisfying (6), namely $n_{h,s}$, is obtained by choosing $n_h = n_{h,s-1}$, so that all the costs $c(r, n_r)$ are equal. We thus obtain a recursion formula and two initial conditions for the function $n_{h,s}$, which define it for positive integral values of h and s,

$$n_{h,s} = \sum_{r=0}^{h} n_{r,s-1}, \qquad n_{0,s} = n_{h,0} = 1. \tag{8}$$

These same formulas define the binomial coefficients $(h+s)!/h!s!$. Hence,

$$n_{h,s} = (h + s)!/h!s!. \tag{9}$$

For given values of n and h, the number of washers costing 2^t moves may be written

$$n_{h,t} - n_{h,t-1} = n_{h-1,t}, \qquad t < s. \tag{10}$$

Each of the last $n - n_{h,s-1}$ washers will cost 2^s moves. Hence, by (4), (10), and (9), the minimum number of moves required to move the n washers is given by the formula

$$m(h, n) = \sum_{t=0}^{s-1} 2^t \frac{(h - 1 + t)!}{(h - 1)!t!} + 2^s \left[n - \frac{(h + s - 1)!}{h!(s - 1)!} \right], \tag{11}$$

where s is the largest integer for which the last term on the right of equation (11) is positive. In the classical case $k = 3$, we have $h = 1$, $s = n - 1$, and $m(1, n) = 2^n - 1$.

II. *Solution by the Proposer.*

It will be shown that the minimum number of moves for $k \geqq 3$ is given by

$$\begin{aligned} {}_kX_n &= 2^{s+1}(n - {}_kQ_s) + \sum_{j=0}^{s} 2^j\, {}_{k-1}Q_j, \\ {}_kQ_s &= \binom{k - 2 + s}{s}, \qquad n \subset {}_kI_s, \quad \text{that is,} \quad {}_kQ_s \leqq n < {}_kQ_{s+1}. \end{aligned} \tag{1}$$

Consider a rectangular array of squares with coördinates (k, n), $k \geqq 3$, $n \geqq 1$.

With each square (k, n) there is a corresponding rectangle made up of squares (k', n') such that both $k' \leqq k$, $n' \leqq n$. The proof is by an induction which assumes the formula (1) true for all the squares of the rectangle except (k, n) and then establishes the formula for this corner square.

As a basis for the induction we use the facts that the theorem is true for $k=3$ and any n; and for any k with $n < {}_kQ_1 = k-1$, which are easily seen directly without the formula.

Essentially any possible best way of moving the washers can be described in three steps: move the n_1 uppermost washers to another peg, using all k pegs; move the n_2 remaining washers to a second peg, using the available $k-1$ pegs; and finally move the n_1 washers to this second peg, once again using k pegs. We define the function

$$(2) \qquad {}_kY_n(n_1) = 2{}_kX_{n_1} + {}_{k-1}X_{n_2}, \qquad n = n_1 + n_2,$$

where n_1 and n_2 are positive integers. At first thought, when $k>4$, an extension of this reasoning ought to be considered, dividing the n washers into three (or more) sets and by the symmetry of the problem examining a function

$${}_kY_n(n_1, n_2) = 2{}_kX_{n1} + 2{}_{k-1}X_{n_2} + {}_{k-2}X_{n_3}, \qquad n = n_1 + n_2 + n_3.$$

But the last two terms, representing a way of moving n_2+n_3 washers using $k-1$ pegs, can best be replaced by ${}_{k-1}X_{n_2+n_3}$.

First if $n \subset {}_kI_s$ we can find $n_1 \subset {}_kI_{s-1}$ such that

$$(3) \qquad {}_kY_n(n_1) = {}_kX_n.$$

Either $n - {}_kQ_s < {}_{k-1}Q_s$, and $n_1 = n - {}_{k-1}Q_s$ will serve; or $n - {}_kQ_s \geqq {}_{k-1}Q_s$, and $n_1 = {}_kQ_s - 1$ will serve; in both cases $n_1 \subset {}_kI_{s-1}$ with either $n_2 \subset {}_kI_s$ or $n_2 = {}_{k-1}Q_{s+1}$. The proof of (3) is made by use of the above formulas for ${}_kQ_s$, where ${}_kQ_0 = 1$ and ${}_kQ_s = 0$ if s is negative, and by use of the relation ${}_kQ_{s+1} = {}_kQ_s + {}_{k-1}Q_{s+1}$.

We prove next that ${}_kY_n(n_1)$, for values of n_1 other than those above, is never less than ${}_kX_n$ by considering the variation of ${}_kY_n(N_1)$ as N_1 increases by unity from an n_1 chosen under the conditions above, and similarly when N_1 decreases.

We can easily show that

$$(4) \qquad \Delta_k X_n = 2^{s+1}, \qquad n \subset {}_kI_s.$$

Then for an increasing N_1 we have

$$(5) \qquad \delta_k Y_n(N_1) = 2\Delta_k X_{N_1} - \Delta_{k-1}X_{N_2-1}, \qquad N_1 + N_2 = n,$$

where N_1, N_2 are positive integers admitting the considered variations. From (4) we have $\Delta_k X_{N_1} \geqq 2^s$, ${}_{k-1}X_{N_2-1} \leqq 2^{s+1}$; hence $\delta_k Y_n(N_1) \geqq 0$. For a decreasing N_1 we have

$$(6) \qquad \delta_k Y_n(N_1) = \Delta_{k-1}X_{N_2} - 2\Delta_k X_{N_1-1}, \qquad N_1 + N_2 = n.$$

Here $\Delta_{k-1}X_{N_2} \geqq 2^{s+1}$, $\Delta_k X_{N_1-1} \leqq 2^s$; hence $\delta_k Y_n(N_1) \geqq 0$.

Thus if the problem of moving n washers on k pegs is solved in the shortest

way, the total number of moves is given by ${}_kY_n(n_1)$ which has been shown to be a minimum for n_1 and n_2 chosen in any one of the ways described in (3). But if the assumption of the induction is applied to (2), then by (3) the theorem is true for ${}_kX_n$, for the squares with coördinates (k, n_1) and $(k-1, n_2)$ are in the rectangle corresponding to (k, n). A basis for the induction has already been noted; hence the theorem is true.

Editorial Note. The analysis in each of the above solutions depends upon a preliminary lemma in the statements above equation (2) in each. It would be desirable to have a brief and rigorous proof of these lemmas. It will suffice to prove the following lemma: If the first n_h washers from the top of the initial peg are placed on a single auxiliary peg, say peg h; the next n_{h-1} on peg $h-1$; and so on until the largest washer is placed with one move on the final peg where it is alone; then, for suitable values of n_i, this plan for the removal of all the washers from the initial peg requires as small a number of moves as any other.

In the second solution the induction may be made in steps from $s-1$ to s. The equation (11) in the first solution may also be written

$$m(h, n) = (-1)^h + 2^s\left[n - \sum_{t=0}^{[h/2]} n_{h-2t,s-2}\right]. \tag{12}$$

This is a slight modification of a similar equation given by Frame in the first draft of his solution. The proposer gave the following results for $n=64$ and values of h in the parentheses: (1) 18,446,744,073,709,551,615; (2) 18,433; (3) 1535; (4) 673; (5) 479; (6) 385; (7) 351. We add two other computations, using equation (12): $m(2, 128)=720,897$; $m(2, 192)=10,485,761$.

The set of integers n_i may be chosen in any way so that $n_{i,s-2} \leqq n_i \leqq n_{i,s-1}$, where on the left we must have at least one inequality so that n is in the given interval. For $h=1$ and any n there is only one choice, and for this reason this case is quite simple. If $n=n_{h,s}$, there is only one choice, but there are cases where there may be a large number of different selections of these integers each giving the same total number of moves.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The president of the Association, Professor R. W. Brink, has appointed the following committee on the Slaught Memorial Papers: Professors R. E. Langer (chairman), N. H. McCoy, W. E. Milne, and C. V. Newsom.

The University of Cincinnati, under the auspices of the Taft Memorial Fund, held a mathematical symposium on November 9, 1940. The principal speakers were Professors G. A. Bliss and Marston Morse.

Dr. C. B. Boyer of Brooklyn College has been promoted to an assistant professorship.

A portrait of Professor Abraham Cohen, who has recently retired from a professorship of mathematics in the Johns Hopkins University, was presented to the university at the Commemoration Day Exercises on February 22, 1941.

Professor B. F. Finkel represented the Mathematical Association at the inauguration of James Franklin Findlay as president of Drury College on November 29, 1940.

Professor G. A. Parkinson of the University of Wisconsin Extension Division at Milwaukee is on leave of absence for a period of duty as an officer in the U. S. Naval Reserve aboard the U. S. S. Gilmer based at Seattle.

Assistant Professor J. H. Roberts of Duke University has been promoted to an associate professorship.

J. L. Stearn, Assistant Mathematician of the U. S. Coast and Geodetic Survey, is now on active duty as a lieutenant in the Ordnance Department, U. S. Army.

E. O. Stephany of Cornell University has been called into service as a Naval Reserve Officer. At Cornell University, Dr. E. H. Hadlock has been appointed instructor, and A. R. Turquette and A. M. Peiser part-time instructors for the second semester.

Dr. H. N. Wright, professor of mathematics at the College of the City of New York, has been appointed acting president of the College in succession to Dr. N. P. Mead.

Dr. G. M. Hayes, assistant professor of mathematics at the College of the City of New York, died January 2, 1941.

Professor R. E. Moritz died December 28, 1940, at the age of 72. He had served as chairman of the mathematics department at the University of Washington for over thirty years. He was a charter member of the Mathematical Association.

G. H. Taber, retired vice-president of the Gulf Oil Corporation of Pittsburgh, Pa., died December 10, 1940. He was a charter member of the Mathematical Association.

SPECIALIZED TRAINING FOR FLYING CADETS

W. L. Hart, University of Minnesota

Bulletin of Information by the War Preparedness Committee,
Sub-Committee on Education for Service

My attention has been called to the following letter from the Office of the Chief of the Air Corps. This letter was inclosed in a letter, dated March 3, originating in the Headquarters of the Seventh Corps Area of the Army. This letter stated that presidents of flying cadet examining boards will conduct examinations for cadets who wish to qualify under the provisions of the following letter. It is my understanding that young men interested in this matter can obtain the necessary additional information from those centers where training under the Civil Aeronautics Program is being given, or from officers of the Air Corps. It appears to me that many young men who are studying college mathematics would be interested to learn of the contents of the following letter:

WAR DEPARTMENT
Office of the Chief of the Air Corps
Washington

SPECIALIZED TRAINING OTHER THAN PILOT FOR FLYING CADETS

The War Department is offering to young men, who meet the prescribed requirements, a course of training to qualify them as aerial navigation officers. Instruction will be given in schools under Army supervision but will include no pilot training.

The status, as well as pay and allowances, *etc.*, of cadets undergoing this specialized non-pilot training in navigation is the same as that for cadets receiving pilot training. They are designated "Flying Cadet" and, upon satisfactory completion of the course, as well as an additional period of training with tactical or other Army Corps units, are eligible for commissions as 2nd Lieutenants, Air Reserve. The entire training period will cover approximately nine months.

Applicants for this course must meet the general requirements for appointment as flying cadet. They must be unmarried citizens of the United States between the ages of twenty and twenty-six inclusive, of good character, sound physique, and in excellent health.

The training as "Navigator" is designed to qualify candidates as navigator-gunner members of combat crews. Students will first undergo instruction in an aerial gunnery school. This will be followed by a course in navigation, attention being given, among other subjects, to day and night navigation flights, the use of instruments, maps and charts, dead reckoning procedure and problems, and celestial navigation theory.

EDUCATIONAL QUALIFICATIONS

1st Priority Classification: Graduates of accredited colleges and universities who have received a degree in Engineering.

2nd Priority Classification: Graduates of accredited colleges and universities who have had, as a minimum, mathematics to include Plane Geometry, College Algebra,

and Trigonometry. In addition, applicants are preferred who have had mathematics to include Analytical Geometry, Spherical Trigonometry, and Differential and Integral Calculus.

3rd Priority Classification: Those who have satisfactorily completed two years of accredited college work and who have had the mathematics outlined in the 2nd Priority Classification above.

In all cases consideration will be given to the quality of the applicant's collegiate scholastic record.

PHYSICAL REQUIREMENTS

The physical requirements are somewhat less rigid than those required for pilot training insofar as the requirements reference visual acuity are concerned; however, in general the high physical standards now applicable to flying cadets (pilot) must be met.

Applications for this training should be submitted in triplicate direct to The Chief of the Air Corps. The regular flying cadet application blanks may be used but notation should be made thereon that "navigation" training is desired. Three letters of recommendation, transcript of college work, and birth certificate, if not previously submitted, should also be furnished.

/s/ J. W. DURANT,
Captain, Air Corps,
Asst. Chief, Personnel Division.

W-3717, Rev. 1/30/41.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 2, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., May 3.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May.
MICHIGAN, Ann Arbor, March 15.
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3 or 4.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, Colo., April 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, March 28–29.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit, May 3.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1938 to Professor G. T. Whyburn for his article "On the structure of continua" published in the *Bulletin of the American Mathematical Society,* volume 42, 1936, pp. 49-73.

As determined more recently by the Trustees, the prize is to be awarded for a noteworthy expository paper. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

VOLUME 48 APRIL 1941 NUMBER 4

CONTENTS

The AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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EDITORIAL

In the present issue of the MONTHLY, Professor Curtiss gives an interesting paper on the professional interests of mathematical instructors in junior colleges. Among other things, he reports on the publication needs of the large group of persons whose work centers in freshman and sophomore college courses.

Recognizing the mathematical interests of this group, the MONTHLY has made efforts to provide facilities to meet their needs. Members of the group have from time to time made use of these facilities, and they are invited to continue to contribute along lines suggested below:

1. Historical papers or notes concerning topics closely related to junior division mathematics.
2. Papers or notes concerning applications of mathematics to science, art, industry, or war.
3. Papers or notes concerning the organization and content of courses in mathematics.
4. Papers or notes pertaining to examinations in mathematics at various levels of attainment in colleges and universities.
5. Reports or notes on experiments in the administration of freshman mathematics, such as grouping of students according to ability, or according to their plans for the future.
6. Reports or notes concerning trends in registration in mathematics in high schools and in colleges.
7. Announcements of meetings of mathematical and scientific organizations, and of programs offered in the universities, in which this group is materially interested.
8. News items regarding college instructors in mathematics, particularly appointments, promotions, special honors, or deaths.

The MONTHLY through its departments of Recent Publications, Clubs and Allied Activities, and Elementary Problems, provides for other interests of teachers who are primarily concerned with freshman and sophomore mathematics.

Good technical papers which may have a wide interest, which are based on a moderate mathematical background, and which are in good expository form, will continue to have a place in the MONTHLY; in fact, good expository papers are especially desirable.

The MONTHLY welcomes material offered for publication, even in the face of an abundant supply. By selection on the basis of merit, with a reasonable regard for the breadth of interests of teachers of college mathematics, it aims to serve all such persons, including those in the junior colleges.

THE PROFESSIONAL INTERESTS OF MATHEMATICAL INSTRUCTORS IN JUNIOR COLLEGES*

D. R. CURTISS, Northwestern University

The title of this address might be variously interpreted. In a narrower sense, the professional interests of such groups as lawyers, doctors, college teachers, are served by bar and medical associations, and the Association of University Professors. Useful as such societies are, they do not enter the field of my discussion; nor do the general and local educational associations which include teachers of all subjects. I shall limit myself to the consideration of what may be done to further the mathematical interests and increase the fitness of those who are teaching mathematics in Junior Colleges.

Much of what I shall have to say is the outgrowth of an investigation by a committee of two, Miss Martha Hildebrandt and myself, appointed by the Mathematical Association of America at the suggestion of its Committee on Association Activities. This committee of two was asked to study particularly the question of how the Association could best serve the needs of instructors of mathematics in Junior Colleges. Miss Hildebrandt, I must confess, did practically all of the work of investigation, and prepared the report with little aid from me. It is only because she was unable to attend this meeting that I am speaking to you in her place. The report has not yet been acted upon by the Association, so I feel free to quote only parts of it. The responsibility for what I am to say rests on my own shoulders, though I am using the materials collected by Miss Hildebrandt and am indebted to her report for the substance of this address.

We are familiar with the rapid growth of the Junior Colleges all over the country. The largest group are the municipal Junior Colleges, administered as part of the city public school system and closely coördinated with the local high schools. There are also many private Junior Colleges, some of which are denominational schools that have found it expedient to come down from a four-year to a two-year program. Many of these private Junior Colleges are, however, extensions of secondary schools.

As might be expected, the teaching personnel of the Junior Colleges has been largely determined by these affiliations with secondary schools. The more experienced and capable secondary teachers have been promoted to Junior College positions. The result has been advantageous in that these teachers are a group selected from the much larger one of all secondary teachers, but this situation has resulted in a lower level of mathematical attainment than would probably have been the case if the Junior College had split off from the four-year college.

In 1938, there were 767 instructors in mathematics in Junior Colleges. The extent of their training may be inferred from the highest degrees which they have earned, which is shown in the following tabulation:

* Presented at the joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics at Baton Rouge on January 1, 1941.

A.B.	75, or 9.8%
B.S.	64, or 8.3%
A.M.	351, or 45.7%
M.S.	134, or 17.5%
Ph.D.	55, or 7.2%
No degrees listed	57, or 7.4%
Other degrees	31, or 4.0%

It was not ascertained how many of these degrees represented majors in mathematics, but two small samplings have shown that three-fourths of one group of Masters of Arts had majored in Education, and that but three out of ten in another group holding Master's degrees had majored in mathematics.

This situation could have been predicted from the way in which the Junior College has been developed. It seems likely that in the future Junior Colleges will become more independent, and a smaller proportion of their staffs will be recruited from the high schools. In consequence, there will be a demand for increased mathematical preparation.

The state of affairs which I have thus outlined urgently requires that everything possible be done by these teachers to increase their mathematical interests, and by existing mathematical organizations to aid them. The National Council of Teachers of Mathematics represents their former affiliations and still serves many of their interests. The Mathematical Association of America, which covers the collegiate field, should be the organization which especially meets their needs. The majority of the 767 teachers I have listed should have belonged to both of these societies. Actually, only a little over ten per cent were members of the Association; I have no figures regarding the Council.

Small as is the average salary in Junior Colleges, it is hardly the two dollars for membership in the Council, and the four dollars for the Association, that account for this low percentage of membership, which may be compared with the thirty per cent membership in the Association of all collegiate instructors in mathematics in this country. It is true that many Junior College teachers are required to be members of national and local educational associations, and that the pressure of summer school and sometimes of evening work, together with heavy teaching loads and guidance and club assignments, leaves them little energy for any further affiliations. The conclusion, however, is almost unavoidable that Junior College teachers of mathematics have not felt that membership in a mathematical society was a pressing need.

Inquiry among Junior College teachers has brought out many criticisms of the Association from their point of view. They feel that its activities and interests are dominated by the group that are also members of the American Mathematical Society, and are largely interested in research. The journal of the Association, the AMERICAN MATHEMATICAL MONTHLY, devotes relatively little space to their especial curricular and teaching problems. Those who are now members feel that their interests are unrepresented at the Association's meetings and that they are unrecognized in the administration of the Association.

The Committee on Association Activities recognized the justice of these criticisms, and recommended that the Association consider these measures:

(1) remission of initiation fees for some specific period;

(2) assignment of space in the MONTHLY for papers dealing with matters of especial interest to Junior College teachers, and editorial representation on the MONTHLY;

(3) provision for the presentation of their programs and the discussion of their problems at the Association meetings;

(4) representation on the Association's Executive Committee.

These recommendations followed a paragraph which I here quote:*

"To merely invite into membership in the Association individuals engaged in the Junior College field, will, we believe, promise but small return. The Association does not seem to this group to be concerned with many matters that are of moment to it. It should not be regarded in this connection as a question of the existing membership of the Association assuming to supply to the Junior College group those things they need to fill their want, but rather of the Association's assuring that group of the fundamental identity of interests and objectives, and with the invitation to participate in membership *en masse* offering it the opportunity and the facilities which the organization has at hand for it to serve its own needs within the Association and with the coöperation of the Association's entire membership."

These suggestions were turned over to the Association's committee of two. They soon concluded that there were other groups whose interests were so close to those of Junior College teachers as to make it seem unwise not to include them. Thus many teachers in four-year colleges, having teaching schedules and other duties which have shut them out from mathematical research, are primarily interested in teaching problems; this is true even of many instructors in freshman and sophomore classes in universities. The same interests are primary in normal schools, teachers colleges, and university and college departments of Education. Some high school teachers who have done graduate work in mathematics and some undergraduate and graduate students who expect to teach may be added to this group. Whatever the Association may do for Junior College teachers would also serve these other groups. It seems advisable, therefore, that any action taken by the Association should include them, as well as the Junior College teachers. Thus the Association would not seek to form a sub-organization for Junior College instructors only, but would provide programs and means of publication for all who are interested in problems centering on the first two years of college mathematics.

The Association has two annual meetings and there are many annual meetings of Sections. The Section meetings have often considered matters of interest to the groups mentioned above, and some Junior College teachers have expressed the opinion that these meetings are more useful to them than are the meetings

* This MONTHLY, vol. 47, 1940, p. 74.

of the whole Association. I believe that the Sections should continue to pay attention to this point, even to the extent of providing sessions entirely devoted to problems centering on the first two years of college mathematics. At the annual meeting of the Association there should also be provision for such programs; there could be simultaneous meetings of two or more groups interested in different classes of problems, as well as joint meetings; this plan is now familiar to members of the American Mathematical Society. None of these sectional meetings should be labeled "for Junior College teachers only." The programs at sectional meetings on teaching problems should so far as possible be organized by the groups interested, and the papers prepared by them, rather than by distinguished mathematicians who never teach a freshman class.

There remains the question of satisfactory publications for these groups. Shall the MONTHLY be enlarged to include new departments, with associate editors who will represent these interests? Would it be better not to alter the MONTHLY, an admirable publication which is satisfactory to the majority of present members of the Association, but instead to provide an alternative journal especially catering to Junior College teachers and related groups? This plan is not new in scientific societies. Members of the Association would then be offered their choice of the two publications, with a possible differential in membership fees if one journal is less costly than the other. There might also be a rate for the two in combination. Possibly some existing independent mathematical magazine might be taken over for the new journal.

To illustrate the scope of the proposed new publication, I shall quote suggestions for departments as listed by Miss Hildebrandt:

1. A department for historical papers on topics such as pre-Newton handling of individual calculus problems, history of polar coördinates, the development of the unit circle and ratio concepts of trigonometric functions.

2. Bibliographies of articles in the undergraduate field already published in the MONTHLY, such as *A Reading List in the Elementary Theory of Equations*, by the late Raymond Garver, vol. 40, 1933, pp. 77–84.

3. Articles concerning applications of mathematics somewhere between the popular presentation for the layman and the text-book, which would presuppose a knowledge of calculus, but not much more than that.

4. A department interested in pedagogical questions concerning themselves with college mathematics, that is, the content of courses of the first two years, notes on teaching these courses, and so on.

5. A problem department similar to that now in operation in the MONTHLY.

6. A department devoted to extra-classroom activities in mathematics.

7. A department which would invite articles on the relation of mathematics to the physical sciences, art, social studies, philosophy, psychology, medicine, engineering, and traffic problems.

8. Humorous mathematics such as the "Work Problem Applied, Important Ambiguous Case" on page 33 of the January, 1940 issue of the MONTHLY, mathematical oddities and amusements, "Trivia Mathematica."

9. Book reviews.
10. News notes.
11. Notices of mathematics meetings both national and sectional, or state.
12. Announcements of summer school programs, and so on.

I have no doubt that a new publication of this sort would attract more members than would an enlarged MONTHLY. It remains to be seen whether the Association will approve, or will feel that it cannot afford the financial hazard involved.

In conclusion let me say explicitly what I have implied in many places in this address, that every Junior College teacher should belong to an organization representing mathematical interests, and that these organizations, on their side, should do their utmost to provide for the special needs of Junior College instructors.

SOLITAIRE ON A CHECKERBOARD

B. M. STEWART, Michigan State College

The game of Solitaire which is to be described herein is played with checkers or pegs upon a field of any convenient number of squares arranged in rows and columns. During the play there can be at most one piece on a square, and at the outset of the game at least one square must be free so that the play, which consists of a series of jumps, may begin. A jump may be made if in three consecutive squares A, B, C of any row or column there are pieces on A and B while C is free. The jump is moving the piece on A over B to C and removing

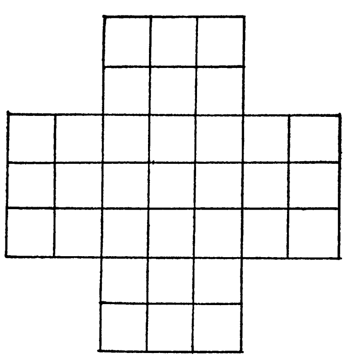

FIG. 1

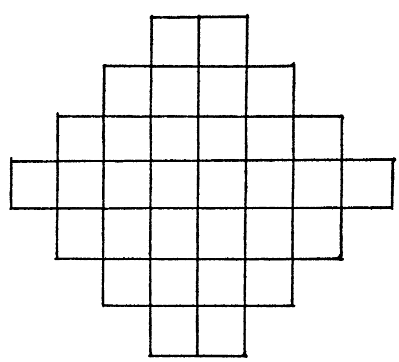

FIG. 2

the piece on B from the play. Jumps along the diagonals of the field are forbidden. The object of the play is to remove all the pieces from the field except one, leaving this final piece on a specified square.

Of ancient origin is that particular form of Solitaire sold in this country under various trade names such as Puzzle-Peg or Chinese Pegs with a field like that of Figure 1. It is in this form that the game first appeared in mathematical literature, having been found a pleasant diversion by Leibniz (1710). A history

of Solitaire and solutions of some of its forms are given in the well known work by W. Ahrens.* The interesting, if elementary, theory underlying the game is attractively presented by G. Kowalewski† and applied to the games possible on the field of Figure 1. It is my intention to review this theory briefly and apply it to a very familiar playing field, that of Figure 2, which the reader will recognize to be a convenient presentation of the actual playing squares of the ubiquitous checkerboard.

Let the playing field be any convenient shape. Let the diagonals that run from lower left to upper right be labelled cyclically red, yellow, and blue so that every square has a color label. Let the original arrangement of pieces be such that upon red squares there are R pieces; upon yellow squares, Y pieces; and upon blue squares, B pieces. Suppose that during the course of play there are r jumps ending upon red squares, y jumps ending on yellow squares, and b jumps ending on blue squares. Observe that a jump ending on a square of a certain color decreases the number of pieces on squares of each of the other colors by one. Thus we may write the following equations:

$$\begin{aligned} R + r - y - b &= R', \\ Y - r + y - b &= Y', \\ B - r - y + b &= B', \end{aligned} \tag{1}$$

where R', Y', B' are the number of pieces remaining on red, yellow, blue squares, respectively, after the $r+y+b$ jumps have been made. In the ordinary version of the game, R', Y', B' are to have the values (1, 0, 0), (0, 1, 0), or (0, 0, 1). Thus a first necessary condition for the game to have a solution is that the original configuration of R, Y, B pieces be such that one of the above choices for R', Y', B' will lead to a set of rational integral solutions for r, y, b in the system (1). This Diophantine condition is easily investigated by solving (1), say by determinants. Thus we have

$$r = \frac{Y + B - (Y' + B')}{2}, \quad y = \frac{B + R - (B' + R')}{2}, \quad b = \frac{R + Y - (R' + Y')}{2}. \tag{2}$$

Hence for any one of the above choices for R', Y', B' it is evident that R, Y, B cannot all be even or all be odd. This criterion is strong enough to show many problems impossible of solution. For example, in the checkerboard problem of Figure 3, R, Y, B are all even, so there is no system of play that will leave a single checker on the board.

The relations (2) afford still further information. Let the one of R', Y', B' with the value 1 be called singular. Let the one of R, Y, B which is even while the others are odd, or which is odd while the others are even, be called singular. Then the relations (2) show that a necessary condition for the game to have a solution is that the initial and final singular colors be the same.

* W. Ahrens, Math. Unterhaltungen und Spiele, Teubner, 1910, vol. I, chap. VIII, pp. 182–210.

† G. Kowalewski, Alte und Neue Math. Spiele, Teubner, 1930, "Das Solitärspiel," pp. 126–145.

Since exactly the same arguments apply if one labels with colors the diagonals running from upper left to lower right, the necessary conditions found above are doubled in strength. With respect to this set of diagonals there must exist an initial singular color, and the initial and final singular colors must be the same.

Suppose that for a given original configuration the necessary conditions are satisfied for each set of diagonals. Then the play may or may not be possible, for of course these necessary conditions are by no means sufficient; but if the play is possible, the final piece *must* be on a square with a definite pair of color labels. The squares with such a pair of labels form a pattern over the field, consisting of every third square in a row or column, counting from any square with the proper labels.

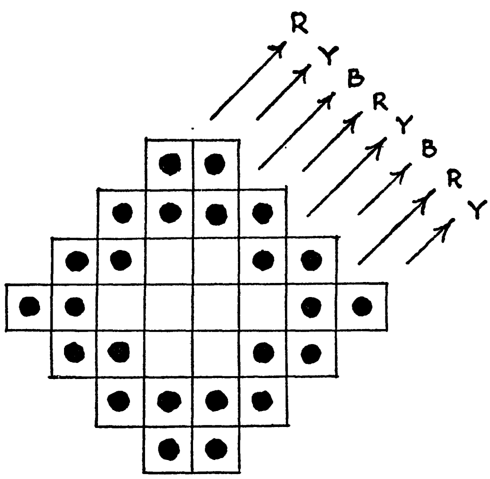

FIG. 3

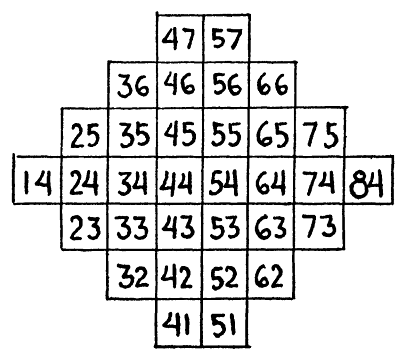

FIG. 4

In the study of the field in Figure 1, Kowalewski has considered the games that arise when every square except one is occupied initially. If we count all the squares, then $R=Y=B=11$ (similarly for the other set of diagonals), so that the necessary conditions are satisfied when just one square is empty, and the color of this square becomes the singular color of the theory. The many cases that arise are reduced in number by the symmetry of Figure 1 and by other considerations. Then by actually recording jump by jump a solution for each case, Kowalewski shows that the necessary conditions are indeed sufficient.

I propose to show the same result in the case of the checkerboard field of Figure 2. Here, if we count all the squares (for either set of diagonals) there is for each color an even number of squares. Thus if initially just one square is left unoccupied, the necessary conditions are satisfied and the colors of the empty square are the singular colors of the theory. Let each square of the field be given a pair of coördinates, as in Figure 4. Then by reason of symmetry, only the

initial squares 14, 24, 34, 44, 23, 33, 43, 32, 42, 41 need be considered. By the color tests for each of these initially empty squares, there is a limited set of possible ending squares. In terms of the coördinates, this ending square must have each coördinate congruent modulo 3 to the corresponding coördinate of the initial square. Thus if we begin with 14, possible ending squares are 14, 41, 44, 47, 74. If the case 14-41 is solved, then 14-47 can be solved by using the symmetry of the figure. If the case 14-41 is solved, then the case 41-14 can be solved by reversing the rôles of occupied and unoccupied squares. By arguments of symmetry and reversal, the total number of cases to be studied is reduced to the following twenty-five:

14-14, 14-41, 14-44, 14-74, 24-24, 24-51, 24-54,
34-34, 34-64, 44-41, 44-44, 23-23, 23-53, 23-56,
33-33, 33-36, 33-63, 33-66, 43-43, 43-46, 32-32,
32-62, 42-42, 41-41, 41-47.

The solution for the difficult case 14-14 is shown graphically in Figure 5. In the remaining cases, to save space a symbolic notation has been adopted by which the interested reader may follow the play on a board. This system describes the play in the first row of Figure 5 by "3414, 3234, 4424, 1434." Each set of four numbers indicates a jump from the square whose coördinates are the first two numbers to the square whose coördinates are the last two numbers. Related cases have been grouped together for still greater economy of space.

Group I. Begin 14. End 14. See Figure 5.

Group II. Begin 14. End 41, 44, or 74.
3414, 3634, 4424, 1434, 5535, 5755, 2545, 5535, 4745, 3555,
6444, 3454, 5456, 7555, 5654, 8464, 6365, 6664, 4363, 4143,
5153, 3234, 5333, 2343, 6444, 3454, 7353, 5452, 6242, 4341.
or 5452, 6242, 4244.
or 4363, 6264, 5474.

Group III. Begin 24. End 24, 54, or 57.
4424, 1434, 6444, 5254, 3252, 4442, 2343, 5153, 5333, 4143,
8464, 7353, 6563, 6264, 4363, 6365, 6664, 4565, 4745, 7555,
4565, 5755, 6545, 6444, 4446, 2545, 3335, 3656, 3555, 5557.
or 3656, 3555, 5654.
or 3634, 4644, 4424.

Group IV. Begin 34. End 34 or 64.
3634, 5535, 4745, 3555, 6545, 5755, 4565, 3335, 2545, 1434,
5333, 5153, 3252, 6242, 4143, 4442, 5452, 2343, 7353, 6563,
8464, 6365, 7555, 4565, 6664, 5254, 6444, 3454, 4244, 5434.
or 4464.

Group V. Begin 44. End 41 or 44.
6444, 6664, 4565, 4745, 3555, 6545, 5755, 5535, 6365, 7555,
8464, 2545, 3335, 1434, 4424, 3634, 2444, 4363, 7353, 4565,
6563, 4143, 5333, 2343, 4424, 5153, 3252, 6343, 6242, 4341.
or 4244.

Group VI. Begin 23. End 23, 53, or 56.
4323, 3533, 1434, 2343, 5333, 4143, 6242, 3252, 5153, 5452,
3353, 5254, 7353, 6563, 8464, 4565, 7555, 4745, 6646, 3656,
5535, 2545, 5355, 5535, 5755, 6365, 6545, 3533, 4543, 4323.
or 3533, 4543, 3353.
or 3555, 3454, 5456.

Group VII. Begin 33. End 33, 36, 63, or 66.
3533, 1434, 3335, 5333, 4143, 6242, 3252, 5153, 4442, 2343,
4244, 5452, 7353, 5254, 6563, 8464, 6365, 6664, 4565, 4745,
5755, 4446, 2545, 3656, 6444, 5654, 7555, 5535, 5434, 3533.
or 5535, 5434, 3436.
or 4565, 4464, 6563.
or 4565, 4464, 6466.

Group VIII. Begin 43. End 43 or 46.
2343, 3533, 1434, 5535, 4745, 6646, 3656, 5755, 4323, 4143,
5333, 2343, 6242, 3252, 5153, 6462, 8464, 6563, 6264, 4363,
7353, 4565, 7555, 2545, 4565, 5355, 3454, 6545, 6444, 4543.
or 4446.

Group IX. Begin 32. End 32 or 62.
3432, 5333, 4143, 6242, 3252, 5153, 1434, 6462, 8464, 6563,
6264, 4565, 4745, 6646, 3656, 5755, 5456, 7555, 5654, 4363,
2343, 7353, 4446, 6444, 3454, 2545, 4644, 4442, 5452, 5232.
or 4262.

Group X. Begin 42. End 42.
6242, 3252, 3432, 1434, 5333, 5153, 2343, 3533, 3234, 4442,
4143, 5535, 4745, 6646, 3656, 5755, 6444, 4446, 2545, 4644,
3454, 5456, 7555, 5654, 8464, 5452, 7353, 5254, 6444, 4442.

Group XI. Begin 41. End 41 or 47.
4341, 6242, 4143, 5452, 5153, 6462, 8464, 6563, 4565, 5755,
4745, 6664, 4565, 7555, 2545, 3335, 1434, 3533, 5535, 3634,
3454, 6444, 4442, 2343, 3252, 5254, 7353, 5452, 6242, 4341.
or 4745, 6264, 4363, 7353, 2343, 3533, 1434, 3335, 5333, 3234,
3454, 6444, 4446, 2545, 3656, 5654, 7555, 5456, 6646, 4547.

The interesting fact that the necessary conditions of Solitaire on the fields of Figures 1 and 2 are also sufficient is not the general fact. Let us describe briefly the situation with some other playing fields. If one square of a 3×3 field be left vacant, the necessary conditions are not sufficient, for the limited playing space soon leads to a stalemate. If one square on a principal diagonal of a 4×4 field be left vacant, the necessary conditions are satisfied for one set of diagonals but not for the others. If for this same field one square not on a principal diagonal be left vacant, the necessary conditions are satisfied and the play is indeed possible, but the singular colors of the theory are not the same as the colors of the initially empty square.

In an example like this there is to the uninitiated some element of black magic in the assertion that a certain arrangement of Solitaire is impossible of solution or that if the game can be solved the final piece must be on a certain square. Being in a sense a professional dealer in black magic, the author hopes his secrets are in understanding hands.

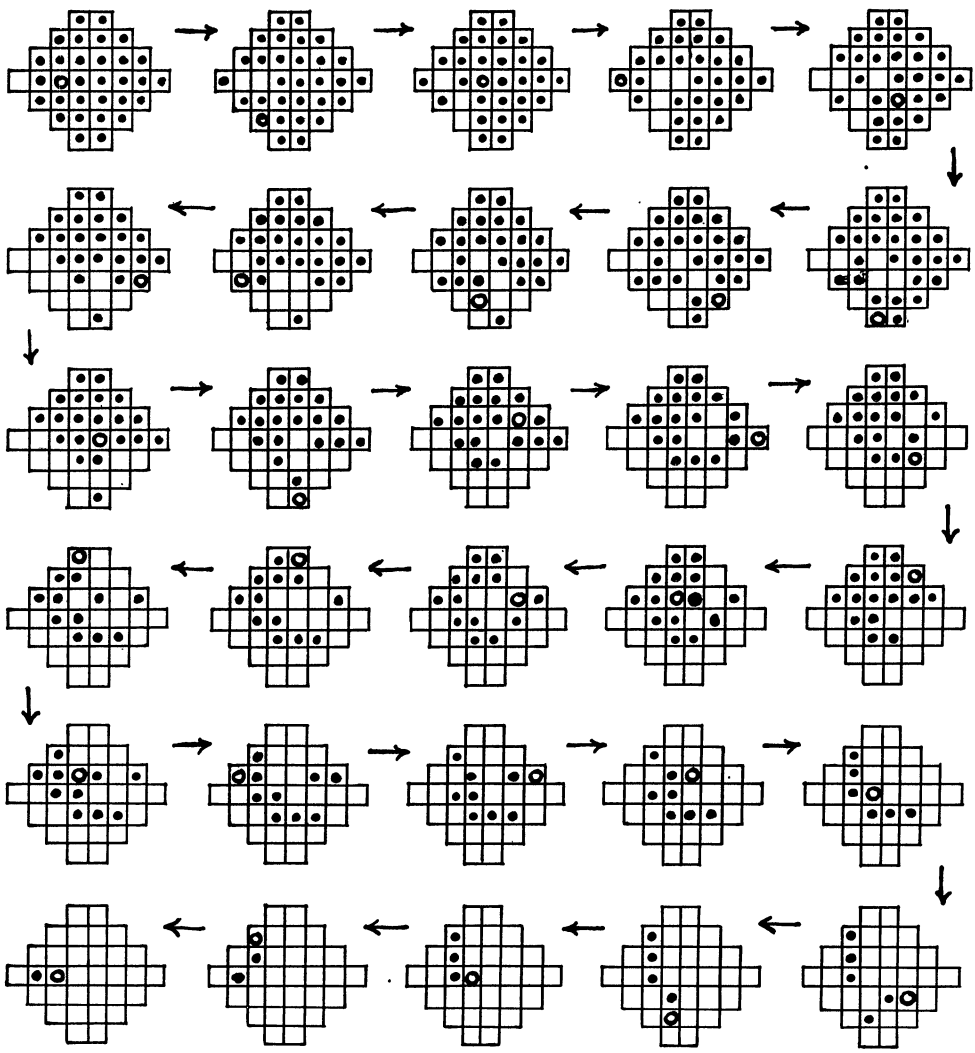

FIG. 5. Begin 14. End 14.

ON UNICURSAL PATHS IN A NETWORK OF DEGREE 4

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Consider a plane network v at all of whose nodes exactly 4 lines meet. By Euler's theorem, it is possible to find unicursal paths in such a network which do not cross themselves, *i.e.*, it is possible to start at any node and trace out a path which takes in every line just once, taking at each node either the right-hand or the left-hand turning, and returning to the original node.

Suppose further that we do not count two such unicursal circuits as different if they differ only in the starting-points, or the sense of description. Then we can calculate the number of unicursal circuits in the form of a determinant, as we shall proceed to show. Let us call this number N and the number of nodes in the network n. If we do make a distinction between different starting-points, and different senses, the number of such non-self-crossing unicursal circuits will be $2n\cdot 2\cdot N=4nN$, since each node is visited twice.

In the network we may give directions to the lines, so that they proceed consistently around each elementary area. This may be done by assigning positive and negative orientations to alternate areas. Let the nodes be numbered arbitrarily $P_1, P_2, \cdots, P_n$. Construct the square matrix (c_{rs}) as follows: let $c_{rr}=2$, 1, or 0 according as the number of lines which merely link the node P_r to itself is 0, 1, or 2, (thus usually $c_{rr}=2$); if $r\neq s$, let $c_{rs}=0$, -1, or -2 according as the number of lines proceeding from node P_r to node P_s is 0, 1, or 2, (thus usually $c_{rs}\neq c_{sr}$). Thus each row and column of (c_{rs}) sums to zero. If $n>1$, all first cofactors are therefore equal; denote their common value by C. If $n=1$ (so that (c_{rs}) has only one element, $c_{11}=0$) we shall put $C=1$.

We propose to show that

$$N = C. \tag{1}$$

We shall proceed by induction. Suppose that (1) is true for all networks of fewer than n nodes. Let P_1 be a node; we may suppose without loss of generality that:

(*Case i*) the four nodes to which it is joined are P_2, P_3, P_4, P_5 in order, or

(*Case ii*) if there is a loop joining P_1 to itself, it is joined to P_2 and P_3.

The line proceeds from P_1 to P_2 in each case. The remaining case is that P_1 is only joined to itself; in this case, relation (1) may be readily proved.

We shall examine the cases in which P_2, P_3, P_4, P_5 are distinct. In other cases the argument proceeds in a very similar manner.

Case i. The matrix (c_{rs}) has the form

$$\begin{array}{c|cccccc}
 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6\cdots \\ \hline
P_1 & 2 & -1 & 0 & -1 & 0 & 0\ \cdots \\
P_2 & 0 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\cdots \\
P_3 & -1 & c_{32} & c_{33} & c_{34} & c_{35} & c_{36}\cdots \\
P_4 & 0 & c_{42} & c_{43} & c_{44} & c_{45} & c_{46}\cdots \\
P_5 & -1 & c_{52} & c_{53} & c_{54} & c_{55} & c_{56}\cdots \\
P_6 & 0 & c_{62} & c_{63} & c_{64} & c_{65} & c_{66}\cdots
\end{array}.$$

From the network v form two others, v' and v'', thus:

For v' remove P_1 and place instead links from P_3 to P_2 and from P_5 to P_4.

For v'' remove P_1 and place instead links from P_5 to P_2 and from P_3 to P_4. (Fig. I.)

Suppose the corresponding matrices and numbers are (c'_{rs}), C', N' and (c''_{rs}), C'', N'', respectively. Then, by the inductive hypothesis, since v' and v'' have only $n-1$ nodes,

$$N' = C', \qquad N'' = C''. \tag{2}$$

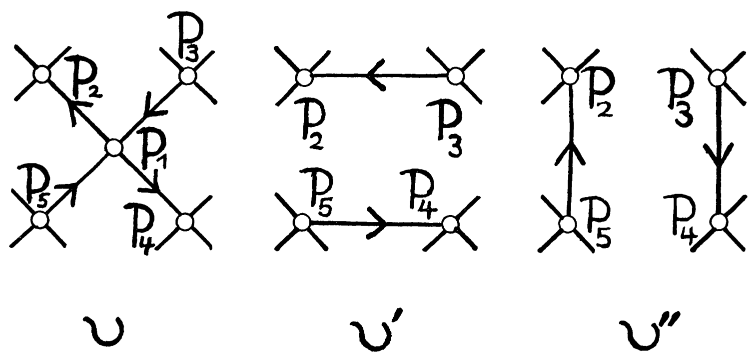

FIG. I

Every non-self-crossing unicursal path in v corresponds either to one in v' or to one in v'', but not both; and every such path in v' and v'' corresponds to one in v. Therefore $N=N'+N''$ and hence

$$N = C' + C''. \tag{3}$$

The matrices are

$$(c'_{rs}) = \begin{array}{c|llll} & P_2 & P_3 & P_4 & P_5 \cdots \\ \hline P_2 & c_{22} & c_{23} & c_{24} & c_{25} \cdots \\ P_3 & c_{32}-1 & c_{33} & c_{34} & c_{35} \cdots \\ P_4 & c_{42} & c_{43} & c_{44} & c_{45} \cdots \\ P_5 & c_{52} & c_{53} & c_{54}-1 & c_{55} \cdots \end{array},$$

$$(c''_{rs}) = \begin{array}{c|llll} & P_2 & P_3 & P_4 & P_5 \cdots \\ \hline P_2 & c_{22} & c_{23} & c_{24} & c_{25} \cdots \\ P_3 & c_{32} & c_{33} & c_{34}-1 & c_{35} \cdots \\ P_4 & c_{42} & c_{43} & c_{44} & c_{45} \cdots \\ P_5 & c_{52}-1 & c_{53} & c_{54} & c_{55} \cdots \end{array},$$

the remaining elements being unaltered. Now C is the cofactor of c_{31} in (c_{rs}). On expanding by the elements of the top row, we have

$$\begin{aligned} C &= \text{cofactor of } c_{34} \text{ in } (c'_{rs}) + \text{cofactor of } c_{32} \text{ in } (c''_{rs}) \\ &= C' + C''. \end{aligned}$$

Hence, by equation (3), $C = N$, as was to be proved.

Case ii. In this case we have the matrix

$$
\begin{array}{c|ccccc}
 & P_1 & P_2 & P_3 & P_4 & \\
\hline
(c_{rs}) = P_1 & 1 & -1 & 0 & 0 & \cdots \\
P_2 & 0 & c_{22} & c_{23} & c_{24} & \cdots \\
P_3 & -1 & c_{32} & c_{33} & c_{34} & \cdots \\
P_4 & 0 & c_{42} & c_{43} & c_{44} & \cdots
\end{array} .
$$

Let v' be the network obtained by removing P_1, and adding a line from P_3 to P_2. (Fig. II.) Let (c_{rs}'), C', N', be the corresponding matrices and numbers.

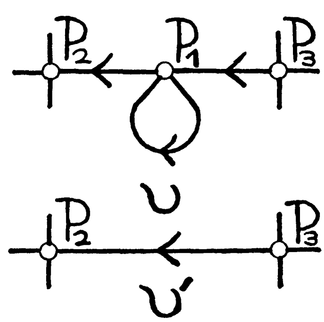

FIG. II

Therefore, since v' has $n-1$ nodes, we have

$$C' = N'. \tag{4}$$

Now to every non-self-crossing unicursal path in v corresponds just one in v', and conversely. Hence,

$$N' = N. \tag{5}$$

Moreover,

$$
\begin{array}{c|ccc}
 & P_2 & P_3 & \\
\hline
(c_{rs}') = P_2 & c_{22} & c_{23} & \cdots \\
P_3 & c_{32} - 1 & c_{33} & \cdots
\end{array} .
$$

But C is the cofactor of c_{31} in (c_{rs}), and on expanding by the top row, we have

$$
\begin{aligned}
C &= \text{cofactor of } c_{32} \text{ in } (c_{rs}') \\
&= C'.
\end{aligned}
$$

Therefore, by equations (4) and (5), $C = N$, which was to be proved.

Hence, if equation (1) holds for all networks of fewer than n nodes, it holds for all networks of n nodes. But when $n = 1$, $C = 1 = N$. Therefore it is true in general.

There is a simple extension of this theorem to any network, plane or non-plane, in which four lines meet at each node. We may assign directions to the lines so that two lines run towards each node, and two away from it. We can then define (c_{rs}) and C as before. Certain unicursal circuits in the network will follow these directions consistently. By the same proof as before, the number of such circuits is C. For the total number of unicursal circuits in the network we sum the quantities C for all possible assignments of directions in the above manmer; we do not distinguish for this purpose between the assignment A and that obtained from A by reversing the direction of every line.

COMPLEX ROOTS OF A POLYNOMIAL EQUATION*

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1. Introduction. Although a graphic interpretation of the complex roots of a cubic polynomial has been given previously by various authors,† it does not seem to have been noticed by any of them that the interpretation given in the case of the cubic is capable of being generalized to the case of a polynomial of any degree. Accordingly the theorems of this paper are believed to be new, and the description given in section 5 of the complex roots of a quartic polynomial with two real roots is also believed to be new.

2. A graphic interpretation of a pair of complex roots. It is well known that if we wish to represent graphically the real roots of a polynomial equation with real coefficients, $F(x) \equiv a_0x^n + \cdots + a_n = 0$, we may plot the graph of $y = F(x)$ on a pair of cartesian axes. If this is done, there corresponds to each real root x_0 of $F(x) = 0$, a point $(x_0, 0)$ in which the curve $y = F(x)$ intersects the x-axis. We shall show that if $F(x) = 0$ has complex roots, this same graph has certain properties which are dependent upon the values of these complex roots.

Suppose then that $n \geqq 2$, and that the complex numbers $a \pm bi$ are roots of $F(x) = 0$.‡ For simplicity we shall let $F(x) = (x^2 - 2ax + a^2 + b^2)f(x)$, where $f(x)$ is a polynomial of degree $n-2$ with real coefficients. We shall next consider some relations between the curve $y = F(x)$ and the curves of the family $y = mf(x)$, where m is a constant.

Each curve of the family $y = mf(x)$ intersects the curve $y = F(x)$ on the x-axis in the $n-2$ real or complex points whose abscissas satisfy the equation $f(x) = 0$,

* Presented to the Upper New York State Section of the Mathematical Association of America at Hamilton, N. Y., May 11, 1940.

† R. E. Gleason, Popular Astronomy, vol. 17, 1909, p. 119. R. E. Gleason, Annals of Mathematics, vol. 11, 1909–10, pp. 95–96. Frank Irwin and H. N. Wright, Annals of Mathematics, vol. 19, 1918, pp. 152–158. E. S. Crawley, this MONTHLY, vol. 25, 1918, pp. 268–269. Garcia Henriquez, this MONTHLY, vol. 42, 1935, pp. 383–384. G. A. Yanosik, National Mathematics Magazine, vol. 10, 1935–36, pp. 139–140.

‡ The discussion may be made to include the case where these two roots are real. This has been done by Gleason, Annals, *loc. cit.*, in the case of the cubic. When the roots are real, the parameter m is negative, b is a pure imaginary, and hence $a \pm bi$ are real numbers.

and also in the two real or complex points $S=(x_1, y_1)$, $T=(x_2, y_2)$, whose abscissas are the roots of $x^2-2ax+a^2+b^2=m$. Let U be the midpoint of the line segment ST. Since $\frac{1}{2}(x_1+x_2)=a$ is independent of m, it follows that the point U is real and lies on the line $x=a$ for any value of m.

If $m<b^2$, the points S and T are complex. But if $m>b^2$, the points S and T are real and distinct. Furthermore, neither of these points is on the x-axis, except when the parameter m has one of that finite set of values for which either x_1 or x_2 is also a root of $f(x)=0$.

While the above remarks hold true for general curves of the family $y=mf(x)$, we are particularly interested in the special curve for which $m=b^2$. In this case, and only in this case, are the roots of $x^2-2ax+a^2+b^2=m$ real and equal. Accordingly the points S and T coincide with the point $R=(a, b^2f(a))$, and the curve $y=b^2f(x)$ is tangent to $y=F(x)$ at that point.

These remarks are summarized in the following theorem. It is easily established that the curve $y=b^2f(x)$ is the only curve of the family $y=mf(x)$ which intersects $y=F(x)$ at a point at which the functions and their derivatives agree in value up to and including the non-vanishing derivative of lowest order, and that the only point of intersection at which $y=b^2f(x)$ and $y=F(x)$ have this property is the point $R=(a, b^2f(a))$.

THEOREM 1. *Let $F(x)=(x^2-2ax+a^2+b^2)f(x)$, where $b\neq 0$, and $f(x)$ is a polynomial with real coefficients. The parameter m may be so chosen that the curve $y=mf(x)$ intersects the curve $y=F(x)$ in two points S and T which are real and not on the x-axis. The abscissa of the midpoint of the line segment ST is a.*

If $f(a)\neq 0$, there is a unique curve of the family $y=mf(x)$ which is tangent to the curve $y=F(x)$ at a real point not on the x-axis. The abscissa of the point of tangency is a, and the parameter m for the tangent curve is b^2.

If $f(a)=0$, but $f(x)=0$ has no real multiple roots except possibly a, there is a unique curve of the family $y=mf(x)$ which is tangent to $y=F(x)$ at a real point on the x-axis. The abscissa of the point of tangency is a, and the parameter m is b^2.

In general, there is a unique curve of the family $y=mf(x)$ which is tangent to the curve $y=F(x)$ at a real point R, having the property that the derivatives of the two functions at the point R are equal from the first to the kth inclusive, where k is the order of the lowest-ordered non-vanishing derivative of $f(x)$ at the point R. The abscissa of the point of tangency is a, and the parameter m for the tangent curve is b^2.

We next consider a few polynomials of low degree with complex roots, and consider the special form which the theorem takes on in each of these cases.

3. Quadratic with two complex roots. Let $F(x)=x^2-2ax+a^2+b^2$. Here $f(x)=1$, and the family of curves is the family $y=m$ of lines parallel to the x-axis. Of these, $y=b^2$ is the only line which is tangent to the parabola $y=F(x)$, the point of tangency being its vertex (a, b^2). Hence the complex roots of a quadratic may be described thus:

The real part is the abscissa of the point of tangency of the horizontal tangent to $y=F(x)$; the coefficient of i is the square root of the distance from the x-axis to the tangent line.

This description is in accord with the vocabulary of Theorem 1. A briefer description would be:

The real part is the abscissa of the vertex of the parabola $y=F(x)$; *the coefficient of* i *is the square root of the ordinate of the vertex.**

4. Cubic with one real and two complex roots. Let $F(x)=(x^2-2ax+a^2+b^2)(x-r)$. Here $f(x)=x-r$, and the family of curves is the family $y=m(x-r)$ of lines passing through the point $(r, 0)$ which corresponds to the real root of $F(x)=0$. Of this family of lines, just one is tangent to the cubic curve $y=F(x)$. If $r=a$, the point of tangency is $(r, 0)$, which is in this case the inflection point of the cubic. The complex roots of a cubic may be described thus:

The real part is the abscissa of the point of tangency of the tangent line passing through the real intersection of the curve $y=F(x)$ *with the x-axis; the coefficient of* i *is the square root of the slope of the tangent line.*†

5. Quartic with two real and two complex roots. Let $F(x)=(x^2-2ax+a^2+b^2)(x-r_1)(x-r_2)$. Here $f(x)=(x-r_1)(x-r_2)$, and the family of curves is the family $y=m(x-r_1)(x-r_2)$ of parabolas passing through the points $(r_1, 0)$, $(r_2, 0)$ which correspond to the real roots of $F(x)=0$. If $r_1=r_2$, the parabolas are tangent to the x-axis at the point $(r_1, 0)$. Of this family of parabolas, just one is tangent to the quartic curve $y=F(x)$. It is either (1) an ordinary tangent at a point R not on the x-axis, if $a\neq r_1$, $a\neq r_2$; or (2) an inflection tangent with three-point contact at $(r_1, 0)$, if $a=r_1\neq r_2$; or (3) a tangent with four-point contact at $(r_1, 0)$, if $a=r_1=r_2$. The complex roots of a quartic with two real roots may be described thus:

The real part is the abscissa of the point of tangency of the tangent parabola passing through the real intersections of the curve $y=F(x)$ *with the x-axis; the coefficient of* i *is the square root of the reciprocal of the latus rectum of the tangent parabola.*‡

6. Note on a curve suggested by C. F. Barr. In Theorem 1, the number a is described as a length, but b is the square root of a parameter. The result given next enables us to describe b also as a length.

Let us consider the relation between the curve $y=F(x)$ and the curve $y=2b^2f(x)$ of the family $y=mf(x)$.§ This curve is the curve of the family which passes through the point $(a, 2b^2f(a))$, whose abscissa is equal to that of R, but whose ordinate is double that of R.

THEOREM 2. *The curve* $y=2b^2f(x)$ *of the family* $y=mf(x)$ *intersects the curve* $y=F(x)$ *in points* S *and* T *whose abscissas are* $a-b$ *and* $a+b$, *respectively.*

This fact may be used in connection with the results of Sections 3, 4, and 5 to give a new description of the coefficient of i in each case. This new description is particularly useful in case $f(a)\neq 0$.

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* Gleason, Popular Astronomy, *loc. cit.*; Irwin and Wright, *loc. cit.*

† *Cf.* the six references given in section 1.

‡ An entirely different description is given by Irwin and Wright, *loc. cit.*, in which various lines are drawn in place of the parabola used here.

§ The use of this curve was suggested by a note of C. F. Barr on the paper by Henriquez, *loc. cit.*

A NOTE ON THE LINEAR DIOPHANTINE EQUATION

D. H. LEHMER, University of California

A traditional non-tentative method of solving in integers (x, y) the equation

$$a_0x - a_1y = K, \tag{1}$$

where we may suppose that $0<a_1<a_0$ and that a_0 and a_1 are coprime, depends on the continued fraction development of the fraction a_0/a_1, a method which was first presented in a systematic way by Lagrange* in 1767. Lagrange's method, it will be recalled, gives a solution of

$$a_0x - a_1y = 1 \tag{2}$$

from which, as is well known, all solutions of (1) are readily obtainable. He begins by applying the Euclid algorithm as follows:

$$\begin{array}{ll} a_0 = q_1a_1 + a_2, & 0 < a_2 < a_1, \\ a_1 = q_2a_2 + a_3, & 0 < a_3 < a_2, \\ a_2 = q_3a_3 + a_4, & 0 < a_4 < a_3, \\ \cdots\cdots & \cdots\cdots \\ a_{n-2} = q_{n-1}a_{n-1} + a_n, & 0 < a_n < a_{n-1}, \\ a_{n-1} = q_na_n, & \end{array} \tag{3}$$

where, as a matter of fact,

$$a_n = 1, \tag{4}$$

since a_0 and a_1 are coprime. This gives us the continued fraction

$$a_0/a_1 = q_1 + \frac{1}{q_2 +}\,\frac{1}{q_3 +}\cdots\frac{1}{+\, q_n}. \tag{5}$$

The two sequences A_ν and B_ν, the numerators and denominators of the successive convergents to (5), are next computed by familiar recurrence formulas:

$$\begin{array}{llll} A_{-1} = 1, & A_0 = q_1, & A_1 = q_2q_1 + 1, & A_\nu = q_{\nu+1}A_{\nu-1} + A_{\nu-2}, \\ B_{-1} = 0, & B_0 = 1, & B_1 = q_2, & B_\nu = q_{\nu+1}B_{\nu-1} + B_{\nu-2}. \end{array} \tag{6}$$

Then, by the theory of continued fractions,

$$A_{n-1} = a_0, \qquad B_{n-1} = a_1, \qquad A_{n-1}B_{n-2} - B_{n-1}A_{n-2} = (-1)^n. \tag{7}$$

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* Oeuvres de Lagrange, vol. 2, 1868, pp. 386–388. For accounts of the many papers on this subject, see Dickson's History of the Theory of Numbers, vol. 2, chap. II.

This exhibits a solution

$$x = (-1)^n B_{n-2}, \qquad y = (-1)^n A_{n-2} \tag{8}$$

of equation (2).

Because equation (7) depends upon continued fractions, this very practical method of Lagrange is frequently omitted from text-books on the theory of numbers. In some texts the method is presented without actually mentioning continued fractions, although requiring the calculation of the two sequences (6).

It is the purpose of this note to point out that Lagrange's method is twice as laborious as necessary, and that the continued fraction (5) is not the best continued fraction to use, if indeed one is to use the notion of continued fractions at all. In fact, a better continued fraction is obtained by reversing* the order of the q's.

The ordinary "binary" continued fraction algorithm has been generalized by Jacobi,† and a non-tentative method closely paralleling that of Lagrange for solving the general equation

$$a_0x_0 + a_1x_1 + \cdots + a_{m-1}x_{m-1} = K \tag{9}$$

based on Jacobi's m-ary continued fraction has been described and illustrated by D. N. Lehmer.‡ In this case also the algorithm is improved by a reversal of its "partial quotient sets," the analogs of the q's in (5). Here the saving in labor is only one mth part, since it is necessary to compute $m-1$ sequences instead of m sequences if one is to obtain the general solution of (9).

The algorithm described in what follows is presented without reference to continued fractions of the ordinary or higher type, and is based on the simplest properties of determinants. We consider the important binary case separately; the general case is illustrated by solving (9) when $m=4$.

THEOREM 1. *The equation* (2) *has as a solution*

$$x = (-1)^n C_{n-1}, \qquad y = (-1)^n C_n, \tag{10}$$

where the C's are defined by

$$C_0 = 0, \qquad C_1 = 1, \qquad C_{k+1} = q_{n-k}C_k + C_{k-1}, \tag{11}$$

the q's being given by the Euclid algorithm (3).

Proof. In view of (4) and (11) we have

$$\begin{vmatrix} a_n & C_0 \\ a_{n-1} & C_1 \end{vmatrix} = 1.$$

* This reversal of order was advocated by Gauss in his Disquisitiones Arithmeticae, §27, 28. That his suggestion has not been generally adopted is perhaps due to the fact that he gives no reason for his variant of Lagrange's method, but remarks that the two methods are equivalent.

† Gesammelte Werke, vol. 6, Berlin, 1891, pp. 355–384.

‡ Proceedings of the National Academy of Sciences, vol. 5, 1919, pp. 111–114.

If we add to each element of the top row the product by q_{n-1} of the corresponding element of the bottom row, and then interchange rows, we obtain from (3) and (11), with $k=1$,

$$\begin{vmatrix} a_{n-1} & C_1 \\ a_{n-2} & C_2 \end{vmatrix} = -1.$$

If we now multiply the bottom row elements by q_{n-2} and add to the elements of the top row, and then interchange rows, we obtain from (3) and (11), with $k=2$,

$$\begin{vmatrix} a_{n-2} & C_2 \\ a_{n-3} & C_3 \end{vmatrix} = +1.$$

Continuing the process, we obtain finally

$$\begin{vmatrix} a_1 & C_{n-1} \\ a_0 & C_n \end{vmatrix} = (-1)^{n-1}.$$

Expanding the determinant we see that (10) is indeed a solution of (2).

It is worth pointing out that the inequalities in (3) are not at all necessary, nor need we restrict the q's to be positive. All that we need is a set of equations of the type (3) in which the sequence of a's finally terminates with* $|a_n|=1$, the sooner the better. In particular, one may use the "least remainder algorithm" in which each q_k is chosen to make $|a_{k+2}|$ a minimum. If one uses any standard computing machine, and always retains the two latest a's in different parts of the product register, only the q's (which appear in the upper dials) need be written down. In forming the C's from (11) in the same manner, nothing need be written down except the answers C_{n-1} and C_n. If the a_0 and a_1 are exceedingly large, as sometimes happens in practice, recourse may be had to the methods described in a recent paper.† The amount of work varies as the logarithm of a_1.

The following example indicates how the work may be laid out in case no machine is used.

To find the solution of $23134x-15257y=1$. We have

		k	q_{7-k}	C_k
(12)	$23134 = 2\cdot 15257 - 7380$	1	3	1
	$15257 = -2\cdot(-7380) + 497$	2	-3	3
	$-7380 = -15\cdot 497 + 75$	3	7	-8
	$497 = 7\cdot 75 - 28$	4	-15	-53
	$75 = -3\cdot(-28) - 9$	5	-2	787
	$-28 = 3\cdot(-9) - 1$	6	2	-1627
		7	—	-2467

* In case $a_n=-1$, the numbers (10) give a solution of $a_0x-a_1y=-1$.

† This MONTHLY, vol. 45, pp. 227–233.

Since $a_7 = -1$, $n=7$, a solution of (12) is

$$x = C_6 = -1627,$$
$$y = C_7 = -2467.$$

If we wish to find all solutions, in a non-tentative way, of the indeterminate equation

$$a_0x + a_1y + a_2z + a_3w = K, \tag{13}$$

in which we can assume that the a's have no common factor, we proceed to write down the analogs of equation (3) as follows:

$$\begin{aligned} a_0 &= q_1a_1 + r_1a_2 + s_1a_3 + a_4, \\ a_1 &= q_2a_2 + r_2a_3 + s_2a_4 + a_5, \\ a_2 &= q_3a_3 + r_3a_4 + s_3a_5 + a_6, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{n-4} &= q_{n-3}a_{n-3} + r_{n-3}a_{n-2} + s_{n-3}a_{n-1} + a_n, \end{aligned} \tag{14}$$

where

$$a_n = (-1)^{\mu}. \tag{15}$$

The partial quotient sets (q_i, r_i, s_i) are selected in any way so that $a_4, a_5, \cdots, a_n$ decrease in absolute value, the more rapidly the better. For example, we can choose q_i, r_i, and s_i to be the nearest integers to

$$a_{i-1}/a_i, \qquad (a_{i-1} - q_ia_i)/a_{i+1}, \qquad (a_{i-1} - q_ia_i - r_ia_{i+1})/a_{i+2},$$

respectively. There is no need of avoiding zero values of q_i, r_i, or s_i.

Having obtained a set of equations (14), the complete solution of (13) is given by the following:

THEOREM 2. *Compute three sequences A_ν, B_ν, and C_ν by means of the recurrence relations and initial values*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \end{pmatrix}, \quad \begin{aligned} A_{k+1} &= q_{n-k}A_k + r_{n-k}A_{k-1} + s_{n-k}A_{k-2} + A_{k-3}, \\ B_{k+1} &= q_{n-k}B_k + r_{n-k}B_{k-1} + s_{n-k}B_{k-2} + B_{k-3}, \\ C_{k+1} &= q_{n-k}C_k + r_{n-k}C_{k-1} + s_{n-k}C_{k-2} + C_{k-3}. \end{aligned} \tag{16}$$

Then all solutions of (13), *and no sets* (x, y, z, w) *that are not such solutions, are given by*

$$\begin{aligned} x &= (-1)^{n+\mu-1}(K\alpha_{41} + L\alpha_{42} + M\alpha_{43} + N\alpha_{44}), \\ y &= (-1)^{n+\mu-1}(K\alpha_{31} + L\alpha_{32} + M\alpha_{33} + N\alpha_{34}), \\ z &= (-1)^{n+\mu-1}(K\alpha_{21} + L\alpha_{22} + M\alpha_{23} + N\alpha_{24}), \\ w &= (-1)^{n+\mu-1}(K\alpha_{11} + L\alpha_{12} + M\alpha_{13} + N\alpha_{14}), \end{aligned} \tag{17}$$

in which L, M, N range independently over all integers, and where α_{ij} is the cofactor of the element in the ith row and jth column of the determinant

$$D = \begin{vmatrix} a_3 & A_{n-3} & B_{n-3} & C_{n-3} \\ a_2 & A_{n-2} & B_{n-2} & C_{n-2} \\ a_1 & A_{n-1} & B_{n-1} & C_{n-1} \\ a_0 & A_n & B_n & C_n \end{vmatrix}. \tag{18}$$

Proof. Consider first the determinant

$$\begin{vmatrix} a_n & A_0 & B_0 & C_0 \\ a_{n-1} & A_1 & B_1 & C_1 \\ a_{n-2} & A_2 & B_2 & C_2 \\ a_{n-3} & A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_n & 0 & 0 & 0 \\ a_{n-1} & 1 & 0 & 0 \\ a_{n-2} & 0 & 1 & 0 \\ a_{n-3} & 0 & 0 & 1 \end{vmatrix} = a_n. \tag{19}$$

Let us now add to the elements of the first row the corresponding elements of the second, third, and fourth rows multiplied respectively by s_{n-3}, r_{n-3}, and q_{n-3}. In view of (14) and (16) with $k=3$ we obtain, after a cyclic permutation* of the rows,

$$\begin{vmatrix} a_{n-1} & A_1 & B_1 & C_1 \\ a_{n-2} & A_2 & B_2 & C_2 \\ a_{n-3} & A_3 & B_3 & C_3 \\ a_{n-4} & A_4 & B_4 & C_4 \end{vmatrix} = - a_n,$$

a determinant which differs from (19) in that the subscripts of the elements in the first column have been decreased by one, while the subscripts of all the other elements have been increased by unity. To continue the process, we introduce now the multipliers s_{n-4}, r_{n-4}, and q_{n-4}, and again permute the rows as before. After $n-3$ such applications we obtain the determinant (18) which is then seen to have the value

$$D = (-1)^{n-3}a_n = (-1)^{n+\mu-1}.$$

If now we multiply equations (17) by a_0, a_1, a_2, a_3 and add, we find

$$a_0x + a_1y + a_2z + a_3w = (-1)^{n+\mu-1}KD = K,$$

which follows from the expansion of D by cofactors of the elements in its first column, and from the well known theorem that the sum of the products of the cofactors of the elements of one column by the corresponding elements of some other column is zero.

Conversely, if (x, y, z, w) is any solution of (13), the system of four equations (17) may then be solved for the integers K, L, M, N, since the determinant of

* In case the number m of variables in (9) is odd, this permutation does not change the sign of the determinant.

the coefficients $|\alpha_{ij}|$, being the adjoint of D, and hence equal to

$$D^3 = (-1)^{n+\mu-1},$$

has the absolute value unity. Thus Theorem 2 is proved.

To illustrate the application of the algorithm, we consider the indeterminate equation discussed by D. N. Lehmer,*

$$99x + 79y + 55z + 33w = K. \tag{20}$$

The equations (14) in this case may be written†

$$99 = 1\cdot 79 + 0\cdot 55 + 1\cdot 33 - 13,$$
$$79 = 0\cdot 55 + 0\cdot 33 - 6\cdot(-13) + 1.$$

Hence $n=5$, and $\mu=0$. The A's, B's, and C's may be expeditiously calculated from the following scheme, using (16):

k	q_{5-k}	r_{5-k}	s_{5-k}	A_k	B_k	C_k
0	—	—	—	0	0	0
1	—	—	—	1	0	0
2	—	—	—	0	1	0
3	0	0	−5	0	0	1
4	1	0	1	−6	0	0
5				−5	1	0

The determinant (18) is therefore

$$\begin{vmatrix} 33 & 0 & 1 & 0 \\ 55 & 0 & 0 & 1 \\ 79 & -6 & 0 & 0 \\ 99 & -5 & 1 & 0 \end{vmatrix} = 1.$$

Evaluating the cofactors and applying Theorem 2, we have as the general solution of (20),

$$\begin{aligned} x &= 6K + 79L - 198M - 330N, \\ y &= -5K - 66L + 165M + 275N, \\ z &= N, \\ w &= -6K - 79L + 199M + 330N, \end{aligned} \tag{21}$$

where L, M, and N are arbitrary integers.

As was to be expected, this solution is not identical with that of D. N. Lehmer,‡ which in our notation is

* *Loc. cit.*, pp. 113–114. Here we have changed his notation slightly.

† These equations illustrate the advantage of having a free choice of partial quotient sets (q_i, r_i, s_i). Using Jacobi's algorithm, D. N. Lehmer's solution contains 10 partial quotient sets.

‡ His solution is actually incorrect; in fact, the sign of each variable should be changed.

$$
(22) \qquad \begin{aligned}
x &= -4K + 8S - 7T + 9U,\\
y &= -5K + 11S - 11T + 11U,\\
z &= -3K + 4S - 4T + 7U,\\
w &= 29K - 57S + 54T - 65U.
\end{aligned}
$$

The two solutions are equivalent, however. In fact, (21) becomes (22) on making the substitutions

$$
\begin{aligned}
L &= 50K - 106S + 101T - 111U,\\
M &= 25K - 49S + 47T - 56U,\\
N &= -3K + 4S - 4T + 7U.
\end{aligned}
$$

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

A COURSE IN FRESHMAN MATHEMATICS

J. H. ZANT, Oklahoma A. and M. College

For the past four years the mathematics department of the Oklahoma Agricultural and Mechanical College has been giving a course entitled *Practical and Cultural Mathematics.** It is required of all Arts and Sciences students who do not major in the physical sciences or music. The class consists of about 125 students per year.

Definite and specific aims used by members of the department in building the course might be enlightening. Primarily we wanted the students to get the true significance of the development of mathematics and its meaning in modern thought and life. To accomplish this we used two types of subject-matter; one sort showing the historical development of certain phases of mathematics, as, for example, the systems of numerals used by the Egyptians, the Babylonians, and the Mayas, and finally the Hindu-Arabic system. The other sort of material deals with the fundamental meaning of certain elementary mathematical concepts, as, for example, the meaning of number, the fundamental operations on numbers, elementary geometry, and the like. All of these were developed from the logical point of view. We attempted to make the whole course historically

* This course was organized by the mathematics department at the request of the Curriculum Committee and the original aim of the course was "to introduce the student to the fundamentally useful principles of everyday commercial mathematics and to present comparatively brief discussions of the history and meaning of the simpler processes of elementary mathematics, as well as the history of some of the important movements and episodes in the development of the science." (A Liberal College in the Making, Bulletin, Oklahoma A. and M. College, vol. 34, no. 11, p. 18.)

and logically sound as well as simple enough to be grasped by beginning students of the type described above.

We also wanted the class to acquire certain knowledge which is described as "practical mathematics." In this section of the course we discussed compound interest and its applications to annuities, insurance, stocks and bonds, and the like. Here again, less stress was put on numerical calculation than on the fundamental meaning, though we tried to include enough practice to give the student a background for understanding the processes.

The class was required to attend two lectures a week and two discussion periods. The lectures were given to the entire group and were based on a manuscript prepared by two members of the department and mimeographed for the students. The discussion groups contained about twenty students each, and the time was devoted to the difficult parts of the lectures and drill work on the various topics.

The results were of course varied. The class was not divided according to ability, and naturally varied widely in ability and previous preparation. After the close of the 1939–40 school year we sent questionnaires to the students who took the course in an attempt to get their reactions to its various phases. Considerable interest was expressed in the history of numbers, the nature of a mathematical system, and compound interest and annuities. The students also suggested in large numbers that the topics of compound interest, life insurance and annuities be developed more fully than they had been this year. Many students mentioned a new idea of the meaning of mathematics and numbers as one of the outstanding values they received from the course.

It is of course doubtful whether all students can profit by exposure to logical discussions of fundamental mathematical processes. Certainly not all of them have done so in these classes. However, with time, we hope to be able to improve our subject-matter and methods so that all may get something out of it. Many do get an entirely new idea of the meaning and purpose of mathematics. The deciding issue here does not seem to be whether they get a highly developed idea of the foundations of mathematics, but whether they get something that is more valuable than that which they would get from some other sort of mathematics course. The majority of the books on "General Mathematics" seem to assume that the students should practice on a variety of skill problems with no more attention to the fundamental meaning of what they are doing or what its values are than is given in the most stereotyped elementary algebra course. For example, in one such book will be found all the rules for operating with positive and negative numbers, but the reader is left entirely in the dark as to why these rules are used. Our contention is that the well educated college student should know that mathematics is a well organized logical body of knowledge with a rich past and a brilliant future. They should not continue to believe that it consists of a mass of non-understandable rules and tricks, valuable perhaps to scholars in certain fields, but meaningless and useless to the man on the street.

MATHEMATICS FOR PROSPECTIVE TEACHERS IN ELEMENTARY SCHOOLS

Ralph Mansfield, Chicago Teachers College

Courses for prospective teachers of secondary school mathematics have been discussed frequently in this journal. The present paper is concerned with a course in mathematics for prospective teachers in the elementary grades. This is not a course in methods of teaching, but one which has been designed to acquaint prospective teachers with the methods, applications, and social values of mathematics.*

Since this course has been designed for students who do not intend to take a major sequence in mathematics, the only prerequisites are the usual one and one-half units of high school algebra and one unit of high school geometry. No text is employed, but an extensive bibliography has been compiled for the use of students who may wish to carry on independent investigations into some of the topics discussed.

The course opens with a discussion of some limitations existing when the results of theoretical mathematics are applied to physical situations. For instance, in high school geometry, the student learns t' at a point has no dimensions, that a line has only one dimension, and that through two points only one line can be drawn. However, every point that the student has ever drawn certainly is of measurable size, even though he may require the aid of a magnifying lens to discern the dimensions of it. Also, every physical line he has seen certainly has a detectable width. And, if the student uses a pencil with a very fine point, with the aid of a magnifying lens, he may be able to draw several lines through the same pair of points.

Again, the student may have learned that in elementary algebra the commutative law holds for certain operations, but perhaps he has never stopped to examine a case where the commutative law does not hold. It does not require any great amount of mathematical knowledge on the part of the student to find out that in considering the class of positive integers, subtraction and division are not always possible and that the commutative law does not apply to these two operations. Proceeding from this introduction, the structure of the number system reveals to the student properties of numbers which were too subtle for his earlier comprehension.

Once the student has been led to consider these questions as an introduction to the methods of mathematics (the above statements do not do full justice to the various questions falling under this topic), he is asked to consider some things of social significance in which mathematics plays a rôle.

The next topic of study examines the evolution of our present concept of the physical universe. The student is shown that the mathematical theories underlying these concepts are valid only insofar as actually observable phenomena fit in with the theory. This does not preclude the fact that it may be possible to

* This course has been in operation for two years now and the results have been very gratifying to the students and faculty of Chicago Teachers College.

develop another theory which may fit the observable phenomena in far better fashion, and if such a theory is developed it will displace the older ones.

The development of our accepted theory of the solar system illustrates the general situation. A study of the theory of the solar system brings the student to the conclusion that, insofar as observed phenomena are concerned, the Ptolemaic and the Copernican systems are equally valid. However, from the standpoint of mathematical interpretation, the Copernican system is easily seen to give a better working hypothesis than previously accepted systems. In this connection, an examination of Newton's work proves very valuable. It is pointed out that Newton's Law of Universal Gravitation must be superseded by Einstein's Theory of Relativity when we come to consider the inequalities in the orbital motion of Mercury. Space does not permit a complete discussion of all the work done with this topic, but any college instructor who has studied astronomy will be able to fill in the details for himself.

Next, the student is shown how man has been able to determine the size of the Earth (Eratosthenes' method of measuring the Earth), and how he locates points on the surface of the Earth by determining their latitude and longitude. This involves the construction of a coördinate system on a sphere (and, digressing, many students are at first amazed to learn that a triangle may contain more than 180 degrees, but this amazement turns to a keener appreciation of the applicability of euclidean geometry to a non-euclidean world). It is interesting to point out, in connection with this topic, how the ancients were able to determine latitude by the use of the gnomon, and also by the determination of the angle of elevation of the pole star. The use of the sextant, chronometer, and ephemeris are demonstrated so that the student may better appreciate the mathematical problems of navigation. Again we curtail this discussion, leaving it to readers to supply details.

At this point the student has not only oriented the Earth in space, but he has also oriented himself with respect to the Earth and he is now prepared to examine some of the applications of mathematics to the daily activities of man on his planet. Naturally, many possibilities present themselves. Extensions of the preceding work may be examined here or postponed somewhat. The policy I have followed is first to examine man's commercial activity, postponing his arts and sciences to a later stage.

The uses of statistics in many social endeavors are pointed out—*e.g.*, polls of public opinion, population problems, cost of living, *etc.* The considerations of percentage, simple and compound interest are brought in at this point. A very good topic in connection with this is the study of the mathematical principles of life insurance, since from the work mentioned in the first sentence of this paragraph the student may easily be led to the statistical construction of mortality tables. The study of insurance proves to be full of interesting social situations, especially if one stops to point out some of the problems connected with teachers' pension systems and social security. Not only does the student learn to discount bills, invest principal at different rates of interest to obtain desired

amounts, and determine the probability of winning on chance events, but he also learns to compute the premiums on the various types of insurance he carries and why some forms cost more than others.

Following the discussion of financial problems comes cultural activity, art and music. Many excellent sources of information are now available in these fields, and the students are urged to make their acquaintance. Perspective, and dynamic symmetry are brought into the course in this connection, and the student is introduced to some basic art forms. The study of music is one that abounds with mathematical relationships, most of which fall within the students' limited mathematical abilities. The structure of major and minor scales, the ratios of the fifth, third, fourth, *etc.*, prove to be of more than passing interest to the student.

The nature of sound waves and their transmission, studied in connection with music leads to the final consideration of the course, mathematics in science. Here mathematics is applied first to the study of sound; and sound waves lead to questions about light waves. Then we introduce some ideas about electron theory and quantum mechanics, remembering always that we do not wish to steep the student in mathematics or physics, but only to give him an appreciation for mathematics. Finally, the student is asked to conceive of society in non-euclidean space, in communities where the velocity of light, the gravitational constant, and the quantum constant have been altered from our accepted values of these quantities,* and to compare such societies with our present society.

This type of course is exceedingly rich in what I have termed "the social aspects of mathematics." There is more than enough material to satisfy the students' need for an orientation course in the use and interpretation of mathematics—and few people can doubt the beneficial effect of such training upon persons who are going out to teach, for it enriches their knowledge and leads them out of the dangerous rut from which similar prospective teachers have looked out in the past and asked, "Why study mathematics?"

* Suggested by Gamow's little book, Mr. Tompkins in Wonderland, Macmillan, 1940.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON PLACING THE LAST DIGIT FIRST

W. B. CHADWICK, Tower Hill School

Our problem is to determine the number with the fewest digits such that when the last digit is placed first the new number is k times the old, k being an integer between 1 and 10.

Let N be such a number of n digits, d being the last digit. Then, by the conditions of the problem,

$$10^{n-1}d + (N - d)/10 = kN, \tag{1}$$

$$10^n d + N - d = 10kN,$$

$$(10k - 1)N = (10^n - 1)d,$$

$$N = (10^n - 1)d/(10k - 1). \tag{2}$$

We shall now show that $d \geqq k$. We have, in succession,

$$N > 10^{n-1},$$

$$(10^n - 1)d > 10^n k - 10^{n-1}, \tag{3}$$

$$10^n d > 10^n k - 10^{n-1}, \tag{4}$$

$$d > k - .1,$$

$$d \geqq k. \tag{5}$$

Now the steps leading to (2) and (5) are reversible; that (4) implies (3) follows from the fact that the left-hand side of (4) must exceed the right-hand side by at least 10^{n-1}; and so (2) and (5) imply (1). Moreover, any integer N defined by (2) is necessarily less than 10^n, since $d < 10 < 10k-1$, and has d for its last integer. Therefore the least integer defined by (2) with d and k satisfying (5) is a solution of our problem.

Since $10k-1 > d$, it follows from (2) that if $10k-1$ is a prime it must divide 10^n-1. Thus in this case the smallest value of N will be obtained by choosing n to be the smallest integer such that 10^n-1 is divisible by $10k-1$. Larger values of N, of n' digits, can be obtained only by choosing n' as a multiple of n.

If $10k-1$ is composite, some of its factors may divide d and the remaining ones divide 10^n-1. If $k=4$, we have $10k-1=39=3\cdot 13$. But if 10^n-1, being a number all of whose digits are 9's, is divisible by 13 it is divisible by 39, so this particular case is not different from the preceding. Similarly, when $k=7$, if 10^n-1 is divisible by 23 it is divisible by 69. When $k=5$, we have $10k-1=49=7\cdot 7$. If $d \neq 7$ the smallest possible value of n is 42, while if $d=7$ then n may be 6. Divi-

sion of 10^6-1 by 7 gives 142857, whereas the division of $10^{42}-1$ by 49, then multiplication of the result by 7, gives a number of 42 digits in which the six figures 142857 are repeated in that order.

Let us illustrate the theory by finding the numbers for $k=2$ by two methods, and those for $k=4$ by a third method.

Method I. We have $d(10^n-1)/(10k-1)=d(10^n-1)/19$. The value of n is not known (see Method III for ways of finding the value of n), but $999\cdots$ is divided by 19 until the division is exact. The quotient thus found is 52631578947368421. Now $d\geqq 2$; multiplication of this quotient by the eight integers from 2 to 9 inclusive gives the required numbers for $k=2$.

Method II. This method is a variation of Method I. Since the only difference when 10^n, instead of 10^n-1, is divided by $10k-1$ is that each remainder is increased by one (*e.g.*, when 10^6-1 is divided by 39, the remainders in order are 9, 21, 24, 15, 3, 0, and when 10^6 is divided by 39, the corresponding remainders are 10, 22, 25, 16, 4, 1), we may use the figures in the repeating decimal obtained by the division of 1 by $10k-1$. When $k=2$, these figures are .052631578947368421. Since it is a well known fact that when 1 is divided by 19 all 18 possible remainders occur, the figures of this decimal may be multiplied by d by inspection. This shows that some of our numbers will be of a periodic nature. (See *Note* below.)

Method III. When $k=2$, we have $10k-1=19$ and $n=18$. The value of n is the same as the number of remainders in the division of 1 by 19 before the remainders start to repeat. Thus a knowledge of the decimal form of the fractions $1/(10k-1)$, as well as 1/13, 1/7, 1/23, will give us our values of n. These results are now given in a table.

k	2	3	4	5	6	7	8	9
$10k-1$	19	29	39	49	59	69	79	89
n	18	28	6	42 or 6	58	22	13	44

The values for n agree with the results which are possible according to the simple Fermat theorem and Fermat's general theorem. Thus when $10k-1$ is a prime, n must be either $10k-2$ or a factor thereof. According to the general theorem, when $k=4$, $10^{24}-1$ is divisible by 39; when $k=7$, $10^{44}-1$ is divisible by 69. But 10 is not a primitive root of 39 or 69 since neither of these two numbers is in the form 1, 2, 4, p^a, $2p^a$, where p is an odd prime. Thus when $k=4$, n is a factor of 24, and when $k=7$, n is a factor of 44. When $k=5$, so that $10k-1=49=p^2$, n must equal $p(p-1)=42$ (or, when $d=p=7$, n must be a factor of 42). See R. D. Carmichael, *The Theory of Numbers*, pp. 30, 48, 52, 62, 63, 65, 71.

Now

$$\frac{10^n-1}{10k-1}=\frac{10^{n-1}}{k}+\frac{10^{n-2}}{k^2}+\cdots+\frac{10}{k^{n-1}}+\frac{1}{k^n}-\frac{1-1/k^n}{10k-1}\cdot \tag{6}$$

When $k=4$, $n=6$. Substituting these values in the first four terms of (6) and adding, we obtain 25641.015625. When the addition of further terms causes no change in the figures to the left of the decimal point, the figures to the right of

the decimal point may be neglected. The number 25641 multiplied by $d \geqq 4$ gives the 6 numbers listed under $k=4$.

There is now given the complete set of required numbers.

$k=2$, (a) 105263157894736842, (b) $157\cdots$, (c) $210\cdots$, (d) $263\cdots$, (e) $315\cdots$, (f) $368\cdots$, (g) $421\cdots$, (h) $473\cdots$.
$k=3$; (a) 1034482758620689655172413793, (b) $137\cdots$, (c) $172\cdots$, (d) $206\cdots$, (e) $241\cdots$, (f) $275\cdots$, (g) $310\cdots$.
$k=4$, (a) 102564, (b) 128205, (c) 153846, (d) 179487, (e) 205128, (f) 230769.
$k=5$, (a) 102040816326530612244897959183673469387755, (b) $122\cdots$, (c) 142857, (d) $163\cdots$, (e) $183\cdots$.
$k=6$, (a) 1016949152542372881355932203389830508474576271186440677966, (b) $118\cdots$, (c) $135\cdots$, (d) $152\cdots$.
$k=7$, (a) 1014492753623188405797, (b) 1159420289855072463768, (c) 1304347826086956521739.
$k=8$, (a) 1012658227848, (b) 1139240506329.
$k=9$, (a) 10112359550561797752808988764044943820224719.

Note. The numbers that are not fully given in the above table are cyclic permutations of the first number in the same group. The reason for this, as well as many other properties of these "cyclic numbers," will be found in an article by Salomon Guttman, *On cyclic numbers*, this MONTHLY, vol. 41, 1934, p. 159.

These results can easily be extended. We might ask for the smallest numbers such that when the last m digits are placed first, the cyclic order of the digits being maintained, the new number is k times the old. The same methods work here also. Equations (2) and (5) become

$$N = (10^n - 1)M/(10^m k - 1), \qquad M \geqq 10^{m-1}k,$$

where M is the number formed by the last m digits of N. For example, if $m=2$ and $k=3$ we can find such numbers whose last two digits form any number from 30 to 99. For the last two digits 46 we have

$$N = (10^n - 1)46/299 = (10^n - 1)2/13 = 153846, \qquad n = 6.$$

Note by the Editor. Chadwick's results also apply to numbers expressed in another scale of notation. As a matter of fact, since the symbol 10 always indicates the base of the system of notation, the steps leading to (2) and (5) are true in any system, and so the general principles are precisely the same. As an example, consider the case for $k=3$ in the base 7. Supposing N to end in 4, we have, in the scale of 7,

$$N = (10^n - 1)4/(30 - 1) = (10^n - 1)4/26 = (10^n - 1)/5 = 1254, \qquad n = 4.$$

Chadwick's treatment differs from most others* in that he does not admit numbers beginning with one or more zeros. While the addition of such extra digits does not affect the value of the number, it has an effect when the digits

* See Guttman, *loc. cit.*

are permuted. It is this tacit restriction that gives the condition $N > 10^{n-1}$ from which (5) is derived.

In working with cyclic numbers, a list of the decimal expansions of the reciprocals of the integers is very useful. The most complete list of this sort to be published is entitled, *A Table of the Circles Arising from the Division of a Unit or Any Other Whole Number by All the Integers from* 1 *to* 1024, published by Henry Goodwyn in London in 1812. A less complete table is given in the second volume of Gauss's *Werke*, p. 412.

R. J. W.

ON THE TREATMENT OF CERTAIN PROBLEMS OF ELEMENTARY PROBABILITY

R. P. BAILEY, Lafayette College

It is well known that the definition of mathematical probability given in most texts on college algebra* is unsatisfactory from many points of view; in particular, it is clearly useless in problems where the number of possibilities is infinite, whether the variation involved be of the discrete or continuous type. For this reason alone it would seem that the discussion of problems of a geometric nature, and others involving a discrete infinity, should be omitted from an elementary course based on the ordinary definition, even if the technical difficulty of such problems was not, in most cases, reason enough for their exclusion.

Nevertheless, it is a fact that problems of this kind *are* given in some very widely used texts.† The following examples are typical:

Problem I. In throwing two dice, find the probability that a *four* will be thrown before a *seven.*

Problem II. A and B take turns throwing a die, A having the first turn. The first player to throw an ace wins the game. Find the probability of winning for each player.

The difficulty is the same in both problems. We are asked to determine the probability that one of two specified events happens *before* the other, when neither event necessarily occurs in any finite number of trials. Since an infinitude of possibilities is involved, there is some question as to whether, on the basis of the definition given, the required probability even exists.

The solutions of these problems are ordinarily given somewhat as follows:

Problem I. For convenience we separate into two sets the cases in which a *seven* is thrown: (a) where the first die has the lower number, and (b) where the second die has the lower number. Let p_1, p_2, and p_3 denote respectively the probabilities of throwing a *seven* of type (a) first, a *seven* of type (b) first, and a *four* first. The three events in question are mutually exclusive and apparently equally likely, since on any given throw there is no more reason to expect a *seven* of type (a) than a *four* or a *seven* of type (b), *etc.* Assuming that the three events in

* The following is typical: "Suppose that, at any trial, a given event can occur in h different ways and can fail to occur in f different ways and that all the $h+f$ different ways are equally likely; then the probability that the event will occur at any given trial is $p = h/(h+f)$."

† See for instance, Brink, Appleton-Century, 1933, p. 354, probs. 18–21 inc.; Rider, Macmillan, 1940, p. 270, probs. 9, 10; Rietz and Crathorne, Holt, 1924, p. 195, prob. 13.

attempt to answer, except to remark that judged from an empirical point of view the definition seems justified. Moreover, in this paper, no attempt is made to apply the fundamental theorems to any except the ordinary probabilities.

The following corollaries are immediate:

COROLLARY I. *If the probabilities of the events $E_1, E_2, \cdots, E_i, \cdots$ happening in certain independent ways be $p_1, p_2, \cdots, p_i, \cdots$, then $p=\lim_{n\to\infty}\prod_{i=1}^{n}p_i$ exists, and the probability of all the events happening in the prescribed ways is p.*

COROLLARY II. *If the probabilities of the events $E_1, E_2, \cdots, E_i, \cdots$ happening in certain mutually exclusive ways be $p_1, p_2, \cdots, p_i, \cdots$, then $p=\lim_{n\to\infty}\sum_{i=1}^{n}p_i$ exists, and the probability of at least one of the events happening in the prescribed way is p.*

Problems I and II now have straightforward and satisfactory solutions. Consider Problem I. The probability of obtaining a *four* on the ith trial, after having failed to get a *four* or a *seven* until that time, is $p_i=1/12(3/4)^{i-1}$. Hence the probability of obtaining a *four* before a *seven* is

$$p = \lim_{n\to\infty} \sum_{i=1}^{n} p_i = 1/3$$

by Corollary II. The second problem is solved in a similar manner.

It is to be noted that technically the solution of Problem I involves nothing more difficult than finding the sum of an infinite geometric progression, and is therefore a perfectly reasonable problem for the student of college algebra. If the usual treatment is supplemented by some such limit definition as we have proposed, there seems to be no reason why problems of the type considered should not be included in any text. They are in fact of special interest to students because of their immediate applicability to many familiar games of chance. It seems, however, very unfortunate and undesirable that such problems should be included without the necessary extensions of fundamental notions.

A SERIES COMPARISON TEST FOR CALCULUS STUDENTS

HENRY SCHEFFÉ, Oregon State College

Elementary calculus texts do not commonly include the following:

SERIES COMPARISON TEST. *Let*

$$u_1 + u_2 + u_3 + \cdots \tag{1}$$

and

$$v_1 + v_2 + v_3 + \cdots \tag{2}$$

be two series of positive terms. Then if

$$\lim_{n\to\infty} u_n/v_n = 1, \tag{3}$$

the series converge or diverge together; *that is, if* (2) *converges, so also does* (1); *and if* (2) *diverges, so does* (1).

The proof is practically obvious. Because of (3), an N exists such that for $n>N$ we have $|u_n/v_n-1|<1/2$, so that $(1/2)<u_n/v_n<(3/2)$, and hence

$$(1/2)v_n < u_n < (3/2)v_n. \tag{4}$$

If (2) converges, use the second of the inequalities (4) and compare (1) with $(3/2)\sum v_n$. If (2) diverges, use the first of the inequalities (4) and the series $(1/2)\sum v_n$.

In spite of the fact that the above test might be generalized in various ways and is more restricted than the usual comparison tests, it suffices in this simple form for most of the problems in the texts, ordinarily solved by the latter tests. The students seem to find the familiar calculation of a limit easier than the manipulation of inequalities. (If facility in such manipulation is an objective of the course, our test may not recommend itself.)

As an illustration, suppose the student has to test the series

$$\sum_{n=1}^{\infty} 64n(n+1)/(2n+1)(2n+3)(2n+5).$$

By inspection he guesses that for large n the terms approach $64n^2/(2n)^3=8/n$ in a percentage sense. This suggests to him the divergent test series $\sum 8/n$, and he has no difficulty in showing that the limit of corresponding terms is unity.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Foundations of Geometry. By G. de B. Robinson. (Mathematical Expositions, No. 1.) Toronto, University of Toronto Press, 1940. 11+167 pages. $2.00.

Displacement, Velocity, and Acceleration Factors for Reciprocal Motion. By L. B. Smith. Hampton, Virginia, Levi B. Smith, 1940. 6+17 pages. $0.40.

Introduction to Algebraic Theories. By A. A. Albert. Chicago, University of Chicago Press, 1941. 8+137 pages. $1.75.

Geometry for Today. By A. J. Cook. Based on a Junior Geometry by A. W. Siddons and R. T. Hughes. Toronto, The Macmillan Company of Canada, Limited, 1940. 10+260 pages. $1.00.

Facsimiles of Two Papers by Bayes. Prepared under the direction of W. E. Deming. (i) An Essay Toward Solving a Problem in the Doctrine of Chances, with Richard Price's Foreword and Discussion; Phil. Trans. Royal Soc., 1763, pp. 370–418. With a Commentary by Edward C. Nolina. (ii) A Letter on Asymptotic Series from Bayes to John Canton; pp. 269–271 of the same volume. With a Commentary by W. Edwards Deming. Washington, D. C., The Graduate School of The Department of Agriculture, 1940. 12+59 pages. $1.00.

REVIEWS

A Treatise on Advanced Calculus. By Philip Franklin. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1940. 14+595 pages. $6.00.

This book is a valuable contribution to the field of advanced calculus. It definitely bridges the gap between formal elementary work and the exacting rigor of modern analysis. Evidence for this is found in the topic headings of the early chapters. The starting-point is the positive and negative integers; then follow the rational numbers, and the irrational numbers by means of the Dedekind partition, the definition of limit point, the Bolzano-Weierstrass and the Heine-Borel theorems. The second chapter gives a careful and not too long discussion of limits, bounds, and continuity; a proof by means of the Heine-Borel theorem of the fact that a continuous function is uniformly continuous; and proofs of such theorems as, a continuous function is bounded and assumes its upper and lower bounds, a non-decreasing variable tends to a limit or becomes positively infinite, proofs that are usually omitted from books and courses in this field. Chapter I ends with a list of thirty-four problems involving applications of mathematical induction, the Dedekind cut, and the Heine-Borel theorem. Chapter II with a list of thirty-five problems involving the definitions of algebraic and transcendental numbers, proves that the algebraic numbers are denumerable,

and that the real numbers are non-denumerable, and gives the definition of Peano's space filling curve.

Thus the pace is set. To put this material in a form acceptable to mature readers is one thing; to place it against a background of the usual undergraduate mathematics in such a way that it has meaning for a student entering a second course in calculus is something else. The latter was the author's aim, and in the reviewer's opinion he has succeeded.

Coming to Chapter III, we find constructive arithmetical definitions of the exponential, logarithmic, and trigonometric functions. The approach here given is new, and of unquestionable interest and value; in fact, to the reviewer, it is the most interesting part of the book. The time and attention to detail that it requires, however, will probably bring about its exclusion from most main-line courses in advanced calculus.

There are three chapters on integration, and two on complex variable. The work on integration is restricted to Riemann integrals, proper and improper. The Darboux theorem is given, and enough of the theory of content and measure is included to permit the proof of the fact that a function of bounded variation is integrable, and the fact that a necessary and sufficient condition for integrability is that the set of discontinuities has exterior measure zero. With this done, the machinery is at hand to give a brief outline of the Lebesgue theory, and one wonders why the author failed to do so. A simple type of Stieltjes integral is discussed for use in physical applications.

The first chapter on complex variable gives constructive definitions of the elementary functions, and the Gauss proof of the fundamental theorem of algebra. This theorem makes possible the decomposition of rational functions into partial fractions before integration. In the second chapter there is the usual development of analytic functions, leading to Morera's theorem, Laurent's series, singular points, and enough of the theory of residues to permit the evaluation of certain definite integrals by means of contour integration. The chapter closes with a section on analytic functions of several complex variables.

In the chapters on series and sequences we find, in addition to the usual topics, sections on infinite products, convergence in the mean, and equi-continuity. The chapter on Fourier series and integrals includes Féjer's theorem, the Weierstrass approximation theorem, Parseval's theorem, Fourier integrals, and Laplace transforms. In the chapter on differential equations there are existence theorems for ordinary differential equations, one existence theorem for a special type of partial differential equation, and theorems on envelopes related to these existence theorems. In the closing chapter there is a detailed discussion of the gamma function; also, sections on Bernoulli polynomials and numbers, Stirling's formula, a determination of Euler's constant, and some of the common non-elementary integrals.

In addition to what has been mentioned there are included all the usual topics of advanced calculus. Sufficient has been said to give an idea of the scope of the book. So far as it goes, and this is a long way in comparison with other

texts in the same field, it is complete. Nowhere do we find the author using theorems of which he says: "The proof is beyond the scope of this book." Nor do we find him referring to more advanced treatises for proofs. This completeness, together with the fact that there are few references of an historical nature or hints of things to come, gives the book an air of finality, which is, perhaps, undesirable. Other features which might be considered undesirable are: (1) there are too few diagrams, twenty-eight as compared with eighty-nine in a recent book in the same field; (2) there are no examples worked in the body of the text, and no discussion of particular cases leading up to general theories; and (3) the long lists of problems at the end of each chapter, while packed with material of the utmost value, are, nevertheless, stated in very general form. But what the book lacks by way of illustration, is, in a large measure, made up in exposition. The reviewer has worked through some sections for class use and finds that little amplification is necessary.

It is clear that the work under review is on a high plane of scope and maturity. Is this plane too high for a second course in calculus? It was the reviewer's experience to go from a none-too-good preparation in sophomore calculus directly to Goursat-Hedrick, *Mathematical Analysis*, vol. I, in company with a class of over twenty mathematicians, physicists, and chemists. The memorable thing about this experience is that none of the students thought, nor did the instructor ever hint, that the demands of the course were unreasonable. The book under review is at about the same level of maturity as Goursat-Hedrick, and it would be difficult to find better exposition at this level. If instructors would take the attitude of expecting their classes to get the work, and refrain from hinting that there is an easier alternative, students would find themselves on firm ground a year earlier, and mathematics generally would be benefited.

R. L. JEFFERY

Cours de Démographie et de Statistique Sanitaire. By M. Huber. I. Introduction à l'Etude des Statistiques Démographiques et Sanitaires, pp. 67. II. Méthodes d'Elaboration des Statistiques Démographiques, pp. 110. (Actualités Scientifiques et Industrielles, Nos. 598, 599.) Paris, Hermann et Cie, 1938.

Every topic that deals in numbers is grist for the mathematical mill. But the mill must be worked with discretion, for whatever comes out cannot be of better quality than what goes in. These two monographs, first in a series of six planned by the French statistician Huber, are devoted practically in their whole extent to a consideration of the sources of vital statistics, the methods of collecting them, and the purposes to which they are applied, without presenting any of the mathematical operations employed in working up the data.

After defining demography as "the application of statistical methods to the study of populations, or, more generally, of human aggregates," and briefly reviewing the history of this field, the author lists and succinctly indicates the nature of some of the methods of collecting observations (direct method, indirect methods, sampling methods, *etc.*). Equally brief enumeration is given of

some of the steps involved in working up the data, such as their scrutiny and checking, classification, tabulation, *etc.* Next follows a section on the utilization of demographic statistics, graphic representations, some of the simple standard statistical measures, such as mean, mode, dispersion, *etc.* Little more than a page is devoted to time series and a few pages to analysis of statistics represented in double-entry tables, including the study of correlation and contingency. The closing paragraphs of the text deal with comparison of related series. The Appendix to Part I gives very brief indications regarding statistical machines.

Part II is a 57-page summary of population censuses, the first 35 pages being devoted entirely to French data. A feature not often found in other works on these subjects is an account of the costs of censuses in different countries. The ensuing sections deal with population movement: vital statistics, migration, and population registers, with cost figures for the last mentioned.

A. J. LOTKA

Traité du Calcul des Probabilités et de ses Applications. By Émil Borel. Tome IV, Fascicule ii. *Applications aux Jeux de Hazard.* By Émil Borel, rédige par Jean Ville. Paris, Gauthier-Villars, 1938. 2+122 pages.

This book, while far from being an exhaustive treatment, gives an interesting and simple exposition of some applications of the theory of probability to certain games of chance. In fairness to the author, it must be stated that the author did not intend this monograph to be exhaustive.

The discussion is divided into five chapters.

In Chapter I the author describes a die, and points out the well known construction of it. It is seen that the numbering on the die assists in the simplification of the solution of certain problems involving games with dice. We examine the frequency of occurrence of various sums when one, two, three, four, and five dice are thrown. There is indicated the problem of combinations with and without repetitions. We can see how the probability of winning is dependent on the number of players involved as well as on the imposition of conditions of play.

In Chapter II the author discusses the celebrated "Problem of Points" concerning the division of stakes between two players who separate without completing their game. This problem, it appears, was proposed to Pascal by Chevalier de Méré. The simple case and a generalization of the problem are discussed. The discussion is with some detail. Also, certain questions related to this problem are presented.

In Chapter III we find somewhat general remarks in regard to games with cards, and some particular problems are discussed as examples. In the reviewer's opinion, the author shows more rigor in the discussion of rather simple examples than one usually finds in other monographs on the same subject.

In Chapter IV we find discussed certain questions where psychology plays an important rôle. The author uses as an illustration a game which the reviewer would call the game of "choices." To illustrate: Let there be two players A and B. Each player chooses a number. Let a and b be the respective numbers

chosen. We consider the sum of a and b. Now A wins if $a+b$ is an odd number and B wins if $a+b$ is an even number. Various ramifications involving two and more than two individuals, and two and more than two choices are discussed. The author presents cases where discrete and continuous variables are introduced. He also presents interesting examples in the game of strategy, particularly military strategy. There is also indicated the case of manœuvre for advantage. It is not often that one finds anything said about such non-chance elements. The reviewer feels that such elements are very important in the consideration of games of chance and that the author is to be congratulated in calling them to our attention.

In Chapter V questions where both chance and psychology are involved are discussed. Here we have a rather detailed presentation of the game of poker. Following Chapter V we find a presentation of an interesting general theory of games in which the competence of players is considered.

In the reviewer's opinion, the book is very clearly written and is very instructive and exceedingly fascinating. This book is highly recommended to those who wish to gain some knowledge about the applications of probability and other factors that may enter into the playing of games of chance.

F. M. WEIDA

Displacement, Velocity, and Acceleration Factors for Reciprocal Motion. By L. B. Smith. Hampton, Virginia, L. B. Smith, 1940. 6+17 pages. $0.40 (three copies for $1).

The present tables are believed to be more complete and to cover a wider range than any earlier ones. The argument is the ratio of the length of the connecting rod to the length of the crank, and the range is from 2 to 6, in varying increments. The range of the crank angle is from 0° to 180°, at intervals of 2° for the first and last 30° of the stroke, and of 5° for the central portion. The functions mentioned are arranged in separate tables, each preceded by an illustrative example worked out in detail. The results of the computation were photographed from type-written sheets, and provided with instructions concerning the position of the decimal point. An appendix provides the detailed derivation of the formulas employed.

VIRGIL SNYDER

CORRECTION

In the March issue of the MONTHLY in the list of New Books Received, the author's name was inadvertently omitted from the following:

A Survey of Methods of Apportionment in Congress. By E. V. Huntington. (Senate Document, No. 304.) Washington, D. C., Government Printing Office, 1940. 41 pages. $0.10.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

NOTICE TO ALL CLUBS AND DEPARTMENTS

Reports of programs and activities carried on by mathematics departments and by mathematics clubs should be sent to this department of the MONTHLY before the close of the calendar of the college year. We welcome full and complete details of the meetings held by any groups, including the date of meeting, name of speaker and title of talk, a summary of the topic discussed, as well as details of other portions of the program, including social activities. Open meetings, contests and their winners, joint meetings with other groups, conferences, honors awarded, and any other activities carried on by any person or group in the department should be reported. Send us copies of any mimeographed material used at club meetings or at social events, magazines or publications, bibliographies of topics used, as well as any material which you have found helpful in your college and which may be of interest to other readers of this department.

CLUB REPORTS, 1939-40

Kappa Mu Epsilon

The fifth biennial convention of *Kappa Mu Epsilon* will be held at the Central Missouri State Teachers College, Warrensburg, Missouri, on April 18–19, 1941. An opportunity will be given for each chapter of *Kappa Mu Epsilon* to report upon its activities. Also, plans will be made for the next biennium.

Mathematics Club, Chicago Teachers College

Guest speakers appearing at club meetings included Dr. Philip Fox, former director of the Adler Planetarium and former director of the Rosenwald Museum of Science and Industry, who spoke on Mathematics—a useful tool in science and industry, and Mr. Jerome Sachs of Wilson Junior College who spoke on Non-euclidean geometry. The question "The Place of Mathematics in Secondary Education" was debated at one meeting by Dr. J. A. Bartky, president of the college, Dr. John deBoer, and Dr. J. T. Johnson. Topics at other meetings included Mathematics and astronomy, by Mr. Ralph Mansfield; Mathematical recreations, by Mr. J. J. Urbancek; and The sociological significance of the introduction of number to early civilizations, by Mr. Jules Karlin. At a joint meeting with the Camera Club, Dr. D. H. West spoke on The mathematics of photography, and at a meeting with the Geography Club the discussion centered on the topic Mathematical geography. A Teachers Day held in May included an exhibit featuring The golden section, Non-euclidean geometry, Introduction to the theory of numbers, Mathematical relationships found in music, Oddities in mathematics, and Mathematical analysis of simple card tricks.

Mathematics Club, Cooper Union

Teams made up of members of the club won honors in the Putnam Competition as well as in the Metropolitan Intercollegiate Mathematics Contest. The club awarded a log log duplex slide rule to Harold Grad for excellence in first-year mathematics. At one meeting, films and slides on the Isograph were shown. Topics discussed included: The slide rule, by B. Lewis; Difference equations, by S. Manson; Relativity, by M. Weiss; and Crypt solving, by H. Oakley. Officers were: President, B. Lax; Vice-President, S. Manson; Secretary, T. Gold; Treasurer, I. Gordon; Faculty Adviser, Professor F. H. Miller.

Newtonian Society, Lehigh University

Topics discussed at meetings included: Mathematical relation to plant growth, by Robert Hammond; Astronomy of the planets, by Professor E. N. Van Arnam; Principles of aerodynamics, by Professor C. A. Shook; Mathematical prodigies, by E. F. Bodine; Magic squares, by G. E. Raynor; and Trick mathematics, by W. C. Walker. Officers were: President, E. F. Bodine; Secretary, R. B. Anderson; Treasurer, R. E. Cullen.

Pi Mu Epsilon, Michigan State College

Entirely faculty conducted meetings were tried during the year and found to be extremely successful. The meetings were alternately open and closed. At the open meeting, the simpler portion of a certain topic was discussed. All students interested in mathematics were invited to attend, especially freshmen. The closed meeting which followed was a continuation of the discussion of the same topic as in the previous open meeting, but directed at students beyond the sophomore calculus. The programs were as follows: The arithmetic of infinites, by Dr. Welmers; Geometry, what is it?, by Professor Grove; Topics in theory of numbers, by Dr. Hill; and Diophantine analysis, by Mr. Nordstrom. The guest speaker for the banquet was Professor L. S. Johnston of the University of Detroit. Officers were: President, Edith Kelso; Vice-President, Perry Schlesinger; Secretary-Treasurer, Ruth Winegar; Faculty Director, John W. Zimmer.

Mathematics Club, Brown University

The twenty-sixth annual program of the club included six regular monthly meetings. At four of these there were two half-hour talks given by members of the club, which were followed by informal discussion and refreshments. The student speakers received advice and assistance in the preparation of their papers from members of the faculty. Titles and speakers were: The game of Nim, by Angela Coffey; Pascal's arithmetical triangle, by Clark Foster; Various proofs of the Pythagorean theorem, by Eugenia Borys; Topics in vector analysis, by Robert Poole; Simpson's rule, by Paul Tamarkin; Regular polyhedra, by Blanche Lunden; Diophantine equations, by Donald Bliss. At one meeting each semester the address was given by a guest speaker. Professor Neugebauer of Brown University spoke in January on Babylonian mathematics, and Mr. H. A. Grout, Associate Actuary of the John Hancock Mutual Life Insurance Company and father of one of the club members, spoke in March on Mathematics in the actuarial field. A special feature this year was an historical talk on Twenty-five years of the Mathematics Club, by Mary Tyrell. Printed programs were distributed in October by the program committee: Professor J. S. Frame, faculty representative, Clark Foster, Ruth Hunt, Gerald Oster, and Irving Twomey. A club picture is taken each year, and added to the collection started in the early days of the club and hung in the Mathematics Exhibit Room of the University.

Freshman Mathematics Club, Hunter College

This club was organized primarily for the freshman students who are specializing in mathematics or in statistical science. The program of the club, at the regular weekly meetings during each semester, was based upon the presentation of the biographies contained in Bell's *Men of Mathematics* by student speakers. Professor L. G. Simons, Professor J. H. Bushey, and Miss C. Eisele were guest speakers. During each semester the members participated in the collection of topics of mathematical interest from newspapers, general periodicals, and miscellaneous sources. These clippings were exhibited on the bulletin board maintained for the purpose in the club meeting room. Later they were compiled by the secretary into the club scrap-book. Officers were: President, Helene Oxhorn; Vice-President, Marie Johnson; Secretary-Treasurer, Patricia Specter; Publicity Director, Wilma Menaker; Faculty Adviser, Dr. Madeline Levin.

Mathematics Club, Colgate University

During the academic year, fourteen bi-weekly meetings were held at which a total of fifty-six papers were presented by the student members of the club. Approximately half of these meetings were followed by an informal social gathering with refreshments being served. Of the papers presented, those of the junior and senior members of the club dealt with topics chosen from or suggested by the following sources: past comprehensive examinations, the problem and club departments of the journals, and the material of the upper-class seminars. The sophomore members coördinated the preparation of their papers with the tutorial program of the college. All members of the mathematics faculty assisted with their advice. Special emphasis was placed upon the careful presentation of the topics. During the year several solutions were submitted by student members to problems suggested in the MONTHLY, some of which were published. Officers were: Chairman, Leonard Reinsmith; Vice-Chairman, Russell Freeston; Secretary, W. R. Crosier; Treasurer, Clark Stowe; Faculty Adviser, Professor C. W. Munshower.

Mathematics Club, Oberlin College

A varied program for the monthly meetings included a demonstration of "Mobius bands and what happens when they are cut, by Alice Rowe; a talk on When can a polygon be retraced without traversing any segment twice and without crossing the part already covered?, by William Hosier; a piano demonstration, by Katherine Fuller, of the applications of mathematics in music discussed in Birkhoff's *Aesthetic Measure*; a discussion of the properties of surfaces as shown in string models, by Professor Wagner; and a talk by Professor Cairns on Mathematical aspects of seismology. The Christmas meeting was modeled on the radio feàture *The Battle of the Sexes*, and over one hundred simple mathematical questions and puzzles were used. Young's *Fundamental Concepts of Mathematics* was used by Rudolph Schmidt and James Russell as a source for discussing cardinal and ordinal numbers; and John Insprucker presented a phase of the Calculus of finite differences, using material gathered from Hall and Knight's *Higher Algebra*. Officers were: President, L. Syckes; Vice-President, R. Schmidt; Secretary, Katherine Fuller; Treasurer, Alice Rowe; Adviser, Professor R. W. Wagner.

Mathematics-Physics Club, College of Saint Teresa

This organization met bi-monthly to discuss topics in mathematics and physics. A study of the slide rule was taken up at several of the first meetings. At subsequent meetings papers on Television, Areas and volumes, Calculating by eights, Squaring four points, Nature and mathematics, and Economy were presented. Recreations included a skit "Magic Math," the "Game of Euclid" (an original mathematical game similar to "Authors"), a valentine project, a treasure hunt, missing word stories, and cross-word puzzles. Officers were: President, Irene Sustman; Vice-President, Eileen Whitty; Secretary-Treasurer, Gertrude Kleiber; Faculty Adviser, Sister M. Leontius.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 466. *Proposed by W. C. Rufus, Observatory of the University of Michigan.*

A is travelling in a restricted zone at two-thirds the speed limit, and B passes him, going twice as fast. Five minutes later a "speed cop," C, passes A and overtakes B. He spends two minutes giving out a ticket, then starts back at speed limit and meets A one mile back. Find at least one practicable solution. Discuss other possible solutions, limiting B's distance to five miles.

E 467. *Proposed by V. Thébault, San Sebastián, Spain.*

In a given triangle, show that the radical axes of the circumcircle with the respective circles whose diameters are the three medians, meet the corresponding sides in three collinear points.

E 468. *Proposed by W. R. Ransom, Tufts College.*

The Fibonacci numbers, defined by $f_1=f_2=1$, $f_{j+1}=f_{j-1}+f_j$, are known to yield a puzzle in which a square of side f_n is cut into four pieces which can apparently be rearranged to form a rectangle $f_{n-1}\times f_{n+1}$. Show that the same four pieces can be rearranged to form a figure which appears to consist of two rectangles $f_{n-1}\times 2f_{n-2}$ connected by a rectangle $f_{n-4}\times f_{n-2}$, the error being again one unit of area.

E 469. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Show that the exradii and circumradius of a triangle satisfy the identity

$$\sum \frac{a^2(b^2-c^2)}{r_a(r_b^2-r_c^2)} = 4R.$$

E 470. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Circle I has its center on another circle J. They intersect at A and C. From any point B on J, draw BC, intersecting I again at D. Prove that $BD=BA$.

E 458 [1941, 148]. *Correction.* The last sentence should read: "Show that the same result holds for any matrix (a_{rs}) in which $a_{r1}=a_{2r}$, $a_{1r}=a_{r2}$, $(r>2)$, and $a_{11}=a_{22}$."

SOLUTIONS

E 428 [1940, 395]. *Proposed by Ruth Mason Ballard, Chicago.*

It can be shown simply that there is only one way of replacing the asterisks by the integers from 1 to 7 so as to make

$$\begin{array}{ccc} * & 9 & * \\ * & * & * \\ 8 & * & * \end{array}$$

a magic square, and that this square cannot be uniquely determined by fewer than two of the nine numbers. Give an analogous scheme for a 4×4 square, using the numbers 1, 7, 11, 16 (and twelve asterisks). Is it possible to determine the 4×4 square by fewer than four of the sixteen numbers?

Solution by D. H. Browne, Buffalo, N. Y.

Here are two of the many possible schemes using the numbers 1, 7, 11, 16:

$$\begin{array}{cccc} * & 7 & * & * \\ * & * & 16 & * \\ * & 11 & * & * \\ 1 & * & * & * \end{array} \qquad\qquad \begin{array}{cccc} * & * & * & * \\ * & * & 16 & * \\ * & 11 & * & * \\ 1 & * & 7 & * \end{array}$$

Yes, a 4×4 magic square can be determined by various sets of *three* numbers, such as those printed in italics below:

$$\begin{array}{rrrr} \mathit{12} & 7 & \mathit{9} & 6 \\ 13 & 2 & 16 & 3 \\ 8 & 11 & 5 & 10 \\ 1 & 14 & 4 & \mathit{15} \end{array} \qquad\qquad \begin{array}{rrrr} 6 & \mathit{11} & \mathit{10} & 7 \\ 4 & 13 & \mathit{16} & 1 \\ 15 & 2 & 3 & 14 \\ 9 & 8 & 5 & 12 \end{array} \qquad\qquad \begin{array}{rrrr} \mathit{16} & 5 & 9 & 4 \\ 2 & \mathit{11} & 7 & 14 \\ 3 & 10 & 6 & 15 \\ \mathit{13} & 8 & 12 & 1 \end{array}$$

The uniqueness is most easily verified by writing each of the sixteen numbers in the form $4a+b$, ($a=0, 1, 2, 3$; $b=1, 2, 3, 4$), and considering the squares formed by the a's and b's separately.

E 432 [1940, 487]. *Proposed by C. W. Trigg, Los Angeles City College.*

If a and b are the radii of two spheres, tangent to each other and to a plane, show that the radius x of the largest sphere which can pass between them is given by the formula

$$x^{-1/2} = a^{-1/2} + b^{-1/2}.$$

Solution by M. W. Fleck, Eastern New Mexico College.

Let A, B, C be the centers of the spheres with radii a, b, x, respectively, where (C) is in the critical position, tangent to the given spheres and plane, and

let A', B', C' be the projections of A, B, C upon the given plane. Then by the theorem of Pythagoras, the distance between the parallel lines AA' and BB' is

$$\left\{(a+b)^2-(a-b)^2\right\}^{1/2}=2(ab)^{1/2}.$$

Similarly, the distances from CC' to AA' and BB' are $2(ax)^{1/2}$ and $2(bx)^{1/2}$, respectively. Since all these lines are coplanar, we have

$$2(ab)^{1/2}=2(ax)^{1/2}+2(bx)^{1/2},$$

which is clearly equivalent to the desired formula.

Also solved by W. E. Buker, W. R. Crosier, William Douglas, B. A. Hausmann, C. W. Moran, Nathan Newman, C. C. Oursler, P. W. A. Raine, Hazel E. Schoonmaker, C. F. Strobel, E. P. Starke, P. D. Thomas, and the proposer. Douglas points out that a similar solution is given by S. I. Jones in *Mathematical Nuts*, Nashville, 1936, p. 173.

Editorial Note. When rationalized, the formula becomes

$$2(x^{-2}+a^{-2}+b^{-2})=(x^{-1}+a^{-1}+b^{-1})^2,$$

which is clearly a special case of Professor Soddy's equation

$$2(x^{-2}+a^{-2}+b^{-2}+c^{-2})=(x^{-1}+a^{-1}+b^{-1}+c^{-1})^2,$$

whose roots are the radii of the largest and smallest spheres which can touch each of three given spheres, of radii a, b, c, in mutual contact. See *Nature*, vol. 137, p. 1021; vol. 138, p. 958; vol. 139, pp. 77–79 (1936–37).

E 433 [1940, 487]. *Proposed by A. A. Bennett, Brown University.*

Two parallel vertical walls, separated by a distance of d feet, have level ground between them. Two ladders, of length a and b feet respectively ($a>b$), abut each against a foot of one of these walls and lean against the other wall, crossing each other at a height of c feet above the ground. Show that a solution in integers is given by

$$\begin{aligned}ka&=(su+tv)(s-t)(u+v),\\ kb&=(sv+tu)(s-t)(u+v),\\ kc&=(su-tv)(sv-tu),\\ kd&=2(stuv)^{1/2}(s-t)(u+v),\end{aligned}$$

where s, t, u, v are any positive integers subject to the three conditions $u>v$, $sv>tu$, and $stuv$ is a perfect square, k being the greatest common divisor of the four right-hand members. What is the simplest particular solution in which a, b, c, d are all odd?

Solution by W. E. Buker, Pittsburgh Public Schools.

We note the relation

$$(a^2 - d^2)^{-1/2} + (b^2 - d^2)^{-1/2} = c^{-1},$$

and observe that this is satisfied by the given expressions for a, b, c, d. The three conditions ensure that $a > b > 0$ and that d is an integer.

Also solved by E. P. Starke, who showed that every solution in integers is of the proposed form, and by B. C. Zimmerman, who showed that the simplest odd solution is

$$(s, t, u, v) = (7, 1, 9, 7), \qquad k = 64, \qquad (a, b, c, d) = (105, 87, 35, 63).$$

E 435 [1940, 488]. *Proposed by David Segal, Kosow Huculski, Poland.*

Show that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

is a necessary and sufficient condition for p to be an odd prime.

Partial Solution by H. W. Brinkmann, Swarthmore College.

We consider the necessity and sufficiency separately.

I. If p is an odd prime, the congruence holds. This was proved by Babbage, *Edinburgh Philosophical Journal*, vol. 1, 1819, p. 46. Wolstenholme showed that, for $p > 3$, it even holds modulo p^3. (See Dickson, *History of the Theory of Numbers*, vol. 1, p. 271.) A still more precise result can be obtained as follows.

We have

$$\binom{2p-1}{p-1} = \frac{(p+1)(p+2)\cdots(p+\overline{p-1})}{(p-1)!}$$
$$= 1 + \frac{A_{p-2}p + A_{p-3}p^2 + \cdots + p^{p-1}}{(p-1)!},$$

where A_ν is the sum of all products of $1, 2, \cdots, p-1$ taken ν at a time. Since it is a familiar fact that, p being prime, $A_\nu \equiv 0 \pmod{p}$ for $0 < \nu < p-1$, Babbage's congruence follows.

In the identity

$$(x-1)(x-2)\cdots(x-\overline{p-1}) = x^{p-1} - A_1x^{p-2} + \cdots - A_{p-2}x + (p-1)!$$

we put $x = p$, obtaining

$$A_{p-2} = A_{p-3}p - A_{p-4}p^2 + \cdots + p^{p-2}.$$

From now on, we assume p to be a prime greater than 3. Wolstenholme's congruence follows from the fact that $A_{p-2} \equiv 0 \pmod{p^2}$. Now, since

$$A_{p-2}/p^2 = A_{p-3}/p - A_{p-4} + \cdots + p^{p-4} \equiv A_{p-3}/p \pmod{p},$$

we have

$$\binom{2p-1}{p-1} \equiv 1 + \frac{p^3}{(p-1)!}\left(\frac{A_{p-2}}{p^2} + \frac{A_{p-3}}{p}\right) \pmod{p^4}$$
$$\equiv 1 + \frac{2p^3}{(p-1)!}\,\frac{A_{p-3}}{p} \pmod{p^4}.$$

Wilson's theorem enables us to replace the $(p-1)!$ by -1; and from a general result of Glaisher's (see Dickson, *op. cit.*, p. 100) we find

$$A_{p-3}/p \equiv \tfrac{1}{3}B_{p-3} \pmod{p},$$

where B_{p-3} is a Bernoulli number as defined by the symbolic identity

$$e^{Bx} = x/(e^x - 1).$$

Thus finally,

$$\binom{2p-1}{p-1} \equiv 1 - \tfrac{2}{3}p^3 B_{p-3} \pmod{p^4}.$$

II. I believe the condition is *not* sufficient. The following argument shows that the proposed congruence holds if $p=q^2$, where q is a prime such that $B_{q-3}\equiv 0 \pmod{q}$. Such a prime must be sought among the *irregular* primes 37, 59, 67, 101, $\cdots$, which are such that q divides at least one of the numbers $B_2, B_4, B_6, \cdots, B_{q-3}$; and although in the first few cases B_{q-3} is not the one that q divides, there does not seem to be any reason why this should not happen eventually.

We define

$$f(1) = 1, \qquad f(m) = \prod_i \frac{m + a_i}{a_i},$$

where a_i runs through the $\phi(m)$ positive integers less than m and prime to it. This is not usually an integer; but for a prime p,

$$f(p) = \binom{2p-1}{p-1}.$$

In general we readily get

$$\binom{2m-1}{m-1} = \prod_{d\mid m} f(d).$$

In particular, if $m=p^2$ we have

$$\binom{2m-1}{m-1} = f(p^2)\cdot f(p) = f(m)\cdot\binom{2p-1}{p-1}.$$

Since, for $m>2$, $\sum a_i^{-1}\equiv 0 \pmod{m}$, we have

$$\begin{aligned} f(m) &= (m^{\phi(m)} + m^{\phi(m)-1}\sum a_i + \cdots)(\prod a_i)^{-1} + m\sum a_i^{-1} + 1 \\ &\equiv 1 \pmod{m^2}. \end{aligned}$$

Hence, by the final result of I above,

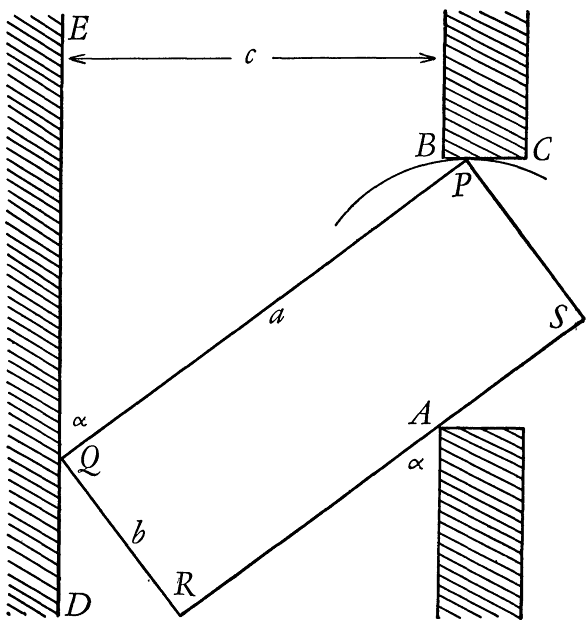
$$\binom{2p^2-1}{p^2-1} \equiv 1 - \tfrac{2}{3}p^3B_{p-3} \pmod{p^4}.$$

Thus, if there is a prime p which divides B_{p-3}, then the square of that prime must satisfy the congruence

$$\binom{2m-1}{m-1} \equiv 1 \pmod{m^2}.$$

E 436 [1940, 569]. *Proposed by E. H. Johnson, Emory University.*

A tall rectangular piece of furniture, of length a and width b, is moved down a hallway of width c, and goes through a door whose width, d, barely allows its passage into an adjacent room. If we neglect the thickness of the wall, it is easily seen by the comparison of similar triangles that $d=ab/c$. If the wall has a thickness h, find the value of d in terms of a, b, c, and h.



Solution by W. B. Carver, Cornell University.

Neglecting trivial cases, we assume $b<c<a$. If the side RS slides past the corner A while the corner Q slides along the wall DE (as in the diagram), the

corner P will describe a curve. Using DE as the y-axis, with the x-axis through A, we let P and Q be (x, y) and $(0, z)$; also, we let α denote the angle PQE. Then we have

$$y - z = a \cos \alpha, \qquad x = a \sin \alpha, \qquad c + z \tan \alpha = b \sec \alpha.$$

Eliminating z and α, we obtain the equation of the curve in the form $y=f(x)$, where

$$f(x) = \frac{ab + (x - c)(a^2 - x^2)^{1/2}}{x},$$

so that

$$f'(x) = \frac{a^2c - x^3 - ab(a^2 - x^2)^{1/2}}{x^2(a^2 - x^2)^{1/2}}$$

and

$$f''(x) = -\frac{a^2(x - c)}{x(a^2 - x^2)^{3/2}} - \frac{2a}{x^3}\left(\frac{ac}{(a^2 - x^2)^{1/2}} - b\right).$$

This last expression is clearly negative for $c \leqq x < a$; hence $f(x)$ has at most one maximum in this range. We now distinguish three cases.

Case 1. If $f'(c) \leqq 0$, so that y decreases as x increases from c to a, the thickness of the wall is immaterial, and we have $d = f(c) = ab/c$. This happens if

$$ab \geqq c(a^2 - c^2)^{1/2}.$$

Case 2. If $f'(c) > 0$ and y attains its maximum for $x < c+h$, the width of the door will be this maximum value of y; that is,

$$d = \{ab + (x - c)(a^2 - x^2)^{1/2}\}/x,$$

where x is the positive root of the equation

$$ab(a^2 - x^2)^{1/2} = a^2c - x^3.$$

This happens if

$$ab < c(a^2 - c^2)^{1/2}$$

and either $a \leqq c+h$ or $ab > \{a^2c-(c+h)^3\}/\{a^2-(c+h)^2\}^{1/2}$.

Case 3. If $f'(c+h) \geqq 0$, so that y increases throughout the interval $c \leqq x \leqq c+h$, the width of the door is $f(c+h)$; that is,

$$d = [ab + h\{a^2 - (c + h)^2\}^{1/2}]/(c + h).$$

This happens if

$$ab \leqq \{a^2c - (c + h)^3\}/\{a^2 - (c + h)^2\}^{1/2}.$$

A rational numerical example for Case 2 is

$$a = 6, \quad b = 2.38, \quad c = 4.5, \quad h > 0.3, \quad x = 4.8, \quad d = 3.2.$$

Also solved (in Case 3 only) by W. E. Buker and C. C. Oursler.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3993. *Proposed by N. A. Court, University of Oklahoma.*

A variable plane passing through a fixed point of the face ABC of the tetrahedron $DABC$ meets the edges DA, DB, DC in the points P, Q, R. Show that the locus of the point U common to the three planes PBC, QCA, RAB is a cone of the second degree.

3994. *Proposed by C. E. Springer, University of Oklahoma.*

If we define a_{ij} by the equations,

$$a_{11} = a_{22} = a_{33} = \sum_{j=1}^{n} \binom{K}{3j} (n-1)^{K-3j},$$

$$a_{12} = a_{23} = a_{31} = \sum_{j=1}^{n} \binom{K}{3j+1} (n-1)^{K-3j-1},$$

$$a_{13} = a_{32} = a_{21} = \sum_{j=1}^{n} \binom{K}{3j+2} (n-1)^{K-3j-2},$$

show that the determinant

$$| a_{ij} | = [(n-1)^3 + 1]^K.$$

3995. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Integrate the partial differential equation

$$y(x+y)z_{xx} - (x^2 - y^2)z_{xy} - x(x+y)z_{yy} + (x-y)(z_x + z_y) = 0.$$

SOLUTIONS

3919 [1939, 363]. *Proposed by Richard Bellman, Brooklyn College.*

Prove that

$$\begin{vmatrix} \frac{x}{1-x} & 1 & 0 & 0 & 0 \cdots & 0 \\ \frac{x^2}{1-x^2} & \frac{x}{1-x} & 2 & 0 & 0 \cdots & 0 \\ \frac{x^3}{1-x^3} & \frac{x^2}{1-x^2} & \frac{x}{1-x} & 3 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{x^r}{1-x^r} & \frac{x^{r-1}}{1-x^{r-1}} & \cdot & \cdot & \cdots & \frac{x}{1-x} \end{vmatrix} = \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

Solution by J. S. Frame, Brown University, Providence, R. I.

If we write $f_k = x^k/(1-x^k)$, then the quantities f_k satisfy the identity

$$f_k f_{r-k}/f_r = f_k + f_{r-k} + 1, \qquad 0 < k < r. \tag{1}$$

Denoting by D_r the determinant obtained from the given one by dividing its successive rows by 1, 2, 3, $\cdots$, r, we are to prove that

$$D_r = f_1 f_2 f_3 \cdots f_r. \tag{2}$$

Expanding D_r by minors of its last row, and multiplying by r, we have

$$rD_r = f_1 D_{r-1} - f_2 D_{r-2} + f_3 D_{r-3} - \cdots + (-1)^{r-1} f_r D_0, \tag{3}$$

where we set $D_0 = 1$, $D_{-1} = D_{-2} = \cdots = 0$.

We shall prove equation (2) by mathematical induction. Noting that $D_1 = f_1$ by definition, we assume $D_k = f_k D_{k-1}$ for $k < r$, and prove this for $k = r$. Multiplying (1) by $f_r D_{r-k-1}$ we have

$$f_k D_{r-k} = f_r(f_k D_{r-k-1} + D_{r-k} + D_{r-k-1}), \tag{4}$$

if we use the induction hypothesis $D_{r-k} = f_{r-k} D_{r-k-1}$. Substituting from (4) in (3) we find that

$$\begin{aligned} rD_r &= f_r(r-1)D_{r-1} + f_r \sum_{k=1}^{r} (-1)^{k-1} D_{r-k} - f_r \sum_{k=1}^{r} (-1)^{k-2} D_{r-k-1} \\ &= f_r(r-1)D_{r-1} + f_r D_{r-1} - f_r D_{-1} \\ &= r f_r D_{r-1}. \end{aligned} \tag{5}$$

Factoring r, we have the conclusion of the induction.

Solved also by the proposer.

Editorial Note. The proposer made use of the known identity

$$\prod_{i=1}^{\infty}(1+x^i t) = 1 + \sum_{r=1}^{\infty}\frac{x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}\,t^r,$$

stating that the coefficient of t^r is the elementary symmetric function formed by the sum of the products of r terms at a time of the infinite sequence $x, x^2, x^3, \cdots$. The symmetric function s_r formed by the sum of the rth powers of these terms is $x^r/(1-x^r)$. Then by Newton's formula we have

$$-p_1+s_1=0, \qquad 2p_2-p_1s_1+s_2=0, \cdots$$
$$(-1)^r r p_r + (-1)^{r-1}p_{r-1}s_1 + \cdots - p_1 s_{r-1} + s_r = 0.$$

Solving this system for p_r we obtain the above result.

The proposer gave merely these formal steps, but with patience and care it may be shown that they are valid, assuming first a certain condition on the x which may be dropped in the final result.

3920 [1939, 363]. *Proposed by F. A. Lewis, University of Alabama.*

Set

$$A_i = \sum_{j=1}^{4} a_{ij}x_j, \qquad B_i = \sum_{j=1}^{4} b_{ij}x_j, \qquad C_i = \sum_{j=1}^{4} c_{ij}x_j, \qquad (i = 1, 2, 3, 4),$$

where the x's represent independent variables, and the determinant of the coefficients of any four of the twelve linear forms does not vanish. Determine the number of distinct identities of the form

$$\alpha\prod_{i=1}^{4}A_i + \beta\prod_{i=1}^{4}B_i + \gamma\prod_{i=1}^{4}C_i \equiv 0$$

in the x's, where α, β, γ are arbitrary constants not all zero, and each coefficient in the various linear forms is zero or an nth root of unity, where n is given in advance.

An example of such an identity is, for $n=2$,

$$(x_1^2-x_2^2)(x_3^2-x_4^2) - (x_1^2-x_3^2)(x_2^2-x_4^2) + (x_1^2-x_4^2)(x_2^2-x_3^2) \equiv 0.$$

An identity with the same coefficients in the various linear forms and $(\alpha, \beta, \gamma) = (k, -k, k)$ is not considered as distinct from the foregoing.

Remarks by the Proposer.

Each such identity implies a desmic system of tetrahedra. See, for example, C. M. Jessop, *Quartic Surfaces*, (i) page 24 and (ii) page 25. Hence there are at least two such identities for $n=2$. J. R. Musselman, *American Journal of Mathematics*, vol. 49, no. 3, July, 1927, p. 363, implies the existence of at least 27 such identities for $n=6$. From the properties of "related" systems (Jessop, page 26),

it seems reasonable to suppose that there are at least 54 such identities when $n=6$.

From a study of the configuration of a certain group of degree four, the writer is inclined to believe that he can produce 64 such identities for $n=8$.

None of these statements have actually been verified, nor has any systematic determination of the actual number of such identities been attempted for a particular or general value of n.

3921 [1939, 364]. *Proposed by V. Thébault, Le Mans, France.*

Let BCA_1A_2, CAB_1B_2, ABC_1C_2 be similar rectangles constructed upon the sides $BC=a$, $CA=b$, $AB=c$ of a triangle ABC of area S, the three rectangles being all directed interiorly or all exteriorly, and $CA_1/a=AB_1/b=BC_1/c=k$. Let A_h, B_h, C_h, be points on A_1A_2, B_1B_2, C_1C_2 such that $A_1A_h/A_1A_2=B_1B_h/B_1B_2=C_1C_h/C_1C_2=\lambda$. The straight lines AB_h, BC_h, CA_h determine a triangle $\alpha\beta\gamma$ of area S' similar to ABC, and

$$S' = (k \cot V - \lambda)^2 S/(k^2 + \lambda^2),$$

where V is the angle of Brocard for ABC.

Problem 3850 [1937, 668] was incorrectly stated. Its last lines should be: Let A_3 be the symmetric of A_1 with respect to A_2, and similarly for B_3, C_3; then AB_3, BC_3, CA_3 meet in a point.

This follows from the present problem by taking $k=1$ for squares directed interiorly and $\lambda=2$.

Solution by Nicholas Christ Scholomiti, New York, N. Y.

In triangles BC_1C_h, AB_1B_h, CA_1A_h we have $BC_1=kc$, $C_1C_h=\lambda c$, $AB_1=kb$, $B_1B_h=\lambda b$, $CA_1=ka$, $A_1A_h=\lambda a$; further, $\measuredangle BC_1C_h=\measuredangle AB_1B_h=\measuredangle CA_1A_h=90°$. Therefore the three triangles are similar right triangles. Hence $\measuredangle C_1BC_h=\measuredangle B_1AB_h=\measuredangle A_1CA_h=\arctan \lambda/k$. The complements of these equal angles must also be equal; thus $\measuredangle C_hBA=\measuredangle B_hAC=\measuredangle A_hCB=\arctan k/\lambda$. It follows now that the lines BC_h, AB_h, CA_h if extended will form a triangle $A'B'C'$ $(\alpha\beta\gamma)$ similar to ABC because the three lines form with c, b, a respectively the same angle, arctan k/λ.

Let us now compute the ratio $A'B'/AB$. Applying the law of sines to triangles $BB'C$ and $BA'A$, we get, respectively, $BB'=(k/\sqrt{k^2+\lambda^2})(a/\sin B)$, where $k/\sqrt{k^2+\lambda^2}=\sin B'CB$, and

$$BA' = \sin A'AB\cdot c/\sin A = (\lambda \sin A - k \cos A)/\sqrt{k^2+\lambda^2}\cdot c/\sin A.$$

Therefore

$$A'B' = BB' - BA' = 1/\sqrt{k^2+\lambda^2}\,[ka/\sin B - c(\lambda - k \cot A)],$$

$$A'B'/AB = 1/\sqrt{k^2+\lambda^2}\,(ka/c \sin B - \lambda + k \cot A).$$

Now

$$a/c \sin B = \sin A/\sin C\cdot\sin B = \sin (B + C)/\sin B\cdot\sin C = \cot C + \cot B;$$

also, $\cot A+\cot B+\cot C=\cot V$, where V is the angle of Brocard. Substituting from the last two equations, we have,

$$A'B'/AB = (k \cot V - \lambda)/\sqrt{k^2 + \lambda^2},$$

whence

$$S'/S = (A'B'/AB)^2 = (k \cot V - \lambda)^2/(k^2 + \lambda^2).$$

In the above solution the rectangles were all taken as directed interiorly. If they are all directed exteriorly we may proceed in exactly the same way to find

$$S'/S = (k \cot V + \lambda)^2/(k^2 + \lambda^2).$$

The equation in the problem as stated is incomplete. The minus sign in the numerator holds for interiorly directed rectangles. For exteriorly directed rectangles the sign is plus.

Solved also by O. J. Ramler and the proposer.

Editorial Note. We shall give a different construction for the triangles $\alpha\beta\gamma$ which yields other information. On AB as a chord construct the circle (AAB) tangent to CA, and similarly the circles (BBC), (CCA). These three circles meet in the Brocard point Ω' inside ABC so that $\angle\Omega'AC=\angle\Omega'BA=\angle\Omega'CB=V$. Take any point α on (AAB); draw αA cutting (CCA) in γ, and αB cutting (BBC) in β. Then β, C, γ are collinear; and it is easily seen that triangles $\alpha\beta\gamma$ and ABC are directly similar, and that they have the Brocard point Ω' in common. Set $\theta=\angle AB\alpha=\angle\Omega'\alpha A$, then $\cot\theta=\lambda/k$. There are similar equations by cyclic permutation. The triangle $\alpha\beta\gamma$ reduces to a point if and only if $\theta=-V$. The triangle $\Omega'A\alpha$ gives easily the ratio of the sides $\alpha\beta/AB=\Omega'\alpha/\Omega'A=\sin(V+\theta)/\sin V=\sin\theta(\cot V+\cot\theta)$. The maximum $\alpha\beta\gamma$ is the antipedal triangle of Ω' with respect to ABC, *i.e.*, we take α so that αA is perpendicular to $\Omega'A$.

3922 [1939, 453]. *Proposed by V. Thébault, Le Mans, France.*

The triangle BAC is right angled at A; the squares CAA_1C_1 and ABB_1A_2 are constructed exteriorly on the sides CA and AB; and M is the foot on BC of the exterior bisector of angle A. (1) Show that the polygon P', the antipedal of M with respect to polygon $P\equiv BB_1A_2CC_1A_1$, is inscribed in the circle Σ, passing through M and concentric with the square constructed interiorly on the hypotenuse BC. (2) Express the radius r of Σ as a function of elements of triangle BAC, and obtain the condition that $r=BC$. (3) Show that the areas of P and P' are equal.

Note. The antipedal triangle of a point M with respect to a triangle ABC is determined by the intersection of the perpendiculars to MB and MC, to MC and MA, and to MA and MB at the respective points B, A, C.

Editorial Note. Denote the polygon P' by $B_1'A_2'C'C_1'A_1'B'$, where B_1, A_2, $C, \cdots$ lie respectively on the sides $B_1'A_2'$, $A_2'C'$, $C'C_1'$, $\cdots$. Set $AC=b\mathbf{i}$ and $AB=c\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors, and $b\neq c$. Then $CB=c\mathbf{j}-b\mathbf{i}$ and $b\mathbf{j}+c\mathbf{i}$ is a normal vector, while $\mathbf{i}+\mathbf{j}$, $\mathbf{j}-\mathbf{i}$ are vectors along the internal and external bisectors of the right angle A. The center S of the square, directed interiorly, on CB is given by $SA=(b-c)(\mathbf{j}-\mathbf{i})/2$. We then find the following vectors:

$$SM = \frac{b^2+c^2}{2(b-c)}(\mathbf{j}-\mathbf{i}),$$

$$(1) \qquad SB_1' = \frac{1}{2(b-c)}\left[(2bc+b^2-c^2)\mathbf{j}+(2bc-b^2+c^2)\mathbf{i}\right],$$

$$SC' = \frac{b^2+c^2}{2(b-c)}(\mathbf{j}+\mathbf{i}).$$

It will be found that the square of each of the three vectors in (1) is

$$(2) \qquad r^2 = \frac{1}{2}\left(\frac{b^2+c^2}{b-c}\right)^2,$$

and SM, SC' are perpendicular. If $B'B_1'$, $A_1'B'$ cut SM in U, V, it will be seen that S is the common midpoint of C_1B_1 and VU, and the hexagon P' is symmetric with respect to SM and SC'. Hence S is the center of a circle of radius r through M and the vertices of P'. In order for r to have the length of CB, we must have $b^2+c^2-4bc=0$, or $b/c=2\pm\sqrt{3}$, which means that one or the other of the acute angles of ABC must be 15°. Also,

$$(3) \quad \text{Area } B_1'A_2'U = B_1'B_1 \times B_1U = \frac{bc}{c-b}(\mathbf{i}+\mathbf{j}) \times \frac{bc}{c+b}(\mathbf{i}-\mathbf{j}) = \frac{2b^2c^2}{b^2-c^2}\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a right-hand set of orthogonal unit vectors; and

$$(4) \quad \text{Area } UC'B' = C'S \times SU = \frac{b^2+c^2}{2(c-b)}(\mathbf{i}+\mathbf{j}) \times \frac{b^2+c^2}{2(c+b)}(\mathbf{j}-\mathbf{i}) = \frac{(b^2+c^2)^2}{2(c^2-b^2)}\mathbf{k}.$$

The area of P' is twice the sum of the two areas in (1) and (2), or $(c^2-b^2)\mathbf{k}$, which is also the area of P. See the solution of 3867 [1940, 247].

3923 [1939, 454]. *Proposed by R. E. Gaines, University of Richmond.*

It is known that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus. Show that the diameter d of the circle is given by $d \sin\alpha \sin\beta \sin\gamma = a$, where α, β, γ are the angles which the tangents make with the axis of the parabola, $y^2=4ax$.

Solution by Huang K'un, Yenching University, Peiping, China.

We write the equation of the parabola in the parametric form

$$(1) \qquad x = as^2, \qquad y = 2as.$$

Let P_i, $(i=1, 2, 3)$, be any three points on it with their coördinates $(as_i^2, 2as_i)$, respectively. Since the slopes of the tangents of the parabola at P_i are $1/s_i = \tan \alpha_i$, we have their equations

$$t_i\text{:} \quad x - s_i y + a s_i^2 = 0. \tag{2}$$

Now, suppose that $T_1 \equiv (t_2, t_3)$, T_2, T_3 are the three points of intersection of these tangents, that θ_i are the angles of the triangle formed by T_i, and that C is the circumcircle with the diameter d; then we have numerically,

$$T_1T_2 = d \sin \theta_3 = d \sin (\alpha_2 - \alpha_1). \tag{3}$$

On the other hand, if we solve (2) for $i=2, 3$ and $i=3, 1$ we readily obtain the coördinates of T_1 and T_2, respectively, $[as_2s_3,\ a(s_2+s_3)]$ and $[as_3s_1,\ a(s_3+s_1)]$. Thus, we have numerically,

$$T_1T_2 = a(s_1 - s_2)(1 + s_3^2)^{1/2}. \tag{4}$$

Using $\tan \alpha_i = 1/s_i$, and hence $\sin \alpha_i = (1+s_i)^{-1/2}$ and $\cos \alpha_i = s_i(1+s_i)^{-1/2}$, we combine (3) and (4), and we have numerically,

$$d(s_1 - s_2)(1 + s_1^2)^{-1/2}(1 + s_2^2)^{-1/2} = a(s_1 - s_2)(1 + s_3^2)^{1/2}, \tag{5}$$

and therefore,

$$d \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 = a. \tag{6}$$

We note that we have not utilized the known property which is mentioned in the problem.

Solved also by J. Barinaga, J. H. Butchart, E. P. Starke, P. D. Thomas, F. Underwood, and the proposer.

Editorial Note. Barinaga and the proposer showed also that the area of the triangle is

$$\frac{a^2}{2}\,\frac{|\sin (\alpha - \beta) \sin (\beta - \gamma) \sin (\gamma - \alpha)|}{\sin^2 \alpha \sin^2 \beta \sin^2 \gamma};$$

and Barinaga remarked that from this it follows that the area of the triangle formed by the points of contact is twice this expression.

3924 [1939, 515]. *Proposed by M. T. Bird, Utah State Agricultural College.*

Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{b \cos x \sinh y + \sinh 2y}{b \sin x \cosh y + \sin 2x}.$$

Solution by Paul D. Thomas, Norman, Okla.

Writing the equation in the form

$$b(\sin x \cosh y\, dy + \cos x \sinh y\, dx) + (\sin 2x\, dy + \sinh 2y\, dx) = 0,$$

and using the integrating factor $(\sin x \sinh y)^{-2}$, we find the general solution

$$b \csc x \operatorname{csch} y + 2 \cot x \coth y = C \text{ (arbitrary constant).}$$

Solved also by J. A. Bullard, J. P. Dalton, D. F. Gunder, J. F. Locke, O. J. Ramler, F. C. Smith, F. Underwood, A. K. Waltz, and the proposer.

Editorial Note. The solutions by Bullard, Dalton, and Locke were similar to the above. Gunder, Ramler, and Underwood introduced complex variables, setting $\sinh y = -i \sin iy$, $\cosh y = \cos iy$. This gives

$$\frac{\sin z\, dz}{b + 2 \cos z} = \frac{\sin \bar{z}\, d\bar{z}}{b + 2 \cos \bar{z}},$$

where $\bar{z}$ is the conjugate of $z = x + iy$. Smith multiplied $\sin 2x$ by $\cosh^2 y - \sinh^2 y = 1$ and $\sinh 2y$ by $\cos^2 x + \sin^2 x = 1$ in the equation of the solution above. Then setting $u = \cos x \cosh y$, $v = \sin x \sinh y$, he obtained the equation $b\, dv - 2\, v\, du + 2\, u\, dv = 0$, *etc.* The proposer set $t = \tan(x/2)$, $s = \tanh(y/2)$, and then $u = t/s$, $v = st$; this separates the variables. Waltz set $u = \operatorname{sech} y$, $v = \cos x$, and then used a power series development of u in terms of v.

3925 [1939, 515]. *Proposed by V. Thébault, Le Mans, France.*

The triangle ABC is right angled at A, and has the inscribed and escribed circles (I), (I_a), (I_b), (I_c) with the radii r, r_a, r_b, r_c. The parallel to BC through I_a, the center of (I_a), cuts AB in N and AC in M; the orthogonal projections of M and N on BC are P and Q. Show that: (1) $MQ - MN = r$, $MQ - MP = (BC)^2/2r$; (2) the circumcircle (ω) of rectangle $MPQN$ is tangent to (I_a), (I_b), (I_c); and (3) if D and E are the other intersections of (ω) with AB and AC, then $MP = DE = QN = r_a$, and the lines MP, DE, QN are tangent to a circle with the center ω.

Solution by J. H. Butchart, William Woods College.

Let the coördinates of A, B, C be $(0, 0)$, $(c, 0)$, $(0, b)$, respectively; then the coördinates of the incenter and excenters are

$$I[bc/2s,\ bc/2s], \quad I_a[bc/2(s-a),\ bc/2(s-a)], \quad I_b[-bc/2(s-b),\ bc/2(s-b)],$$
$$I_c[bc/2(s-c),\ -bc/2(s-c)], \qquad 2s = a + b + c.$$

The equation of the straight line through I_a parallel to BC now gives us $M[0,\ b(b+c)/2(s-a)]$ and $N[c(b+c)/2(s-a),\ 0]$. By similar triangles we then compute the coördinates of $P[-b^2c/2a(s-a),\ b(ab+ac-c^2)/2a(s-a)]$ and of $Q[c(ab+ac-b^2)/2a(s-a),\ -bc^2/2a(s-a)]$. By the midpoint formula we find $\omega[c(ab+ac-b^2)/4a(s-a),\ b(ab+ac-c^2)/4a(s-a)]$. Properties (1) and (2) may now be proved by the usual elementary methods with occasional use of $a^2 = b^2 + c^2$ although the second part of (2) is laborious. Statement (3) follows from evident relations of symmetry.

Editorial Note. Part (1) may be proved by comparison of the similar right triangles ABC, PMC, QBN, ANM. From the first three we have $CP = bs/c$,

$BQ=cs/b$; and then $NM=QP=a(b+c)s/bc$. Then $(QM)^2=s^2+(MN)^2$, or $QM=s(a^2+bc)/bc$. Hence $QM-NM=s(a-b)(a-c)/bc=(s-a)=r$; and $QM-PM=sa^2/bc=a^2/2r$. As stated in the solution, (3) is obvious from the figure. Also, since A, B, M, P are concyclic, PM and AB are antiparallel; similarly, ED and BC, QN and CA are antiparallel pairs. Hence the circle (ω) is a Tucker circle.

The proposer stated that (2) and (3) result from a general theorem, which he will publish in *Mathesis*, which says that the circle tangent interiorly to the three escribed circles of any given triangle is a Tucker circle.

3926 [1939, 515]. *Proposed by V. Thébault, Le Mans, France.*

With the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 form a number with ten figures with no repetition of digits such that, if this number is increased by unity, the sum is a perfect square.

Note by E. P. Starke, Rutgers University.

Two such numbers are known already from E414[1940, 711], *viz.*, 7319658024 $=(85555)^2-1$ and $9560341728=(97777)^2-1$. Other values may be found by noting that if N^2-1 is such a number, then N is of one of the forms $9k\pm1$. Squares of sufficiently large numbers of this form may be looked up in tables of squares, and only those whose values, within the limits of the tables, show no duplicate digits need be multiplied out in detail. Such a value is 1265794083 $=(35578)^2-1$. Since the proposal requires only "a number," it seems undesirable to canvass all the possibilities.

Editorial Note. The proposer gave the following indications of a solution: If M is one of the desired numbers, then $M=(N-1)(N+1)$; and, since the sum of the ten figures of M is 45, the factor $N-1$, or $N+1$, must be divisible by 9. The first figure of N must be 3, 4, 5, 6, 7, 8, or 9. By means of easy trials we find M; six examples are

$$4061257983 + 1 = 63728^2, \quad 6971248035 + 1 = 83494^2,$$
$$5871390624 + 1 = 76625^2, \quad 7319658024 + 1 = 85555^2,$$
$$6150794328 + 1 = 78427^2, \quad 9560341728 + 1 = 97777^2.$$

3928 [1939, 601]. *Proposed by J. R. Musselman, Western Reserve University.*

If O is the circumcenter of triangle $A_1A_2A_3$, and B_i is the image of A_i in the side A_jA_k, show that the circles A_1OB_1, A_2OB_2, A_3OB_3 meet in that point which is the inverse in the circumcircle of the isogonal conjugate point of the nine-point center.

Generalization by R. Goormaghtigh, Bruges, Belgium.

The considered property is equivalent to the following one: the intersections of the sides A_2A_3, A_3A_1, A_1A_2 with the perpendiculars to A_1O, A_2O, A_3O at their midpoints are on a straight line, and the image of O in that line is the inverse, as to the circumcircle, of the isogonal conjugate of the nine-point center.

We will now prove this more general theorem:

THEOREM. *If C_1, C_2, C_3 are the points on A_1O, A_2O, A_3O such that*

$$C_1O:A_1O = C_2O:A_2O = C_3O:A_3O = \lambda,$$

the intersections of the perpendiculars at C_1, C_2, C_3 on A_1O, A_2O, A_3O with the sides A_2A_3, A_3A_1, A_1A_2 are on a straight line Δ; if P is the projection of O on Δ and if Q is the point such that

$$OQ/OP = 1/\lambda,$$

then Q is the inverse, as to the circumcircle, of a point J on the Euler line, such that

$$OJ/OH = 1/(2\lambda + 1),$$

H being the orthocenter.

Proof. In complex coördinates, having as base circle the circle $A_1A_2A_3$ with unit radius, let t_1, t_2, t_3 be the turns corresponding to A_1, A_2, A_3, and call σ_1, σ_2, σ_3 their symmetric functions, *i.e.*,

$$\sigma_1 = t_1 + t_2 + t_3, \qquad \sigma_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad \sigma_3 = t_1t_2t_3;$$

then, if $\bar{a}$ is the conjugate to a, we have

$$\bar{\sigma}_1 = \sigma_2/\sigma_3, \qquad \bar{\sigma}_2 = \sigma_1/\sigma_3, \qquad \bar{\sigma}_3 = 1/\sigma_3.$$

The equations of A_2A_3 and the perpendicular at C_1 to A_1O are

$$x + \bar{x}t_2t_3 = t_2 + t_3,$$
$$x + \bar{x}t_1^2 = 2\lambda t_1;$$

their intersection is

$$[(2\lambda + 1)\sigma_3 - \sigma_2 t_1]/(t_2t_3 - t_1^2)$$

and therefore Δ has for its equation

$$[\sigma_1^2 - (2\lambda + 1)\sigma_2]x + [\sigma_2^2 - (2\lambda + 1)\sigma_1\sigma_3]\bar{x} + (2\lambda + 1)^2\sigma_3 - \sigma_1\sigma_2 = 0.$$

Hence the point Q is

$$\frac{\sigma_1\sigma_2 - (2\lambda + 1)^2\sigma_3}{2\lambda[\sigma_1^2 - (2\lambda + 1)\sigma_2]}\,.$$

But J is the point $\sigma_1/(2\lambda+1)$; and, as the isogonal conjugate of a point x is

$$y = (x - \sigma_1 + \sigma_2\bar{x} - \sigma_3\bar{x}^2)/(x\bar{x} - 1)$$

(F. and F. V. Morley, *Inversive Geometry*, p. 196), the isogonal conjugate of J is

$$2\lambda\frac{\sigma_2^2 - (2\lambda + 1)\sigma_1\sigma_3}{\sigma_1\sigma_2 - (2\lambda + 1)^2\sigma_3},$$

the inverse of Q as to the circumcircle, for two inverse points u and v are such that $u\bar{v}=1$.

It may be noted that the fact that the intersections of A_2A_3, A_3A_1, A_1A_2 with the perpendiculars at C_1, C_2, C_3 to A_1O, A_2O, A_3O are on a straight line Δ is well known; the theorem was given in Neuberg's *Mémoire sur le Tétraèdre*, 1884, where it is also shown that the envelope of Δ is the inscribed parabola having the Euler line as directrix (Kiepert's parabola).

When $\lambda=1/2$, J is the nine-point center, and we have the theorem of Problem 3928.

When $\lambda=1$, J is the centroid and Δ the Lemoine axis; we find a theorem equivalent to the well known property that the Lemoine axis is the polar of the Lemoine point as to the circumcircle.

Solved also by E. F. Allen and Frank Ayres, Jr., both solutions using inversive geometry as above.

3929 [1939, 601]. *Proposed by J. R. Musselman, Western Reserve University.*

The perpendiculars to the sides of triangle $A_1A_2A_3$ from any point T on the circumcircle of the triangle cut the circle again in the points B_1, B_2, B_3. Show that the image lines of B_i cut the sides A_jA_k in three collinear points. The line of these points is Δ_2 in the problem 3758 [1937, 668].

Editorial Note. For the definition of image line, see the article by the proposer in this MONTHLY, 1938, p. 421, entitled *On the line of images.*

I. *Solution by R. Goormaghtigh, Bruges, Belgium.*

Let P be the homothetic to T in the homothety having $-1/2$ as the ratio and the centroid as center. Since the line of images of B_1 is parallel to A_1T and passes through the orthocenter, the considered intersections are those of the sides with the parallels drawn through the orthocenter to the joins of a point P on the nine-point circle to the midpoints of the sides, and therefore [1937, 668] these intersections are on the Δ_2 line corresponding to P.

II. *Solution by Frank Ayres, Jr., Dickinson College.*

Let the coördinates of the points A_i be denoted by the turns t_i satisfying $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$, and that of the point T by the turn T. Then the coördinate of the point B_i is $-t_jt_k/T$, and the equation of its image line is

$$-\frac{t_jt_k}{T}x-\sigma_3\bar{x}=-\frac{t_jt_k}{T}\sigma_1-\sigma_2.$$

This line intersects the side A_jA_k in the point $x=t_jt_k(\sigma_1+T)/(t_jt_k-t_iT)$, which lies on the line of equation

$$x(T\sigma_2+\sigma_3)+\bar{x}T\sigma_3(\sigma_1+T)-(T\sigma_2+\sigma_3)(\sigma_1+T)=0.$$

Solved also by E. F. Allen and J. W. Clawson in the same way as II.

3930 [1939, 601]. *Proposed by V. Thébault, Le Mans, France.*

Three forces are applied at any point P of the circumcircle of the triangle ABC directed toward the vertices A, B, C. Show that: (1) if the three forces are equal, their resultant passes through a fixed point; (2) if the forces are proportional to the lengths of the sides BC, CA, AB, they are in equilibrium.

Solution by C. M. Sparrow, University of Virginia.

The theorems as stated are not quite accurate, being subject to qualifications which will be noted presently. They are, moreover, only special cases of a much more general theorem which will be proved here, namely:

THEOREM. *If $A_1, A_2, \cdots, A_n, P$ are points on a circle, and forces of constant magnitudes $F_1, F_2, \cdots, F_n$ act along $PA_1, PA_2, \cdots, PA_n$, these forces have a resultant of constant magnitude R, along a line PQ, where Q is a fixed point on the circle, for all positions of P lying between two neighboring points.*

Proof. Take the circle of unit radius, and denote the positions of $A_1, \cdots, A_n$, P by the angles $2\theta_1, \cdots, 2\theta_n, 2\phi$, measured from an arbitrary polar axis. For brevity we will write s_k, c_k for $\sin\theta_k$, $\cos\theta_k$, and s, c for $\sin\phi$ and $\cos\phi$. Let $PA_k=r_k$; then $r_k=2\sin(\theta_k-\phi)=2(s_kc-c_ks)$. We have then

$$F_1r_1 + F_2r_2 + \cdots + F_nr_n = 2c\sum F_ks_k - 2s\sum F_kc_k. \tag{1}$$

If we put $F_1s_1+F_2s_2+\cdots+F_ns_n=R\sin\psi$, $F_1c_1+F_2c_2+\cdots+F_nc_n=R\cos\psi$, the right-hand member of (1) becomes $R\cdot 2\sin(\psi-\phi)$. If Q is defined by the angle 2ψ, and $PQ=r$, we have, therefore,

$$F_1r_1 + F_2r_2 + \cdots + F_nr_n - Rr = 0. \tag{2}$$

But this is the potential energy of the system of forces $F_1, \cdots, F_n, -R$, acting along $PA_1, \cdots, PA_n, PQ$. Since this potential energy is constant, the forces are in equilibrium.

Theorem (1) is obviously a special case of this. Theorem (2) is a special case where $R=0$, but is true only when the force toward that vertex of the triangle which is separated from P by one side of the triangle is reversed in sign. That is, if P is on the arc BC, the force along PA must be from P to A. In this case, equation (2) becomes

$$AC\cdot r_2 + AB\cdot r_3 = BC\cdot r_1,$$

which is nothing but the well known theorem regarding four cyclic points A, B, C, P.

We see, however, that the expression for r_k, namely $r_k=2\sin(\theta_k-\phi)$, changes sign when P passes through A_k. If the point Q is to remain fixed when this happens, the sign of F_k must also be reversed. Possibly this theorem is well known. It is a rather curious property of n points on a circle that when the multipliers $F_1, \cdots, F_n$ are given, a value of R and of ψ can be found to satisfy (2) independently of the position of P. Another way of stating the theorem is as follows:

If forces of constant magnitude directed from a point P toward fixed points on a circle with P are in equilibrium, this equilibrium is astatic.

Solved also by E. F. Allen, R. Goormaghtigh, V. W. Graham, and F. Underwood.

Editorial Note. Goormaghtigh gave a generalization similar to the above for $n=3$ with a different proof. Graham stated that part (2) is exercise 13 in Minchin's *Statics*, vol. 1, fifth edition, p. 19, and that this exercise is followed by the remark that Ptolemy's theorem results from it. There is also a statistical proof of Brianchon's theorem in the appendix of this text. Underwood stated that (2) is part of example 43 in Dobb's *Elementary Geometrical Statics*, p. 162, and that similar problems are given in exercises 41–46, pp. 161, 162. The proposer stated that the four fixed points, resulting from the possible directions of the forces, equal in magnitude, are the Feuerbach points for the anticomplementary triangle; and that this statement results from his theorem, which will appear in *Mathesis*.

3931 [1939, 601]. *Proposed by V. Thébault, Le Mans, France.*

What must be the base of a number system such that numbers of the form *abcabc*004004 and 4004*abcabc* are the squares of numbers of the form *defdef* and *ghighi*?

Solution by E. P. Starke, Rutgers University.

Let B be the desired base, put $L=aB^2+bB+c$, $M=dB^2+eB+f$, $N=gB^2+hB+i$. Then by hypothesis we have $(LB^6+4)(1+B^3)=M^2(1+B^3)^2$ and $(4B^6+L)(1+B^3)=N^2(1+B^3)^2$, or

$$LB^6+4=M^2(1+B^3), \qquad 4B^6+L=N^2(1+B^3). \tag{1}$$

From the above relations, we see that B^3+1 divides $LB^6+4=L(B^3+1)(B^3-1)+L+4$; hence B^3+1 divides $L+4$. But L is less than B^3. Therefore (for $B>1$) we must have $L+4=B^3+1$, and from (1) we obtain

$$L=B^3-3, \qquad M=B^3-2, \qquad N^2=4B^3-3. \tag{2}$$

If now B be so chosen that $4B^3-3$ is a square, we have a solution. By trial, $B=7$ is such a value. Then $L=664$ (written to base 7), $M=665$, $N^2=3664$ or $N=052$. Thus $665665^2=664664004004$ and $052052^2=4004664664$. (Note that the first digit g of N must be 0 whatever the value of B, for $g\geqq 1$ implies $ghighi\geqq B^5$ or $ghighi^2\geqq B^{10}$, but $4004abcabc<B^{10}$.) A general solution in integers of $N^2=4B^3-3$ presents much greater difficulty.

Solved also by the proposer.

Editorial Note. The proposer stated that this problem is an application of the general properties of squares of the form $a_1a_2\cdots a_n\, b_1b_2\cdots b_n=(c_1c_2\cdots c_n)^2$ which he discussed at the last Congrès International des Récréations Mathématiques, Paris, 1937. The value of B is given by the solution in integers of $B^3=t^2+t+1$, and he gave the particular solution $B=7$ with the results as above.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

R. D. Dorsett of the University of Oklahoma has been promoted to an assistant professorship.

Professor R. A. Fisher of University College, London, will be visiting professor of experimental statistics at North Carolina State College during the coming summer session.

Assistant Professor F. H. Hodge of Purdue University has retired.

Dr. K. C. Schraut of the University of Dayton has been promoted to an assistant professorship.

Assistant Professor S. B. Townes of the University of Oklahoma has been promoted to an associate professorship.

Dr. G. C. Webber of the University of Delaware has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
University of Illinois: Dr. R. S. Pate
Iowa State Teachers College: Dr. H. C. Trimble
Oregon State College: Dr. P. C. Hammer

Professor G. A. Knapp of Maryville College, Maryville, Tennessee, died in November, 1940.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1941.

University of California at Berkeley. The following advanced courses will be offered: By Miss Levy: Coördination in mathematics for secondary schools. By Mr. Goldsworthy: Mathematical theory of aeronautics.

University of California at Los Angeles. In addition to the usual courses through the Calculus, the following courses will be offered: By Dr. P. G. Hoel: Statistics. By Dr. A. E. Taylor: Partial differential equations of mathematical physics. By Professor W. M. Whyburn: Calculus of variations. By the staff: Special problems in mathematics.

Catholic University of America. In addition to the usual undergraduate courses, the following advanced courses will be offered: By Dr. Daly: Fundamental concepts. By Professor Rice: Advanced calculus. By Professor Finan: Theory of groups. By Professor Ramler: Advanced analytic geometry.

University of Chicago. In addition to courses in the Calculus, the following courses will be offered: By Professor Bliss: Ballistics. By Professor Dickson:

Theory of numbers. By Professor Lunn: Lattices and crystal groups, Relativity. By Professor Barnard: Higher algebra, Modern theories of integration. By Professor Logsdon: Theory of functions of a complex variable, Projective geometry of hyperspace. By Professor Steenrod: Elementary topology, Differential equations. By Professor Sanger: Synthetic projective geometry. By Dr. Schilling: Theory of equations, Seminar on topics in algebra.

Cornell University. The following advanced courses will be offered: By Professor B. W. Jones: Teachers course, Analytic projective geometry. By Professor W. A. Hurwitz: Modern algebra, Mathematical recreations. By Professor R. P. Agnew: Elementary differential equations. By Professor J. F. Randolph: Advanced calculus.

Duke University. First term, June 10 to July 21. By Professor Carlitz: Projective geometry, Theory of equations, Thesis seminar. By Professor Gergen: Advanced calculus, Real variable, Thesis seminar. By Professor Murnaghan of Johns Hopkins University: Modern developments in mathematics, Vector analysis. By Professor Rankin: The teaching of mathematics. *Second term, July 23 to August 30.* By Professor Roberts: Fundamental concepts in algebra, analysis, and geometry, Projective geometry, Thesis seminar. By Professor Thomas: Advanced calculus, Theory of equations, Thesis seminar.

University of Illinois. In addition to the usual elementary courses, the following advanced courses will be offered: By Dr. E. L. Welker: Elementary statistics. By Dr. I. M. Niven: Introduction to higher algebra. By Professor P. W. Ketchum: Graphical and mechanical methods, Differential geometry and vector analysis. By Professor J. L. Doob: Theory of probability, Topology. By Professor H. R. Brahana: Theory of groups, Algebra. By Professor W. J. Trjitzinsky: Advanced calculus, Complex variable.

The State University of Iowa. In addition to the usual elementary courses, the following advanced courses are to be offered: By Professor Woods: Modern geometry. By Professor Craig: Matrices and determinants, Statistics. By Professor Chittenden: Infinite series. By Professors Chittenden and Ward: Seminar in analysis.

University of Michigan. June 30 to August 22. In addition to elementary courses and the standard courses in Differential equations, Theory of equations and determinants, Advanced solid analytic geometry, and Advanced calculus, the following courses will be offered: By Professor Artin: Representation theory of groups and algebras. By Professor Ayres: Theory of functions of a real variable, General spaces. By Professor Beckenbach: Harmonic and sub-harmonic functions. By Professor Carver: Finite differences, Theory of statistics II. By Professor Churchill: Fourier series and applications, Methods in partial differential equations. By Professor Copeland: Statics, Introduction to the foundations of mathematics. By Professor Craig: Theory of statistics I, Advanced theory of statistics II. By Professor Dushnik: Graphical methods. By Professor

Karpinski: Teaching of geometry, History of arithmetic and algebra. By Dr. Nesbitt: Theory of probability. By Professor Poor: Vector analysis. By Professor Rainich: Algebraic theory, Topics in higher geometry. By Dr. Rainville: Elementary course in complex variable. By Dr. Thrall: Analytic projective geometry. In addition, there will be an orientation seminar, a seminar in pure mathematics, and a seminar in statistics conducted by Professor Craig.

University of Minnesota. First term, June 18 to July 25. In addition to the usual elementary work, the following courses will be offered: By Professor Jackson: Mathematics of exterior ballistics, Fourier, Legendre, and Bessel series. By Professor Underhill: Intermediate calculus. By Professor Gibbens: Modern analytic geometry. By Professors Jackson, Underhill, Gibbens, and Dr. Olmsted: Tutorial courses in senior college and in advanced mathematics. *Second term, July 28 to August 29.* By Professor Carlson: Theory of numbers. By Professor Carlson and Dr. Campaigne: Tutorial courses in senior college and in advanced mathematics.

University of North Carolina. First term, June 12 to July 19. In addition to the regular courses through the Calculus, the following courses will be offered: By Professor Hoyle: College geometry, Advanced calculus. By Professor Lasley: Differential geometry. By Professor Henderson: Differential equations, Analytic geometry of space, Theory of functions of a complex variable. *Second term, July 21 to August 27.* By Professor Garner: Differential equations (continued). By Professor Mackie: Advanced calculus (continued), Theory of functions of a complex variable (continued). By Professor Browne: Analytic geometry of space (continued).

Northwestern University. In addition to courses in Analytic geometry and the Calculus, the following advanced courses will be offered: By Professor Moulton: Vector analysis. By Professor Davis: Teaching of mathematics, Functions of a complex variable. By Professor Wall: Geometry for teachers. By Professor Garabedian: Analytical mechanics.

Ohio State University. In addition to elementary courses in Algebra, Analytical geometry, Statistics, and the Calculus, the following advanced courses will be offered: By Professor Radó: Integral equations, Projective geometry. By Professor Rasor: Differential equations, Functions of a complex variable. By Professor Morris: Mathematical statistics. By Dr. Youngs: Infinite series and products.

University of Pennsylvania. In addition to the usual elementary courses, including the Calculus, the following courses will be offered: By Professor Babb: Modern analytic geometry. By Professor Caris: Advanced calculus. By Professor Shohat: Theory of probability, Theory of functions of a complex variable.

University of Southern California. In addition to elementary courses, the following courses will be offered: *Six weeks division, June 28 to August 7.* By Pro-

fessor Ames: Theory of probability and statistics, Modern higher algebra. By Professor Fagerstrom: Seminar (topic to be announced later). By Professor Steed: Differential equations. *Four weeks division, August 7 to August 30.* By Professor Butter: Second course in analytic geometry, Infinite series.

Stanford University. By Professor Pólya: Elementary mathematics from the higher point of view. By Professor Hille: Selected topics in the theory of analytic functions. By Professor G. T. Whyburn: Selected topics in combinatorial topology.

Teachers College, Columbia University. July 7 to August 15. By Professor Bradley: Navigation, professionalized subject-matter for teachers. Professor Clark: Teaching algebra in secondary schools, Teaching geometry in secondary schools. By Dr. Lazar: Teaching algebra in junior high schools, Logic for teachers of mathematics. By Mr. Mirick: Demonstration class in plane geometry, Elementary mechanics for teachers in secondary schools. By Professor Reeve: Teaching and supervision of mathematics in junior high schools, Teaching and supervision of mathematics in senior high schools. By Professor Shuster: Modern business arithmetic, methods of teaching in junior and senior high schools, Field work in mathematics. By Dr. R. R. Smith: Mathematics for the eleventh year of the high school, Professionalized subject-matter in geometry. By Miss Sutherland: Teaching arithmetic in primary grades, first three (July 7 to 25), Teaching arithmetic in intermediate grades, fourth, fifth, and sixth (July 28 to August 15), Professionalized subject-matter in junior high school mathematics.

University of Texas. In addition to the regular elementary courses, the following advanced courses will be offered: *First term, June 5 to July 16.* By Professor R. L. Moore: Introduction to the foundations of geometry, Theory of sets. By Professor E. L. Dodd: Probability, Theory of functions of real variables. By Professor H. S. Vandiver: Introduction to the foundations of algebra, Topics in modern algebra. By Professor P. M. Batchelder: Advanced calculus. By Professor E. G. Keller: Advanced applied mathematics. By Professor R. N. Haskell: Dynamics. *Second term, July 16 to August 25.* By Professor H. J. Ettlinger: Theory of functions of real variables, Calculus of variations. By Professor R. G. Lubben, Non-euclidean geometry, Projective geometry. By Dr. F. B. Jones: Advanced calculus. By Professor A. E. Cooper: Theory of functions of a complex variable, Group theory of differential equations.

Texas Technological College. First term, June 4 to July 14. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor Michie: Theory of equations. By Dr. Wakerling: Synthetic projective geometry, Research and reading course for master's thesis. *Second term, July 15 to August 22.* By Professor Michie: Analytic projective geometry, Research and reading course for master's thesis.

University of Virginia. First term, June 16 to July 25. By Professor Hedlund: Differential geometry. By Professor McShane: Foundations of geometry. *Second term, July 28 to August 29.* By Professor Hedlund: Differential geometry, Functions of a real variable.

University of Washington. Advanced euclidean geometry, Probabilities and statistics, Advanced analysis, Finite groups, and Potential theory. The last named course will be given by Professor G. C. Evans, who will be the Walker-Ames Professor of Mathematics for 1941.

University of Wisconsin. In addition to the usual elementary courses, the following advanced courses will be offered: *6-week courses.* By Mr. Goheen: Theory of equations. By Mr. Trump: College geometry. By Professor Evans: Topics in vector analysis, Theory of probability. By Professor Langer: Topics in modern analytic geometry. By Professor Ingraham: Mathematics of educational statistics, Critique of elementary and collegiate mathematics, Quaternions. *8-week courses* (can be taken for six weeks). By Professor Evans: Advanced calculus. By Professor Langer: Theory of analytic functions, Calculus of variations.

University of Wyoming. The following advanced courses will be offered: *First term.* Fundamental concepts of mathematics, Functions of a complex variable, The teaching of secondary mathematics. *Second term.* History of mathematics, Functions of a complex variable.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 2, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., May 3.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 10.
MICHIGAN, Ann Arbor, March 15.
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, April 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, April 4–5.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit, May 3.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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CONTENTS

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THE THIRD ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The third annual meeting of the Northern California Section of the Mathematical Association of America was held at the San Francisco Junior College, San Francisco, California, on Saturday, January 25, 1941. Professor Sophia H. Levy, chairman of the Section, presided at both morning and afternoon sessions. During the noon recess, luncheon was served for members and visitors at Kay's Restaurant near the college campus.

The attendance at the two sessions was approximately sixty, including the following twenty-one members of the Association: H. M. Bacon, T. J. Bass, Jr., H. F. Blichfeldt, G. C. Evans, M. A. Heaslet, Emma V. Hesse, V. F. Ivanoff, D. H. Lehmer, Sophia H. Levy, A. L. McCarty, F. R. Morris, W. H. Myers, E. B. Roessler, Marcus Skarstedt, Ethel Spearman, Pauline Sperry, Falka G. Sturges, Ruth G. Sumner, Gabor Szegö, Harriet A. Welch, Fredrick Wood.

The following officers were elected for the coming year: Chairman, F. R. Morris, Fresno State College; Vice-Chairman, Fredrick Wood, University of Nevada; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education*. By invitation of the Section, Professor Pauline Sperry of the University of California gave an hour's address during the morning session.

The following papers were read:

1. "On a local solution of a differential equation of infinite order" by Dr. A. C. Burdette, University of California at Davis, introduced by Professor Roessler.

2. "A college course in survey of elementary mathematics" by Professor W. H. Myers, San José State College.

3. "The gyroscope and its uses in aviation and navigation" by Professor Pauline Sperry, University of California.

4. "On Descartes's rule of signs" by Professor J. V. Uspensky, Stanford University, introduced by the Secretary.

5. "Mathematics in Civil Service examinations" by Mrs. Falka G. Sturges, Humboldt Evening School, San Francisco.

6. "On certain equations all of whose roots are real" by Professor J. H. McDonald, University of California, introduced by the Chairman.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Dr. Burdette exhibited a solution for a class of differential equations of infinite order with constant coefficients in which the known function is analytic in a finite region. The class of equations studied consists of those equations for which the characteristic function is of finite exponential type and possesses the further property of being bounded away from zero on an infinite sequence of expanding closed contours.

2. A course designed to meet the needs of students entering San José State

College with not more than one year of high school algebra has used a text written by Professors Myers and Minssen. Issued in 1932 in mimeograph form, reissued in 1936 and photolithographed in 1940 by the Stanford University Press, the material covers arithmetic, algebra, geometry, intermediate algebra, trigonometry, and brief introductions to analytic geometry and calculus. The course "Survey of Mathematics" meets five days per week although only three units of credit are given. Experience shows that this survey course provides the student with a better foundation for subsequent college mathematics courses than does a course in algebra alone. The student who does recommendable work in this course is permitted to enroll in trigonometry, while work which is passing but not recommendable leads the student into a course in intermediate algebra before trigonometry.

3. The gyroscope, invented by the French physicist Foucault some ninety years ago to demonstrate the earth's rotation, remained for half a century little more than a toy. Today in the air, on land and sea and beneath them, it is doing man's work more efficiently than he can do it himself. The mathematical principles underlying the properties of the gyroscope were explained by Professor Sperry and experiments illustrating the following applications of it were made.

Underlying the gyrocompass we find the principles of rigidity and precession as well as the earth's rotation and gravitation. A mercury ballistic attached to the casing of the rotor of the gyroscope keeps the axle always in the meridian and horizontal. Thus it furnishes an indication of true North without being subject to any of the adverse forces affecting the magnetic compass. It has proved its efficiency to within seven degrees of both the North and South Poles and in submarines, places where the magnetic compass is useless. The compass on ships actuates the gyropilot which keeps the ship always on its course with never more than two degrees of variation, by automatically applying just the right amount of rudder in each case and making hourly corrections to insure great circle sailing.

The gyro stabilizer for ocean-going vessels is a very heavy rotor mounted amidships at the water line, which in response to the small sensitive gyro that feels the wave almost before it reaches the ship, starts precessing in just the amount necessary to iron out the effect of the wave at its very beginning, so that no roll is allowed to build up.

The directional and horizon gyros, two very small free airspun gyros, rotating with great rapidity, mounted in the airplane with their axles horizontal and vertical respectively, indicate to the pilot flying blind just what his altitude and direction are at all times and in such a way as to simulate by the artificial horizon the impression he would receive were he able to see outside the cockpit. If the plane deviates from its course or its horizontal position, the gyroscope through appropriate mechanisms sets the rudder, ailerons, or elevator in motion to make just the necessary correction. The directional gyro came out of the air to direct Byrd's "Snow Cruiser" on a straight course over the snowy wastes in the Antarctic night. Today it charts the curves on the nation's highways.

4. Professor Uspensky showed how Descartes's rule of signs, if properly applied to equations without multiple roots, can be used for the exact determination of the number of positive (or negative) roots. If, after substitutions $x=1+y$ or $x=(1+y)^{-1}$, the transformed equations have not more than one variation, the question is settled by Descartes's rule of signs. Otherwise we continue to apply similar substitutions until we arrive at transformed equations with one or no variations, which necessarily must happen by virtue of a theorem of Vincent, published more than a hundred years ago, but seemingly soundly forgotten.

5. Mrs. Sturges, a teacher of Civil Service, stated that the City of San Francisco Civil Service Commission reports that for all groups of workers except the professionals, mathematics enters into tests only in the form of arithmetic. In the professions, as, for instance, in statistics or engineering, either appropriate university courses or degrees are required, or both. The same plan, with slight variations, holds true for the State. The Federal Government at present is working under emergency conditions, and practically no written examinations are being held. Ordinarily a few arithmetic questions appear in Law Enforcement Tests, but, where knowledge of mathematics is not essential, in no position is mathematics given emphasis. In the professional group, requirements are approximately the same as those of City and State.

6. Professor McDonald noted that, while the determination of the number of real roots of a given equation can be effected by methods associated with the names of Sturm and Hermite, the construction of irreducible equations whose roots have an assigned character of reality is not altogether easy. It was the object of the communication to present classes of equations of some generality whose roots are all real, and to exhibit some of their properties.

H. M. BACON, *Secretary*

MATHEMATICS IN THE DEFENSE PROGRAM*

MARSTON MORSE, Institute for Advanced Study,
AND
WILLIAM L. HART, University of Minnesota

1. Introduction. The American Mathematical Society and the Mathematical Association of America number some 5000 members. The American Mathematical Society is devoted primarily to the development of research in mathematics, and the Mathematical Association of America to the teaching of mathematics. About a year ago these societies appointed a committee known as the War Preparedness Committee, to prepare the two societies to be useful to our

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* Presented by Professor Morse before the National Council of Teachers of Mathematics at Atlantic City on February 21, 1941. The part with special reference to the secondary field, and certain other sections, comprise the essential portion of an address on "Mathematics for National Service" presented by Professor Hart before the National Council at Baton Rouge on January 1, 1941.

nation in time of war. The ways and means of doing this were not prescribed, but were left to the committee. Before I give you details about our organization and aims, it will be helpful to make a few remarks concerning the rôle of science in defense in general.

The most effective employment of science in a defense program must include the use not only of the facts of science, but also of the methods and men. In time of war, science must be resourceful and inventive and capable of *quick* analysis of emergency problems. The defense against the magnetic mine by the English is a magnificent example of the immediate application of theory to practice. Theoretical science cannot be neglected; for it is the reservoir of general methods, any one of which may be needed. But theoretical science should be in a form in which it can be quickly applied. We should further develop the technique of making applications.

This is particularly true of mathematics. North America leads the world in pure mathematics. We are also strong in the simpler applications appearing in ordinary engineering or industrial practice; but we have preferred experiment to theory and have tended to use the laboratory to obtain results which might have been predicted. This is in contrast to the situation in Europe, where tradition as well as material necessity have produced engineers with greater theoretical knowledge and training. This state of affairs should be remedied; for in time of war we cannot take the time to experiment.

We are beginning to correct this situation. In this we are aided by a number of European experts of great talent and ability. Several of the leading authorities on aerodynamics of Germany are now refugees in this country. The leading mathematical authority on ballistics of Italy is also a refugee and is lecturing in this country. In addition, there are a few Americans who are well trained in these fields. But these authorities are in such demand from industry for immediate purposes that they have little time for teaching or general education and research.

I have given you one reason why this bottleneck has arisen. There is another reason which goes very deep. It is our national suspicion of theory, on the part of the general public. We are perilously lowbrow. This is dangerous in a democracy where the great motivating forces must come from the people. One result has been a lack of coöperation between the theoretically-minded scientist and the practically-minded scientist. The pure scientists have intensified their study of science for science's sake, and the applied scientists have adhered to "common sense" and the laboratory. It is one of the problems of education to show that the more mature and socially-minded way is to respect both theory and practice, and particularly their combination.

In this connection I wish to refer you to a pamphlet on *Science in War*, written by twenty English scientists during the last year. This book is in the Penguin series, costs twenty-five cents, and may be ordered from New York at any bookstore. It is an illuminating account of the success of science when used in the English defense, and of the difficulties in getting science used. Here are

discussed the problems of nutrition, of agriculture, of stock-breeding and planting, and of the reactions of the Civil Service and tradition to these problems. There is the problem of rationing, of the hours of labor, of the care of the wounded and prevention of disease, the dispute between the artists and naturalists over camouflage, the uses of mechanical science, the problems of morale and propaganda. On reading this book one sees clearly the necessity in a democracy of an adequate understanding of science by the general public, and as a corollary the fundamental need of education in the methods and aims of science, as well as in the facts.

2. Organization. With the foregoing in mind, I shall now describe the aims and organization of the War Preparedness Committee.

Our objectives may be listed under five heads:

1. *Research.* The solution of mathematical problems essential for military or naval science, or rearmament.

2. *Preparation for Research.* The preparation of professional mathematicians for such research.

3. *Education for Service.* The strengthening of mathematical education in our schools and colleges to the point where it affords adequate preparation in mathematics for military and naval service or rearmament.

4. *Military and Naval texts.* The study by a large group of mathematicians of the current routine military texts and sources wherein mathematics is involved, —to obtain *certain* knowledge of what should be taught in the schools and colleges, and in order that mathematicians may be able to aid in the revision of these texts if and when their aid is needed.

5. *Roster of Personnel.* The collection of specialized information concerning mathematicians, similar to that in the national roster but more detailed as to mathematical training; and the making of this information available to all scientific or military committees or organizations aiding in the defense.

To carry out these objectives, three sub-committees were appointed with the following titles:

1. Research.
2. Preparation for Research.
3. Education for Service.

It is the last committee, on Education for Service, in which you are naturally most interested, but I shall first tell you about the other two committees.

3. Committee on research. This committee is headed by Professor Dunham Jackson of the University of Minnesota. It is ready to receive mathematical problems important for the national defense, and will seek to solve these problems. To aid this committee, we have appointed consultants in each of six fields. These fields are as follows:

Aeronautics.

Ballistics.

Computation (numerical, mechanical, electrical).
Cryptanalysis.
Industry.
Probability and Statistics.

The chief consultant in *aeronautics* is Professor Harry Bateman of the California Institute of Technology. This is perhaps the most difficult of all the fields, and one of the most important. Thousands of hours of mathematical labor go into the design of each new type of aeroplane. There is the problem of the flow of air by moving objects, and the problem of the determination of surfaces of least resistance and greatest lifting power. The problem of flutter is a very troublesome one, but nevertheless admits a mathematical approach. An essential tool here is the theory of conformal mapping. Those who wish further details may refer to a paper entitled *The engineer grapples with non-linear problems*, by Theodore von Kármán, in the Bulletin of the American Mathematical Society of 1940.

The chief consultant in *ballistics* is Professor John von Neumann of the Institute for Advanced Study at Princeton, N. J. The Government maintains its proving grounds at Aberdeen, Maryland, and Dahlgren, Virginia, and has several able mathematicians at work in this field. These men are charged with the proper design of guns and projectiles, with their testing, and the making of tables. The problem of bombsights is also referred to them. An interesting discovery of the last few years is the close connection between the theory of projectiles and that of high speed aeroplanes. High speed projectiles move at a velocity somewhat greater than that of sound, while the maximum speed of aeroplanes is now nearly two-thirds that of sound. It is therefore natural that ballistics and aerodynamics should be intimately related. The speed of sound is critical for bodies moving in the air. The tremendous resistance met at this speed seems to indicate that the maximum velocity at which aeroplanes can fly is fast being approached.

Professor Norbert Wiener of the Massachusetts Institute of Technology is the chief consultant in *computation*. A great deal of the computational work at the Aberdeen Proving Ground is now done by mechanical means by the so-called Bush Analyser. This is an intricate and expensive machine occupying a large room and capable of giving the numerical solutions of an important class of differential equations. Since the original machine was set up at Massachusetts Institute of Technology some ten years ago, several larger and better ones have been built. In the whole world at the present time there are not more than ten such machines. Professor Wiener is working on the problem of using this machine or similar machines to solve partial differential equations. If accomplished, this would be an important aid for applied mathematics. In spite of the existence of these machines, much computation still has to be done in the old-fashioned way. Fortunately for this country, we have a number of experts on numerical computation.

Cryptanalysis is the science of the making and solving of codes and cyphers. There is ample literature on the subject, and by virtue of its intriguing nature it might appeal to students of high school age. The chief consultant is Professor H. T. Engstrom of Yale. Professor Engstrom is an officer in the Naval Reserve, and with his aid a number of able young mathematicians are making an intensive study of cryptanalysis. It is possible to use the latest and most powerful algebras to make codes that are unbreakable. The catch is that complex codes are difficult to transmit without mutilation. Ordinary code theory involves a use of frequency tables and much ingenuity. It was only during the last war that the Germans discovered that it was better to employ statisticians than philologists in this branch of the military service. Here is a field in which mathematicians are very useful.

The chief consultant in *industry* is Dr. Thornton C. Fry, mathematical research director of the Bell Telephone Laboratories. Dr. Fry states that there are now more than fifty corporations employing more than 100 mathematicians. He finds that integral equations are used in prospecting for oil, matrix algebra in studying the vibration of aircraft wings and in electric circuit theory, the calculus of variations in improving the efficiency of relays, the theory of numbers in the design of reduction gears and in splicing telephone cables, and topology in the classification of electric networks. He points out that there is no place in this country where a mathematical consultant for industry can be trained as such. Such a man studies as an engineer, or a physicist, or a mathematician, and must be partially self-trained to serve as a mathematician in industry. Dr. Fry's plea for better training in the field confirms the emphasis of our committee on training in applied mathematics. Moreover, in this field the demand for men exceeds the supply.

Professor S. S. Wilks of Princeton University is the chief consultant in *probability and statistics*. I shall quote Professor Wilks as follows:

"In a war emergency the greatest service which can be rendered by probability and statistics is of the nature of routine and practical applications. Because of the extreme importance of mass production techniques in modern warfare, the feeling is very general that statistical methods of quality control such as those used by Shewhart in the Bell Telephone Laboratories would be valuable. Another main technique is that of sampling surveys and their application to the problem of stores and supplies, personnel selection, transportation, communication, *etc*. There is also the problem of statistical analysis of data obtained in bombing practice and in range firing."

4. Preparation for research. The second main sub-committee, on Preparation for Research, is headed by Professor Marshall H. Stone of Harvard University. It is concerned with the professional education of mathematicians to the end that they may be available for research on mathematical problems of the defense. Up-to-date expositions of ballistics, aerodynamics, and hydrodynamics are not available. This committee is concerned with this lack. It seeks to en-

courage the giving of special courses on applied mathematics in the various graduate schools, and a number of these courses are now being given. Bibliographies need to be published, and special seminars on the mathematics of defense need to be arranged at various scientific gatherings. This is a work of great importance, but one that will take time. It is an essential part of the proposed development of applied mathematics.

5. Education for service. The third sub-committee, and the one in which you are undoubtedly most interested, is on Education for Service. Its chairman is Professor William L. Hart of the University of Minnesota. At my suggestion, his committee embarked on a vigorous campaign of investigation of mathematical education in the secondary schools and of undergraduate mathematical education in the colleges, in relation to the national defense. The objectives as formulated by his committee are as follows:

1. To investigate what mathematics is of prime utility in industry and in the Army and Navy in the national defense.
2. In accordance with the results of this investigation, to make useful recommendations in regard to mathematical curricula at both the secondary and college levels.
3. To determine in what ways mathematicians may aid in the preparation of text-book material and in the teaching of those who will have mathematical duties in industry or as enlisted men or officers.

In order to orient the work of his sub-committee, Professor Hart conferred with the officers in charge of the Reserve Officers Training Corps of the Army and of the Navy at the University of Minnesota, with teachers of aeronautical engineering, and with teachers in ground school courses in the Civil Aeronautics Program. Also, he has drawn on his experience as a major of heavy artillery during 1917–19 and on his recollections of fairly recent visits to several warships and some major coast defenses of the Army. To obtain fresh acquaintance with the industrial side of the applications of mathematics of an intermediate stage, he has examined the workings of a major aircraft plant and has made a thorough inspection of the mathematical parts of the curricula offered by a prominent vocational school, which operates largely above the high school level. At my recommendation, his committee obtained for its consideration representative text-books of a mathematical nature employed in the R.O.T.C., in ground school courses for pilots, and in various service schools maintained by the Army. I shall continue with a reading of parts of a recent address* by Professor Hart which summarizes some of the information collected by his committee and which presents certain conclusions at which he has arrived, with particular emphasis on those related to the field of secondary mathematics:

"The sub-committee on Education for Service is giving no consideration at present to the mathematical training at West Point and Annapolis because the

* Presented at Baton Rouge, as cited previously.

officers from these schools are exceptionally well prepared for their duties. The committee is primarily interested in the mathematical aspects of the training of all others, officers or enlisted men, who eventually will enter the military service of the nation, and of all the men and women who will enter industry.

"In an approach to the military side of its activities, the committee and several coöperating mathematicians reviewed a representative sample of the text-books of a mathematical nature which are used by the Army and the Navy in its training activities. One object of these reviews was to learn at first hand what mathematics is a minimum essential for the study of the texts and for the performance of field duties by the officers and enlisted men in the various branches of the military services. As a second object, we wished to observe the nature of the exposition of mathematical material in the texts, with the possibility in mind that mathematicians might aid in the construction of any future editions of the books. In formulating opinions concerning the industrial side of the situation, not only at the level of collegiate mathematics but also at more elementary levels, our main problem, I believe, is merely to interpret and to emphasize long established conclusions in the light of the present national emergency. In this brief sketch of the basis of my future remarks, I take pleasure in acknowledging assistance received by me from Miss Mary Potter, president of the National Council of Teachers of Mathematics, in connection with viewpoints for the secondary field and mathematics appropriate for skilled industrial workers.

"I shall now summarize some of the evidence at my disposal and then, later, I shall draw certain conclusions, with particular attention paid to the field of secondary mathematics. I trust that you will keep it in mind that, in the remainder of my remarks, I am speaking from a subjective view-point colored by my own experiences and opinions, and that, as yet, my sub-committee has not had an opportunity for group action in the directions which concern me in this address.

"Permit me to be very brief on the non-military side, in spite of its importance not only during the present emergency but, also, from a long range view-point. It appears to me that the aircraft and munitions industries, with their demands for skilled workers and draftsmen, the drain on the national supply of skilled workers due to Army and Navy calls for enlisted specialists, and the statistical work associated with the activities of government agencies and industry, will operate to require largely increased numbers of men and women who have appropriate training in mathematics. It would be desirable if skilled industrial workers had substantial secondary mathematics, through the stage of computational trigonometry, with at least an intuitional knowledge of solid geometry, and with emphasis on applications at all possible stages. For these non-military activities, as many women as possible should be trained at least through substantial secondary mathematics; a more select group should be trained through the stage of elementary college mathematical statistics to create a reservoir of computers for government and industry.

"I evaluate the pure mathematical needs of the various Army and Navy branches as follows, if we eliminate the requirements for those exceptional officers whose work can be designated as military research:

"*First*, the Infantry, motorized or not. Even this supposedly non-technical branch places demands on mathematics. All enlisted men in the Infantry find use for arithmetic and intuitional geometry. The officers, non-commissioned officers, and privates first-class should have familiarity with elementary geometry to permit map reading, map construction, appreciation of contour designations on maps, and the use of coördinate systems. These men also should be able to understand the complicated mechanical drawings and the internal workings of the rifles, light anti-aircraft guns, and other materiel assigned to them. In brief, for these men I would specify elementary algebra and geometry as taught in modern courses for grades nine and ten. In addition, the officers should have some acquaintance with the notions of probability and probable error as met in elementary statistics.

"*Second*, the Coast Artillery Corps. This exceedingly mathematical branch of the Army includes all artillery for seacoast defense, all high altitude anti-aircraft artillery, and all mobile artillery of heavy caliber. The officers of this corps have to perform the duties of surveyors on some occasions, and they deal with very complex optical instruments, motorized machinery, and complicated guns. These men should have very strong training in mathematics—in fact, they should be engineering graduates as the most desirable specification. But, as a minimum, they must know mathematics through computational plane trigonometry, and elementary spherical trigonometry, with some background in solid geometry. They should also have an acquaintance with the notions of probability and probable error as met in elementary statistics in order to appreciate the theory of gunfire. All enlisted men should have a background of geometric and algebraic knowledge equivalent to that which is suitable for skilled industrial workers. In addition, about 25% of the enlisted men should be as well qualified as the officers.

"*Third*, the Field Artillery. This branch of the Army has charge of all light mobile artillery. We can make the same minimum stipulations for mathematical training as in the Coast Artillery Corps, with omission of mention of spherical trigonometry for the officers and with less insistence on the desirability of mathematical training for the enlisted men.

"*Fourth*, the Signal Corps of the Army. The officers should be electrical engineers and the enlisted men should have the mathematical training suitable for skilled industrial workers.

"*Fifth*, the Ordnance Department. It needs various specialists, both officers and enlisted men, with highly mathematical backgrounds such as possessed by engineers or college majors in mathematics.

"*Sixth*, flying officers in the Air Corps of the Army or the Navy, and all other officers of the Navy. They require at least the minimum training specified previously for officers in the Coast Artillery Corps, because of the necessity for

studying navigation in all present cases, aerodynamics and meteorology for officers in the Air Corps, and numerous other technical subjects. In fact, it bewilders a civilian, who has inspected a warship, to conceive of any officer in the Navy who is not a trained engineer. These officers of the Air Corps and the Navy should have substantial knowledge of solid geometry and spherical trigonometry, far beyond what is satisfactory in the heavy artillery service.

"*Seventh*, the ground service of the Air Corps. It requires many graduate engineers, men with college mathematics and physics especially for the meteorological section, and a large number of enlisted men with mathematical backgrounds suitable for skilled industry.

"*Eighth*, enlisted men in the Navy. All of them should have the mathematics suitable for skilled industrial workers. A substantial number of them should be as well qualified as stipulated in the description of minimum mathematics for the officers.

"In summary, I believe that the preceding specifications of mathematical training for officers give minimum levels if our Army and Navy are to be well led. The training which I specified for various types of enlisted men may exceed the *true minimum* but probably is the *desirable* level if it can be attained. I hazard the guess that, without special effort on the part of the high schools, colleges, and centers for adult training, the nation will *not* have a proper reservoir of men with the mathematical knowledge necessary for the needs of industry and the military services.

"Now let me present recommendations for view-points and actions in the secondary field as a consequence of the nature of the probable mathematical needs which I have just enumerated.

"*Item* 1. In the secondary field, it would be very undiplomatic and harmful if the national emergency were taken as a crude excuse for a violent attack on certain curricular trends, even though it is possible that weaknesses of some features of these trends may become apparent when analyzed under the searchlight of our present national requirements. I recommend that, initially, we should make our proposals and state the mathematical objectives in the preparedness program *without* any stipulation as to the pedagogical details involved in attaining the objectives.

"*Item* 2. The National Council of Teachers of Mathematics and organized groups of mathematics teachers at all levels should advertise the utility of mathematics in industry and military service. In high schools it should be advertised that Army and Navy R.O.T.C. units in colleges *require* trigonometry and *should require* solid geometry and spherical trigonometry.

"*Item* 3. Every club of secondary teachers of mathematics should promptly hold a special meeting devoted to a discussion of the rôle of mathematics in the present national emergency and to a discussion of possible local actions in the high schools.

"*Item* 4. I recommend that, in the junior and senior high schools, every boy and girl of sufficient mathematical aptitude should be *urged* by the high school

advisors to take as much mathematics as possible, through the stage of trigonometry and some solid geometry, as a national service.

"*Item* 5. I suggest that a *new definition of socialized mathematics* be adopted in the curricula for students of *all* ability levels, where we would recognize that, for boys, *mathematical content with military and industrial uses* is the most socialized variety of mathematics to which they can be exposed at present.

"*Item* 6. The military and industrial utility of spherical trigonometry, space intuitions, and three-dimensional diagrams, leads me to recommend that the high school course in solid geometry be given much more emphasis than in recent years. I suggest that it be modified by replacing some of the classical content with a treatment of the elements of spherical trigonometry, thus giving a combined course in solid geometry and spherical trigonometry. In fact, this combination appeals to me on purely mathematical and pedagogical grounds apart from the requirements of the preparedness program.

"*Item* 7. I strongly urge that a single set of courses be used for secondary students of *ability* in attaining desired ends, rather than separate curricula, some designed to fit men for industry and some planned for boys and girls who will proceed later into college mathematics, because I believe that the same content can be justified for both categories of students.

"*Item* 8. As a temporary measure, I suggest that boys of good intelligence now in grades 11 and 12, who previously have omitted substantial mathematics, should be offered an abbreviated treatment of logarithms, plane trigonometry, intitutional solid geometry, and a little spherical trigonometry, to permit these students to train themselves rapidly for their entrance into skilled industry, the Army, or the Navy.

"*Item* 9. I advance the opinion that a severe shortage of men with engineering training is at hand. This should be brought to the attention of interested boys of mathematical ability in the high schools.

"*Item* 10. As a final recommendation for the secondary field, I urge the National Council to appoint a special committee on 'Mathematics for National Service' to coördinate and direct appropriate activities in the secondary field."

The applications of mathematics in the national defense will be made by men in all branches of the national service and in the various scientific professions. Some of the men contributing in this way will be mathematicians. The one thing for which mathematicians are mainly responsible and in which they have the greatest influence, is the education in mathematics for this service. I know that we can count on the teachers of mathematics for the fullest aid.

ON THE INFLEXIONAL ELASTICA

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1. Introduction. The "elastica," in its simplest form, is the curve assumed by a perfectly flexible rod subjected to longitudinal thrusts, the ends of the rod being hinged to its supports. Under these conditions, the curve may have points of inflexion not only at its extremities, but at certain intermediate points as well. For this reason, the curve is sometimes referred to as the "inflexional elastica."*

As early as 1694, James Bernoulli gave a discussion of the flexure of an elastic lamina and was led to consider the differential equation of the elastic curve. Some fifty years later, Daniel Bernoulli obtained an expression for the potential energy stored, so to speak, in a bent rod or lamina, and suggested that the rod would assume such a shape as to render this energy a minimum. Euler, to whom this suggestion had been made, took up the problem in this form and solved it as an application of his then recently invented method for dealing with isoperimetric problems in the calculus of variations. The precise problem which Euler solved is stated by him in the following form: "That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of $\int_A^B ds/R^2$ be a minimum."† Here ds is the element of arc along the curve and R is its radius of curvature.

Euler commences the discussion of this problem "with a statement which is of extreme interest as showing the theologico-metaphysical tendency which is so characteristic of mathematical investigations in the 17th and 18th centuries. It was assumed that the universe was the most perfect conceivable, and hence arose the conception that its processes involve no waste, its 'action' was always the least required to effect a given purpose."‡ Euler's remarks prefacing his mathematical discussion of the problem may, therefore, be regarded as having been made in support of the ideas then being introduced by Maupertuis.§ Today, physicists do not attempt to justify their use of variational principles by referring them to a pietistic metaphysics; these principles are now regarded as being primarily useful and compact algorithms, whose final justification, as scientific laws, is to be found in the agreement between results obtained by their use on the one hand, and experimental evidence on the other.

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* The so-called "nodal elastica" is obtained if certain couples are applied at the ends of the rod.

† Euler's results in connection with this problem were published in 1744 in an extensive appendix to his treatise on the calculus of variations. This appendix has been translated into English from the Latin by W. A. Oldfather, C. A. Ellis, and D. M. Brown; it was published in Isis, vol. 20, 1933. The above quotation is taken from p. 79 of this translation. See also, Bolza, Variationsrechnung, p. 536.

‡ Todhunter and Pearson, History of the Theory of Elasticity, vol. 1, p. 34.

§ Maupertuis, Mem. de l'Acad. de Paris, 1740; Mem. de l'Acad., Berlin, 1745. For critical comments, see Lindsay-Margenau, Foundations of Physics, 1936, p. 133. See also, E. T. Bell, The Development of Mathematics, 1940, chap. 17.

A consequence of Euler's work on the elastica, which is of some practical importance, was his discovery of the formula for "critical" loads, which forms the basis of some discussions on column theory. If this "critical" load is exceeded, the rod or column under compression will begin to fail through lateral buckling. This result of Euler's is probably the first one to be met in connection with problems involving elastic stability, and is of a sufficiently practical nature so that it may be found in any of our elementary texts on strength of materials. It is true, of course, that this result was, in Euler's day, of no particular practical use. "The principal structural materials then in use were wood and stone and their relatively low strength necessitated stout structural members for which the question of elastic stability is not of primary importance. It is only when steel began to be used that the question of buckling became of practical importance."* Thus, we have here another example of theory anticipating the practical needs of a later time.

The preceding remarks have been intended to bring out the fact that the problem of the elastica is of some historical importance. The problem is one of the earlier landmarks in

(a) the application of a variational principle to a problem in mechanics;

(b) the solution of the class of isoperimetric problems in the calculus of variations;

(c) the solution of problems involving elastic stability;

(d) the solution of problems involving the notion of characteristic numbers and functions.†

It may also be noted that the differential equation of the elastica occurs in other fields of applied mathematics, for example, in the theory of capillarity, in fluid dynamics, and in the theory of motion of a rigid body with a fixed point.‡

2. The differential equation of the elastica. The fundamental equation underlying the theory of the deflection of beams and columns (rods) is the well known relation,

$$M = EI/R, \tag{1}$$

where M is the bending moment, E is the modulus of elasticity, I is the moment of inertia of the cross-section about the neutral axis, and R is the radius of curvature of this axis.

Relative to the problem of the elastica, this equation takes the form:

$$\frac{d^2y}{dx^2} = -\lambda^2 y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}, \qquad \lambda^2 = \frac{P}{EI}, \tag{2}$$

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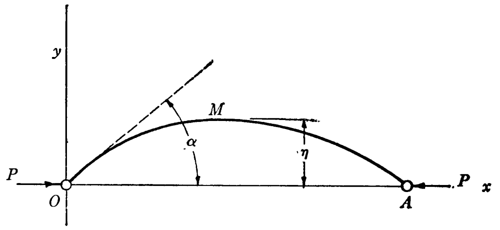
* Timoshenko, Elastic Stability, preface.

† This aspect of the problem seems to have escaped Euler. It appears to have been first noticed by Lagrange in 1770. See his Sur la figure des colonnes, Oeuvres, vol. 2.

‡ See Kirchhoff, J. für Math., vol. 56, 1859, for the kinetic analog of the general elastica. Also, W. Hess, Math. Annalen, vol. 23, 1885.

where P is the axial load. This is one of the simpler types of the non-linear equations of mechanics.*

In the elementary treatment of this problem, the simplifying assumption is made that both y and dy/dx are so small that the term $(dy/dx)^2$ may be neglected, thereby making the problem linear and soluble in terms of the elementary functions. With this assumption, it is still possible to determine the magnitude



of the critical load, but the magnitude of the deflection remains undetermined.† Hence, if we are interested in the force-deflection relation beyond the buckling load, it is necessary to integrate equation (2), taking into account the exact value of the radius of curvature. This leads to a solution involving elliptic integrals and functions. From the point of view of numerical calculation it is preferable, as is usually the case, to express the solution in terms of the Legendre functions $F(\phi, k)$ and $E(\phi, k)$.‡ However, the solution and discussion of the problem in terms of the Weierstrass functions $\wp(z)$, $\zeta(z)$ is not without interest.§ It can be handled in a fairly simple and direct manner and furnishes an excellent exercise in the use of these functions. In what follows we proceed to discuss this aspect of the problem.||

3. The parametric equations of the elastica. In obtaining a first integral of the differential equation (2), use may be made of either of the following initial conditions:

(a) $\qquad dy/dx = 0 \quad \text{for} \quad y = \eta = \text{maximum deflection ordinate, or}$

(b) $\qquad dy/dx = \tan\alpha \quad \text{for} \quad y = 0.$

We obtain,

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* Th. von Kármán, Bull. Am. Math. Soc., vol. 46, 1940, pp. 627–629.

† See any elementary text on Strength of Materials.

‡ For this form of the solution we refer to (*i*) F. Tricomi, Funzioni Ellittiche, 1937, pp. 226–235, and (*ii*) Timoshenko, Elastic Stability, 1936, pp. 69–75. Incidentally, this form of the solution suggests a variety of exercises which will give the student practice in the full use of such tables as those of Jahnke and Emde. Legendre's own tables are available in two recent editions, one edited by Wittwer and Emde, Stuttgart, 1931, and the other edited by Karl Pearson, Cambridge, 1934.

§ The German script p is generally used instead of $\wp$.

|| For an alternative but somewhat more complicated discussion, see Appell-Lacour, Fonctions Elliptiques, pp. 306–317.

$$\frac{dy}{dx} = \frac{\sqrt{1 - \left[1 - \frac{\lambda^2}{2}(\eta^2 - y^2)\right]^2}}{1 - \frac{\lambda^2}{2}(\eta^2 - y^2)} = \frac{\sqrt{1 - \left[\cos\alpha + \frac{\lambda^2}{2} y^2\right]^2}}{\cos\alpha + \frac{\lambda^2}{2} y^2}, \tag{3}$$

from which it may be inferred that the relation between α and η is

$$\eta = \frac{2}{\lambda}\sin\frac{\alpha}{2} \tag{4}$$

It is found that the transformation

$$u = \frac{2}{3} - \frac{\lambda^2}{6}(2\eta^2 - 3y^2) = \frac{2}{3}\cos\alpha + \frac{\lambda^2}{2} y^2, \tag{5}$$

will reduce (3) to the form,

$$\frac{dx}{du} = -\frac{i\lambda}{\sqrt{2}} \cdot \frac{u + \frac{1}{3}\left(1 - \frac{\lambda^2}{2}\eta^2\right)}{\sqrt{R(u)}} = -\frac{i\lambda}{\sqrt{2}} \cdot \frac{u + \frac{1}{3}\cos\alpha}{\sqrt{R(u)}}, \tag{6}$$

in which,

$$\begin{aligned} R(u) &= (u - e_1)(u - e_2)(u - e_3), \\ e_1 &= \frac{1}{3}\left(2 + \frac{\lambda^2}{2}\eta^2\right) = 1 - \frac{1}{3}\cos\alpha, \\ e_2 &= \frac{2}{3}\left(1 - \frac{\lambda^2}{2}\eta^2\right) = \frac{2}{3}\cos\alpha, \qquad (e_1 > e_2 > e_3), \\ e_3 &= -\frac{1}{3}\left(4 - \frac{\lambda^2}{2}\eta^2\right) = -1 - \frac{1}{3}\cos\alpha. \end{aligned} \tag{7}$$

As is well known, the constants e_j determine the periods $2\omega_1$ and $2\omega_2$ with which to construct the Weierstrass function $\wp(v)$ which inverts the integral

$$v = \int \frac{du}{\sqrt{R(u)}} \cdot \tag{8}$$

Since the e_j are real, it is possible to choose the periods $2\omega_1$, $2\omega_2$ so that the first of these is real and the other a pure imaginary (see equation (17) below), and such that $0 < \arg \omega_2/\omega_1 < \pi$.

If now we let $u = \wp(v)$, equation (6), after some reductions, takes the form

$$\frac{dx}{dv} = \frac{i\sqrt{2}}{\lambda}\left[\wp(v) + \frac{e_2}{2}\right] = \frac{i\sqrt{2}}{\lambda}\left[-\zeta'(v) + \frac{e_2}{2}\right] \tag{9}$$

It follows from (5) that

$$\frac{\lambda^2}{2} y^2 = \wp(v) - e_2. \tag{10}$$

Now, we have $y=0$ when $x=0$; but for $y=0$, the parameter v must satisfy the equation

$$\wp(v) = e_2,$$

and hence $v=\omega_1+\omega_2 \equiv \omega_3$ = sum of the half-periods. Therefore, from (9) we have

$$x = \frac{i\sqrt{2}}{\lambda}\int_{\omega_3}^{v}\left(-\zeta'(v) + \frac{e_2}{2}\right)dv,$$

so that

$$x = \frac{i\sqrt{2}}{\lambda}\left[\eta_3 + \frac{e_2}{2}(v - \omega_3) - \zeta(v)\right], \tag{11}$$

where, in the usual notation,

$$\eta_3 = \zeta(\omega_3) = \eta_1 + \eta_2, \qquad \zeta(\omega_1) = \eta_1, \qquad \zeta(\omega_2) = \eta_2.$$

It remains to investigate the range of values for the parameter v. It has been shown that, since at the initial point O, $y=0$, the corresponding value of v is $\omega_1+\omega_2$. To obtain the value of v corresponding to the point M of maximum deflection, set $y=\eta$ in (10). With the aid of (7) it is found that

$$\wp(v) = e_1, \tag{12}$$

so that the value of v is ω_1. It follows that the curve OM is traced by letting v take on values of the form $v=\omega_1+i\mu$, $0 \leqq \mu \leqq \omega_2/i$. It can readily be shown from the properties of the functions $\wp(v)$ and $\zeta(v)$, that for these values of v, the corresponding values of x and y are real, as of course they should be.

We have, therefore, shown that the parametric equations of the elastica are:

$$\begin{aligned} x &= \frac{i\sqrt{2}}{\lambda}\left\{\eta_3 + \frac{e_2}{2}(v - \omega_3) - \zeta(v)\right\}, \\ y &= \frac{\sqrt{2}}{\lambda}\sqrt{\wp(v) - e_2}, \\ v &= \omega_1 + i\mu, \qquad 0 \leqq \mu \leqq \omega_2/i. \end{aligned} \tag{13}$$

4. The Euler critical load. The preceding results may be used to derive the expression for the Euler critical or buckling load. Thus, it has been shown that the maximum deflection η corresponds to the value $v=\omega_1$; hence the abscissa of M is given by

$$x_M = \frac{\sqrt{2}}{\lambda}\left(i\eta_2 + \frac{e_2}{2}\,\frac{\omega_2}{i}\right). \tag{14}$$

It is clear that as the maximum deflection η tends to zero, the value of x_M tends to $\frac{1}{2}L$, where L is the length of the rod. We are, therefore, led to evaluate

$$\lim_{\eta\to 0} \frac{\sqrt{2}}{\lambda}\left(i\eta_2 + \frac{e_2}{2}\,\frac{\omega_2}{i}\right). \tag{15}$$

This evaluation is, apparently, most conveniently carried out if (15) is translated into the Legendre notation. To do this, construct the Legendre complete elliptic integrals corresponding to the modulus k^2, where,

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = 1 - \frac{\lambda^2\eta^2}{4} = \cos^2 \frac{\alpha}{2}\cdot$$

In particular, we have

$$E' = \int_0^{\pi/2} \sqrt{1 - k'^2 \sin^2 \phi}\, d\phi$$

$$K' = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}}, \qquad k'^2 = \frac{\lambda^2\eta^2}{4} = \sin^2 \frac{\alpha}{2}\cdot$$

It is known* that

$$E' = \frac{i}{\sqrt{e_1 - e_3}}(\eta_2 + e_3\omega_2),$$

$$iK' = \sqrt{e_1 - e_3}\,\omega_2.$$

Using equations (7), it follows that

$$i\eta_2 = \sqrt{2}\,E' - ie_3\omega_2,$$

$$\omega_2/i = K'/\sqrt{2}.$$

Hence,

$$\lim_{\eta\to 0} \frac{\sqrt{2}}{\lambda}\left(i\eta_2 + \frac{e_2}{2}\,\frac{\omega_2}{i}\right) = \lim_{k'\to 0} \frac{1}{\lambda}(2E' - K') = \frac{\pi}{2\lambda^*},$$

where λ^* is the value of λ corresponding to the critical value P^* of P. It is therefore found that P^* is given by the formula

$$P^* = \pi^2 EI/L^2, \tag{16}$$

which is Euler's formula.

5. The load-deflection curve. From the parametric equations of the curve, it readily follows that

$$\left(\frac{ds}{dv}\right)^2 = \left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2 = -\frac{2}{\lambda^2},$$

* Hancock, Elliptic Functions, pp. 461, 495.

so that, on integrating from $v=\omega_3$ to $v=\omega_1$, we obtain

$$\frac{L}{2} = -\frac{i\sqrt{2}}{\lambda}\omega_2.$$

Hence, it follows that

$$2\omega_2 = iL\sqrt{\frac{P}{EI}}, \tag{17}$$

so that the purely imaginary period of $\wp(v)$ is readily calculated.

From (17) and (16), we have

$$P = -\frac{8EI}{L^2}\omega_2^2 = -\frac{8\omega_2^2}{\pi^2}P^*, \tag{18}$$

whence by elimination of ω_2 it follows that

$$P = \frac{4}{\pi^2}K'^2P^* \tag{19}$$

Now let p^2 be defined by

$$p^2 = \sin^2\frac{\alpha}{2} = \frac{\lambda^2\eta^2}{4},$$

so that

$$\eta = pL/K'. \tag{20}$$

If, now, we denote by $K(p)$ the complete Legendre integral corresponding to the modulus p^2, and let the reduced load and deflection be represented by Π and ν respectively, we may write,

$$\begin{aligned} \Pi &= \frac{4}{\pi^2}K^2(p), \quad \Pi = P/P^*, \\ \nu &= \frac{p}{K(p)}, \qquad p = \sin\frac{\alpha}{2}, \qquad \nu = \eta/L. \end{aligned} \tag{21}$$

These may be regarded as the parametric equations of the load-deflection curve. For its graph, refer to the paper of von Kármán.* It may easily be constructed with the aid of tables for $K(p)$.

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* *Loc. cit.*, p. 628. See also, Southwell, Theory of Elasticity for Engineers and Physicists, Oxford, 1936, pp. 429–436.

REMARKS ON GROUPS OF HOMEOMORPHISMS*

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1. Introduction. Groups of homeomorphisms have been extensively studied from many points of view and with several degrees of generality. In the groups of ordinary analytic and projective geometry the transformations are given by linear functions of the coördinates. A more general situation in which differentiability conditions play a predominant part was studied by Lie and his followers. This paper is concerned with some of the things which can be said about transformations and groups when not even differentiability conditions are assumed. It is probably superfluous to say that there will only be time to summarize a few of the results in this general direction. The bibliography at the end is far from complete, but it contains everything referred to here.

2. Metric spaces. Since the concept of homeomorphism is inextricably bound up with that of space, it is desirable to begin with a few remarks about spaces. For the most part the spaces mentioned are the familiar ones of experience and *all* the ones mentioned are special instances of the metric spaces of Frechet. In order to provide a basis for one or two later uses, a definition of these spaces is included. This definition, as well as some of the ones further on, can be shortened a trifle, but the economy of statement might be at the expense of clarity.

A set of elements M is called a metric space if with each two elements x and y of M there is associated a non-negative real number $d(x, y)$ called the distance from x to y, this distance being required to satisfy the following conditions:

1. $d(x, y) = d(y, x)$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) \leqq d(x, z) + d(y, z)$, (triangle axiom).

Lines, planes, and three-dimensional spaces all with distance as customarily defined are examples of metric spaces. Another example is n-dimensional euclidean space where the distance between two points $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ is defined to be $[(x_1-y_1)^2 + \cdots + (x_n-y_n)^2]^{1/2}$. Any sub-set of a metric space is again a metric space and this establishes the fact that all the figures of elementary geometry, spheres, triangles, cubes, and so on are also metric spaces

3. Transformations and homeomorphisms. If two metric spaces M and N are given, a transformation of M into N is a relation which associates with each point of M a point of N. If x is a point of M let the point of N corresponding to it be $g(x)$; the symbol $g(x)$ is commonly used to denote the transformation. According to the definition, $g(x)$ is defined for each point of M and is a unique point of N. There may, however, be a point y of N for which no x in M satisfies the equality $y = g(x)$; that is, the transformation may take M into a part of N and not the whole of N. A trivial instance of this arises, provided N contains more than one point, if for each x, $g(x)$ is the same point of N.

* Presented at the Baton Rouge meeting of the Mathematical Association of America on January 2, 1941.

The transformation $g(x)$ is said to be a one-to-one transformation of M into N if for each point y in N there is one and only one point x in M such that $y=g(x)$. In this case the inverse transformation $x=g^{-1}(y)$ is defined and is a one-to-one transformation of N into M. Speaking roughly, the transformation $g(x)$ is said to be continuous at x_0 if for all points x near x_0 it is true that $g(x)$ is near $g(x_0)$. More precisely, $g(x)$ is said to be continuous at x_0 if for each positive number ϵ, there exists a positive number δ such that

$$\text{if} \quad d(x, x_0) < \delta, \quad \text{then} \quad d[g(x), g(x_0)] < \epsilon.$$

The transformation is called *continuous* if it is continuous at each point.

If $g(x)$ is a one-to-one transformation of M into N and if $g(x)$ and its inverse $g^{-1}(y)$ are both continuous, then g is called a homeomorphism of M into N. There is nothing in what has been said to prevent M and N from being the same space and indeed this will be the case of most interest in the succeeding paragraphs. If $g(x)$ and $h(x)$ are two homeomorphisms of a space into itself, then the combination $g[h(x)]$, frequently written $gh(x)$, may be formed. It may or may not be the same as $h[g(x)]$. The word transformation is frequently used in place of homeomorphism even though it is in reality a more general term. For example, the expression "group of transformations" is frequently used in place of the expression "group of homeomorphisms."

4. Equivalence of homeomorphisms. As before, let M and N be two metric spaces. Let g_1 be a homeomorphism of M into itself and let h be a homeomorphism of M into N. These two homeomorphisms may be used to define a homeomorphism of N into itself in the following way. The points $h(x)$ and $hg_1(x)$ are in N. Let g_2 be the homeomorphism of N into itself which takes $h(x)$ to $hg_1(x)$. From this definition of g_2 it follows that $g_2h(x)=hg_1(x)$. Applying h^{-1} to both sides of this equation yields the equation

$$g_1(x) = h^{-1}g_2h(x). \tag{1}$$

The two homeomorphisms g_1 and g_2 are called topologically equivalent, and we are thus led to the general definition of topological equivalence, or briefly equivalence, of homeomorphisms. A homeomorphism g_1 of M into itself and a homeomorphism g_2 of N into itself are called equivalent if there exists a homeomorphism h from M to N satisfying the equality (1) above. Here again there is no reason why M and N should not be the same space. In case they are, h will be a homeomorphism of the space into itself.

An illustration may help to bring out the meaning of this concept a little more clearly. The spaces M and N are both to be planes. Points in M are denoted by x and have coördinates (x_1, x_2); those in N are denoted by y and have coördinates (y_1, y_2). Each point of the plane M lies on a unique curve of the family

$$\frac{x_1^2}{4a^2} + \frac{x_2^2}{a^2} = 1. \tag{2}$$

Define $g_1(x)$ as follows. If x is the origin, $g_1(x)=x$. If x is not the origin, use will be made of the parametric representation

$$x_1 = 2a \cos \phi, \qquad x_2 = a \sin \phi \tag{3}$$

of the family (2). Choose an integer at random, say the integer 20, and assume a and ϕ fixed so that the equations (3) yield the point x. Then $g_1(x)$ is defined to be the point with coördinates

$$[2a \cos (\phi + 2\pi/20),\ a \sin (\phi + 2\pi/20)]. \tag{4}$$

Let $h(x)$ be the homeomorphism from M to N given below:

$$y_1 = x_1/2, \qquad y_2 = x_2.$$

By means of this homeomorphism, the ellipses of the family (2) are carried into the family of circles

$$y_1^2 + y_2^2 = a^2.$$

And under this transformation h, the points (3) and (4) go into points in N with coördinates

$$(a \cos \phi,\ a \sin \phi) \quad \text{and} \quad [a \cos (\phi + 2\pi/20),\ a \sin (\phi + 2\pi/20)].$$

The homeomorphism g_2 which takes the first of these points to the second is a rotation through an angle of $2\pi/20$ radians. Therefore the transformation g_1 which moves points by short jumps around the curves of a family of ellipses is equivalent to a rotation which, of course, moves points around circles.

5. Groups of homeomorphisms. A set G of elements of any kind is called a group if there is given a rule of combination of every two elements g_1 and g_2 in G which satisfies the following conditions:

1. g_1g_2 is in G.
2. $g_1(g_2g_3)=(g_1g_2)g_3$.
3. There is an element e, called the identity, which is such that for any g in G, $ge=eg=g$.
4. For each g there is an inverse element g^{-1} having the property that $gg^{-1}=g^{-1}g=e$.

If each g in G is a homeomorphism of M into itself and if the rule of combination and the rule for finding inverses are the usual ones for homeomorphisms mentioned above, then G is called a group of homeomorphisms or a transformation group of M. The element e must take every point into itself, that is, it must be the identity homeomorphism.

A homeomorphism of M into itself is called periodic if there is an integer p such that $g^p(x)=x$ for all x, where $g^p(x)=g \cdots g(x)$, the dots indicating that g is repeated p times. The smallest integer p satisfying this condition is called the period of g. If g has period p, the set of homeomorphisms

$$e(x),\ g(x),\ g^2(x),\ \cdots,\ g^{p-1}(x)$$

forms a group. Such groups are called cyclic.

If G_2 is a transformation group of a metric space N and h is a homeomorphism of M into N, then $h^{-1}G_2h(x)$ is the symbol used to denote all homeomorphisms of M into itself of the form $h^{-1}g_2h(x)$, where g_2 is in G_2. This set of elements is a transformation group G_1 of M into itself. Any transformation group G_1 which can be obtained in this way is called equivalent to G_2. It is a fundamental problem to determine what kinds of transformations and what kinds of transformation groups are equivalent. Some of the results stated in the remainder of this paper are concerned with problems of this kind.

6. Periodic transformations of the plane. Let M be the interior and circumference of a circle in the plane, and let g be a homeomorphism of M into itself of period p. Then (see references in bibliography to Brouwer, Kerékjártó, and Eilenberg) the structure of g has been completely determined. It is known that g is equivalent to a rotation through a certain angle or, as a second possibility, in case $p=2$, to a reflection about a diameter. Analogous results hold in case M is a plane or the surface of a sphere. The example in section 4 is an illustration of these facts. It is also known (Kerékjártó) that periodic transformations and even more general transformations of the torus and other 2-dimensional manifolds are equivalent to simple types of transformations.

7. Periodic transformations in higher dimensions. Many people have pointed out the desirability of finding out whether or not any conclusions of the type mentioned in the last section can be found for spaces of higher dimensions. For example, if g is a homeomorphism of period $p>2$ of 3-dimensional space into itself, must it be equivalent to rotation around an axis? The answer is known neither for this specific question nor for any similar question, and apparently these are questions of considerable difficulty. Nevertheless, a great deal is known about periodic transformations in spaces of dimension 3 and higher (see the papers by Eilenberg, Newman, Richardson, and Smith).

Throughout this section, M will be assumed to be an n-dimensional connected metric space which is locally euclidean. A space is called locally euclidean if around each point of the space there is a set homeomorphic to the interior of a sphere in n-dimensions,* and, to speak intuitively, a space is called connected if it is all in one piece. Occasionally further restrictions will be imposed on M, but if the reader is not concerned with detailed conditions he will do well to assume that M is n-dimensional euclidean space or that it is 3-dimensional euclidean space. All the results stated are valid in particular for these cases, and furthermore, most of the theorems lose little of their force and are of interest and importance even in these restricted cases.

Let L denote the set of fixed points of M under a homeomorphism g of period p. A point is called interior to L if it cannot be approached by means of a sequence of points no one of which is in the set L. *A priori* there is no reason why L might or might not contain inner points. But Newman has shown that if

* A sphere in n-dimensions, called the $(n-1)$-sphere, is the set of points whose coördinates satisfy the equation $x_1^2+x_2^2+\cdots+x_n^2=1$

L does contain an inner point, then L contains every point of M. In other words, if L has an inner point, then the periodic homeomorphism g leaves every point of M fixed and is the identity homeomorphism. This theorem is a very useful one in many circumstances. It has been proved by Smith under conditions relating purely to homology rather than to the locally euclidean character of the space. The suspicion that many periodic homeomorphisms may be equivalent to simple geometric transformations tends to be confirmed rather than contradicted by this theorem, and the same might be said of many of the theorems to follow.

This result of Newman's can be used, as the writer once showed, to conclude that certain transformations are really periodic although at first glance they might appear not to be. Let g be a homeomorphism of M into itself such that for each fixed x there is some integer n such that $g^n(x)=x$, but where it is not supposed that there is any n satisfying this equality for all x. Then it can be shown that there must be an n satisfying the above equality for all x. Briefly, then, it may be said that if g is periodic on each point of the space, then it must be periodic on the space as a whole. As Smith once remarked in conversation, it is true, in view of Smith's extension of Newman's theorem, that this result may also be made to depend on certain homology conditions.

Newman proved a second very interesting theorem about periodic transformations which may be stated as follows: For any given space M and any given integer $p>0$ there is a number $r>0$ such that any homeomorphism of M into itself of period p must move some point a distance greater than r. Smith has extended this theorem in two directions. In the first place he has proved it, as he did the one before, on the basis of pure homology assumptions. In the second place he has shown, by considering orbits, how it is possible to omit the period in the statement of the theorem. The orbit of a point x under a group is the set of all points $g(x)$ for g in G. In the case of a periodic transformation the orbit of a point x is the set of points

$$x,\ g(x),\ g^2(x),\ \cdots,\ g^{p-1}(x).$$

A corollary of Smith's extended theorem could be stated as follows, where as always in this section M is connected and locally euclidean: For any space M there exists an $r>0$ such that under any periodic transformation (not the identity) the orbit of some point must have a diameter greater than r. The diameter of a set is the least upper bound of distances between points in the set.

As Smith points out, it would be very valuable to know whether or not the orbits of more general groups must satisfy a similar theorem. It can be concluded from the above that any finite group or in fact that any compact Lie group (see the next section) does satisfy a similar theorem. This is because such groups must contain periodic elements.

If it is assumed that M is an open connected sub-set of n-space and if the images in M of each sphere of certain dimensions can be shrunk to a point, then every periodic homeomorphism of prime period has a fixed point. This theorem,

first proved by Smith, has been extended and given an interesting new proof by Eilenberg. His conditions are in terms of homology rather than in terms of the possibility of shrinking certain spheres. Smith himself has also recently extended the theorem and simplified the proof.

8. Periodic homeomorphisms of the three-sphere. There are a great many further results on periodic homeomorphisms. We shall have to be content here to cite just one of these which was proved by Smith and which completely specifies the topological structure of the set L of fixed points when M is a 3-sphere.

THEOREM. *If M is a 3-sphere and g is a periodic homeomorphism of M into itself, then the set L is null, or consists of two points, or is a simple closed curve, or is homeomorphic to a 2-sphere.*

As a consequence of this theorem it can be said that if M is 3-space, then L is a point, or homeomorphic to a line, or homeomorphic to a plane.

9. Topological groups. A set G of elements is called a topological group if the following conditions are satisfied:

1. The elements form a group.
2. The elements form a space, for present purposes a metric space.
3. The combination of two elements is a continuous function on the space, and the inverse of an element is a continuous function on the space.

Real numbers under addition and complex numbers of absolute value 1 under multiplication are examples of topological groups. The latter group is the circle group and it is essentially the same group as the group of all rotations of a plane about a point.

There are a number of ways in which a topological transformation group may be defined, but the following one has often been used. A topological group G is called a topological transformation group of a space M, or briefly a transformation group, if each element g of G is a homeomorphism of M into itself and if the following conditions are satisfied:

1. $g(x)$ is simultaneously continuous in g and x.
2. $g_1[g_2(x)]=(g_1g_2)(x)$.
3. The identity of the group is the identity homeomorphism.

This definition does not assume that all homeomorphisms of G are distinct; this freedom is occasionally of advantage. If every two homeomorphisms are distinct, the group is called effective. An effective transformation group is an ordinary transformation group with the additional requirement that $g(x)$ be simultaneously continuous in g and x.

If G_1 is an effective transformation group of a metric space M_1 and G_2 is an effective transformation group of a metric space M_2, then G_1 and G_2 are said to be equivalent if there is a homeomorphism h of M_1 into M_2 such that

$$G_1 = h^{-1}G_2h.$$

The topological groups considered here will be compact, where a compact space

is one in which every infinite set of points has a limit point. The circle group is compact, but the group of real numbers is not. Topological groups have been studied in considerable detail in recent years. Much more is known about the compact ones than the others, one important result (von Neumann) being that if a compact group is locally euclidean, then it is a Lie group, which means that coördinates may be so introduced that the group operations become differentiable.

There are at least three levels of generality on which transformation groups of locally euclidean spaces might be studied. The first, considered by Lie as mentioned in the introduction, is where G is a Lie group and its associated transformations satisfy certain differentiability conditions. The second is where G is a Lie group but where for the associated transformations no differentiability is assumed. The third is where G is a general topological group and the associated transformations are subject to no conditions.

In the second case the theorems on periodic homeomorphisms yield some useful information since a compact Lie group must contain many periodic elements. Thus it can be concluded from the first-mentioned result of Newman that if a compact Lie group acts on a connected locally euclidean metric space M and if the set of fixed points contains an inner point, then every point of M is fixed.

On the third level of generality a very important problem to which reference has already been made is the question of whether or not general compact groups can have orbits in locally euclidean spaces which are arbitrarily small in diameter.

10. Transformation groups in three-space. Let M be euclidean 3-space. Let G_1 be the group of all rotations about an axis in M and let G_2 be the group of all rotations about a point in M. Then in connection with the third level of generality, mention may be made of the following theorem proved by Leo Zippin and the author:

THEOREM. *Every compact connected transformation group of 3-space is equivalent to either G_1 or G_2.*

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MATHEMATICAL EDUCATION

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This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

WHAT ARE THE ADMINISTRATIVE AND GUIDANCE USES OF MATHEMATICS EXAMINATIONS?*

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A consideration of the administrative and guidance uses of mathematics examinations roughly is equivalent to posing the proposition: To apply the mathematical processes of mensuration, logical reasoning, synthesis, and analysis to the measurement of student acquisition of knowledge, student attainment of proficiencies and skills, student retention of mathematical techniques and concepts.

It would appear that examinations given by the early American colleges were regarded solely as administrative devices—for entrance to college, for advance from class to class, and for graduation. Examinations were oral, then oral and written, and finally almost entirely written. The changes through which examinations passed indicate a growing yet unconscious appreciation of the value of subjects to the student.

The development of scientific measurement in education has come about within the memory of men still living. Beginning largely in a quest for the components of intelligence, applications of the new science have been extended rapidly to include achievement, special skills, and aptitudes, attitudes, interests

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* A résumé of the paper presented at the joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics at Baton Rouge on January 1, 1941.

and relatively unique differential traits of personality. Yet, the measurement program well-nigh attained to technical maturity before serious investigation was given to the problem of interpreting or grading the results. With but few exceptions, the first technically good examinations were constructed independently of one another, by different persons, at different times, and in different places, with no view to their possible uses in combination in an integrated program. The norms for the examinations were similarly independently established, each for a different group of schools, at a different time, and under different conditions; often they differed in kind. The situation is not yet radically changed in terms of gross testing figures, but the rise of school, city, state, regional, and national measurement programs has paved the way for the establishment of dependable and comparable interpretative scales. It *is* possible for schools and colleges in almost every part of the country to procure a highly integrated set of examinations, collectively capable of describing the intellectual achievement or educational development of the high school pupil or college student in nearly all of its important aspects, and in terms of measures which are highly comparable from examination to examination. The importance of state or regional measurement programs does not reside merely in the examinations themselves nor in the number and variety used. They are significant, also, because they are gradually doing for *examining* what has already been accomplished for *examinations*. Coöperative effort has injected a genuinely democratic element into the scope and the content of the examinations; and coöperative programs of using examination results are lending educational and social significance to the meaning of guidance and curriculum building.

The administrator, the departmental chairman, and the classroom teacher are perhaps most interested in formal instruction. Their concern is twofold: on the one hand, discovery of the knowledge and proclivities of individual students constitutes the first point of contact between student and teacher; on the other hand, adjustment and appropriate enrichment of the curriculum characterizes effective instruction. To these ends, examination questions on all measures used for pre-, intermediate-, and post-study purposes ought (1) to be directed toward the subject-matter area of which the academic course is itself an abstraction, (2) to be proportioned in some quantitative fashion in accordance with the implicit and explicit objectives sought, (3) to provide for refined and differential measurement of student achievement toward each objective, (4) to be so correlated that neither excessive nor disjunctive measurement infringe upon teaching time.

In mathematics, even more strikingly than in other subject-matter fields, there is a progression of topics to be demonstrated to the student and presumably *mastered* by him. It would seem, therefore, that against this ladder of progress we could mark out the achievement of the student at each of the educational levels from the kindergarten to the graduate school. However, there is no absolute measure of achievement. The nearest approximation to an absolute measure results from the wide and consistent use of ladder-type examinations. Such

examinations are available in forms which overlap from as many as three to five grades progressively throughout the entire school system. When ladder-type examinations are carefully prepared and the questions chosen by precise technical refinements, the results are both meaningful and significant. It is to be observed that grade or class differentials do obtain, but it is also to be observed that sharp demarcations between grades and classes do not obtain.

The administrative and guidance uses of mathematics examinations extend to other than mathematics teachers. The problems of student guidance require comparable measures of student achievement in each course of study pursued by an individual; the selection of students for advanced study in colleges and universities requires demonstrated ability and achievement in a sufficiently varied subject-matter pattern to insure a reasonable rate of academic progress.

"The Mathematical Association of America is in a position to assume leadership in the adaption of (measurement) techniques to the teaching of . . . mathematics as well as to exert influence otherwise."*

COLLEGE GEOMETRY FOR SECONDARY SCHOOL TEACHERS

A. L. O'Toole, Mundelein College

In recent years much dissatisfaction has been expressed with the secondary school mathematics curriculum [1, 2], especially the geometry program. Attempts are being made to modernize high school geometry by emphasizing the nature of proof and by bringing the study of geometry into closer relationship with life problems [3, 4, 5, 6, 7]. Keyser [8] has said that the chief significance of Euclid's work is methodological rather than geometrical, that Euclid's great contribution was not to geometry but to a method of thinking that is applicable in all fields of thought. Pure logic is abstract, but geometry makes logic concrete by applying the thinking to diagrams. However, this should not be interpreted as meaning that knowledge of geometric facts and relationships is unimportant. The movement to relate the study of geometry more closely to the lives of students is laudable, provided only that in their enthusiasm teachers do not make the geometry purely incidental. Geometry courses should not become courses in social problems taught by mathematicians who have had little, if any, preparation in the social studies. To teach geometry should always be an important objective of any geometry course.

Probably, in the long run, the most fruitful approach to the improvement of secondary school mathematics is through the extension of the mathematical background of the high school teachers and prospective teachers of mathematics. At present many teachers of high school geometry have never studied any geometry beyond the course they are teaching, except analytic geometry. The colleges place much more emphasis on algebra than on geometry, although there is no justification for offering poorer preparation in one branch of mathe-

* Report of the Committee to Review the Activities of the Mathematical Association of America, this Monthly, vol. 47, February, 1940, p. 83.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A SIMPLE SOLUTION OF THE GENERAL QUARTIC

J. E. HACKE, JR., University of Georgia

The general quartic

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

can be reduced to the form

$$y^4 + Py^2 + Qy + R = 0, \tag{1}$$

where

$$\begin{aligned}
y &= x + b/4a,\\
P &= (-\,3b^2 + 8ac)/8a^2,\\
Q &= (b^3 - 4abc + 8a^2d)/8a^3,\\
R &= (-\,3b^4 + 16ab^2c - 64a^2bd + 256a^3e)/256a^4.
\end{aligned}$$

Now for any value of z,

$$(y^2 + z)^2 = y^4 + 2y^2z + z^2.$$

Substituting for y^4 from (1), we obtain

$$\begin{aligned}
(y^2 + z)^2 &= -\,Py^2 - Qy - R + 2y^2z + z^2\\
&= (2z - P)y^2 - Qy + (z^2 - R).
\end{aligned} \tag{2}$$

The left member of (2) is a perfect square for all values of z. The right member becomes a perfect square if the discriminant in $y = 0$; thus

$$4(2z - P)(z^2 - R) - Q^2 = 0,$$

$$8z^3 - 4Pz^2 - 8Rz + 4PR - Q^2 = 0. \tag{3}$$

Equation (3) is the resolvent cubic for the reduced quartic (1). Any root z_1 of equation (3) will make both sides of (2) a perfect square, and we have

$$(y^2 + z_1)^2 = K^2y^2 - 2KLy + L^2, \tag{4}$$

where

$$K^2 = 2z_1 - P, \qquad L^2 = z_1^2 - R, \qquad 2KL = Q.$$

Extracting the square root of (4), we obtain

$$\pm\,(y^2 + z_1) = Ky - L$$

which yields two resolvent quadratics,

$$y^2 - Ky + z_1 + L = 0,$$
$$y^2 + Ky + z_1 - L = 0,$$

the roots of which are

$$y = \frac{K \pm \sqrt{K^2 - 4(z_1 + L)}}{2}, \quad y = \frac{-K \pm \sqrt{K^2 - 4(z_1 - L)}}{2}.$$

Consider the following quartic as a simple numerical example:

$$x^4 - 4x^3 + 7x^2 - 10x + 3 = 0.$$

The reduced quartic is

$$(1') \qquad y^4 + y^2 - 4y - 3 = 0, \qquad y = x - 1.$$

We have, in succession,

$$(2') \qquad (y^2 + z)^2 = (2z - 1)y^2 + 4y + (z^2 + 3),$$
$$4(2z - 1)(z^2 + 3) - 16 = 0,$$
$$(3') \qquad 2z^3 - z^2 + 6z - 7 = 0.$$

By inspection, $z_1 = 1$ is a root of (3′). Substituting $z = 1$ in (2′) we have

$$(4') \qquad (y^2 + 1)^2 = y^2 + 4y + 4,$$
$$y^2 + 1 = \pm (y + 2),$$
$$y^2 - y - 1 = 0, \quad \text{and} \quad y^2 + y + 3 = 0;$$

hence

$$y = (1 \pm \sqrt{5})/2, \quad \text{and} \quad y = (-1 \pm i\sqrt{11})/2.$$

Since $x = y + 1$, we find

$$x = (3 \pm \sqrt{5})/2, \quad \text{and} \quad x = (1 \pm i\sqrt{11})/2.$$

As a comparison with Ferarri's method, consider his resolvent cubic

$$Z^3 - BZ^2 + (AC - 4D)Z - A^2D + 4BD - C^2 = 0$$

for the general quartic

$$(5) \qquad Y^4 + AY^3 + BY^2 + CY + D = 0.$$

If $A = 0$, $B = P$, $C = Q$, $D = R$, equation (5) reduces to (1) in form, and Ferarri's resolvent cubic takes the form

$$Z^3 - PZ^2 - 4RZ + 4PR - Q^2 = 0,$$

whose roots are twice the roots of (3).

BRIDGE HANDS

ANONYMOUS

The frequency of bridge hands according to suit distribution is as follows:

The total number of possible hands is 635,013,559,600.

From the following table we observe that over 90% of all hands are of ten standard types, having no suit of more than 6 cards, no void suits, and not more than one singleton. These may be regarded as normal, the others as freaks.

For simplicity, commas are omitted in this table.

Distribution	Number of hands	Approximate frequency per 1000 hands
4 4 3 2	136 852 887 600	214
5 3 3 2	98 534 079 072	155
5 4 3 1	82 111 732 560	129
5 4 2 2	67 182 326 640	106
4 3 3 3	66 905 856 160	105
6 3 2 2	35 830 574 208	55
6 4 2 1	29 858 811 840	47
6 3 3 1	21 896 462 016	34
5 5 2 1	20 154 697 992	31
4 4 4 1	19 007 345 500	30
(Total frequency for these ten hands is 906 per 1000)		
7 3 2 1	11 943 524 736	occurs once in 53 hands
6 4 3 0	8 421 716 160	75
5 4 4 0	7 895 358 900	80
5 5 3 0	5 684 658 408	112
6 5 1 1	4 478 821 776	142
6 5 2 0	4 134 297 024	154
7 2 2 2	3 257 324 928	195
7 4 1 1	2 488 234 320	256
7 4 2 0	2 296 831 680	276
7 3 3 0	1 684 343 232	380
8 2 2 1	1 221 496 848	520
8 3 1 1	746 470 296	850
7 5 1 0	689 049 504	920
8 3 2 0	689 049 504	920
6 6 1 0	459 366 336	1380
8 4 1 0	287 103 960	2200

9	2	1	1	113	101	560	5600
9	3	1	0	63	800	880	10000
9	2	2	0	52	200	720	12200
7	6	0	0	35	335	872	18000
8	5	0	0	19	876	428	32000
10	2	1	0	6	960	096	92000
9	4	0	0	6	134	700	100000
10	1	1	1	2	513	368	250000
10	3	0	0		981	552	640000
11	1	1	0		158	184	4 million
11	2	0	0		73	008	9 million
12	1	0	0		2	028	300 million
13	0	0	0			4	160 billion

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Fundamental Mathematics. By D. C. Harkin. New York, Prentice-Hall, Inc., 1941. 15+434 pages. $3.00.

Plane Geometry. By H. B. Kingsbury and R. R. Wallace. Milwaukee, Bruce Publishing Company, 1941. 9+484 pages. $1.68.

Fourier Series and Boundary Value Problems. By R. V. Churchill. New York and London, McGraw-Hill Book Company, Inc., 1941. 9+206 pages. $2.50.

Introduction to Logic and to the Methodology of Deductive Sciences. By Alfred Tarski. Translated by Olaf Helmer. Enlarged and revised edition. New York, Oxford University Press, 1941. 18+239 pages. $2.75.

British Association for the Advancement of Science. Mathematical Tables. Volume IX. Table of Powers Giving Integral Powers of Integers. Initiated by J. W. L. Glaisher; extended by W. G. Bickley, C. E. Gwyther, J. C. P. Miller, E. J. Ternouth on behalf of the Committee for the Calculation of Mathematical Tables. Cambridge, University Press; New York, Macmillan Company, 1940. 12+132 pages. $4.25.

Essentials of Algebra. First Course. By W. W. Hart. Boston, D. C. Heath and Company, 1941. 7+439 pages. $1.28.

Essentials of High School Algebra. By W. W. Hart. Boston, D. C. Heath and Company, 1941. 9+582 pages. $1.60.

Factor Analysis to 1940. By Dael Wolfle. (Psychometric Monographs, Number 3.) Chicago, University of Chicago Press, 1940. 7+69 pages. $1.25.

REVIEWS

Report of the Sixth Annual Research Conference on Economics and Statistics, July 1 to 26, 1940. Colorado Springs, Colorado: Cowles Commission for Research in Economics. 99 pages.

This report gives abstracts of the 38 lectures which were presented during the sixth annual research conference on economics and statistics of the Cowles Commission.

Mathematical methods were applied in at least 17 of the papers presented. In 8 of these, well known mathematical theory played an essential rôle in the development of economic theory. There were at least 3 papers which presented new results in mathematical statistics and 4 which were primarily expository in character. And there were 2 mathematical papers which discussed subjects which are essentially neither economics nor statistics.

The following titles of these 17 papers indicate their general nature; brief abstracts are available in the conference report.

1. *Mathematical Economics.*

Dynamics of the Business Cycle, Harold T. Davis.
Some Observations on Business-Cycle Theory, Charles F. Roos.
A Comparison of Several Statistical Forecasting Methods, Merrill M. Flood.
The Stability of Equilibrium, Paul A. Samuelson.
The Problem of Testing Economic Theories by Means of Passive Observations, Trygve Haavelmo.
Some Uses of Iso-Outlay Curves in Economic Analysis, Francis McIntyre.
The Study of Business Fluctuations by Means of Economic Models, Francis W. Dresch.
The Theory of Technological Unemployment, Oskar Lange.

2. *Mathematical Statistics.*

Sampling in Production Inspection, Walter Bartky.
A New Foundation of the Method of Maximum Likelihood in Statistical Theory, Abraham Wald.
The Problem of Assigning a Length to the Cycle to be Found in a Simple Moving Average of Chance Data, Edward L. Dodd.
Elementary Analysis of Variance: The Theoretical Background, Burton H. Camp.
Telephone Trunking: A Problem in Economics, Edward C. Molina.
The Probability Theory of Compatible Events, Hilda Geiringer.
The Use of Weighted Regressions in the Analysis of Economic Series, John H. Smith.

3. *Other Mathematical Topics.*

Certain Aspects of the Theory of Genetic Equilibrium, Mark H. Ingraham.
Mathematical Aspects of Some Sociological Problems, Nicolas Rashevsky.

This report again provides a sample of modern econometrics and mathemati-

cal statistics which will be of interest to the mathematician as well as to the economist or statistician.

M. M. FLOOD

Elements of Calculus. By Abraham Cohen. New York, D. C. Heath and Company, 1940. 5+583 pages. $3.50.

Those familiar with the author's revised edition of *Elementary Treatise on Differential Equations*, 1933, will welcome this very scholarly work.

Despite the number of pages the book does not seem so large and as the author says, "it is more satisfactory to omit, if necessary, certain parts of a comprehensive text than to have to supply any missing material from outside sources."

To the reviewer this text appeals admirably as a formal classical treatment adapted to those wishing to specialize in mathematics and to the best students who desire to use calculus in the sciences.

It would be highly desirable if every student of the calculus had the time to equip himself with all the material in this book. It would fill the gaps which have crept into too many courses in college algebra, trigonometry, and analytic geometry where brief courses have been given for lack of time.

In view of the fact that integration is postponed till page 187 and the application to areas is given first on page 287, those desiring to adapt the text to the needs of engineering students might find it a bit difficult. For such students, the author's *Differential and Integral Calculus*, 1925, might prove more suitable, since integration is there introduced on page 88.

Evidently the author does not regard the *Elements of Calculus* as a revised edition of this earlier work since it is not referred to anywhere in the new book. However, a careful examination will show an immense amount of common material. Nevertheless they differ in the following respects: the order of material; amount of emphasis placed on limits, infinite series, curve sketching; and the inclusion of supplementary material.

In the present work fourteen pages on limits are placed in the appendix after an unusually full treatment in the first chapter. Curve sketching is made the heading of the fourth chapter after those on differentiation of algebraic functions and applications. Next come applications to mechanics, differentiation of transcendental functions, geometric applications, and special curves before integration is taken up in the eighth chapter. Then two additional chapters on methods of integration precede that on the definite integral and two more on applications. The remaining chapters treat partial derivatives, envelopes, multiple integrals, infinite series, and miscellaneous theorems and applications.

Hyperbolic functions are fully treated in the appendix along with thirty-two pages of solid analytic geometry. Formulas and theorems for reference from algebra, mensuration, trigonometry, and analytic geometry are followed by a table of some two hundred integrals.

Only two typographical errors were noticed: on page 180, line four from the bottom, the numerator should be $x'y''-x''y'$; and on page 185, line thirteen, $n-2$ should be replaced by $2n-2$.

The word "flex" is used for a "point of inflexion." The vertical bar $|_a^b$ is used instead of the usual bracket $]_a^b$ for evaluating a definite integral. Duhamel's theorem is avoided by several clever devices on pages 290, 315, and 329.

The theorems of Pappus are stated, but Newton's method of solving equations is omitted. The calculations of π and natural logarithms are not fully given, although Simpson's rule is suggested as giving approximations with no statement regarding the error.

The treatment of e as a limit is unusually clear and short. It appears to the reviewer that there is more attention than necessary devoted to partial derivatives. Nevertheless, the usual formula for the area of a general surface is not given. Other omissions seem to be the mention of centrifugal force in connection with a_n, and the relation between kinetic energy of rotation and moment of inertia.

In treating the limit of $\sin \theta/\theta$, the use of areas rather than lengths seems preferable to the reviewer.

Exercises are so abundant that alternate ones may be used in different years. An answer book of 35 pages is available.

An excellent innovation is the printing of the chapter and section numbers for reference at the top of each page. The type is clear and there is a good index.

One is struck with the painstaking care in the wording of definitions and theorems and the treatment of exceptional cases. This is also shown in the hints and practical rules laid down for the student.

There are 190 figures and an unusual number of illustrated exercises worked out. Many historical references are included which are apt and stimulating.

C. C. CAMP

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

TO ALL READERS OF THIS DEPARTMENT

It is at times difficult for us to tell just how we can best serve the interests of clubs and of our readers. We solicit the opinion of each of our readers as to the nature of the material he wishes to see presented in the coming academic year. The following topics are suggested for comments:

1. More detailed or shorter club reports.
2. One or more plays each year.
3. Lists of books for the Mathematics Club Library.
4. Descriptions of mathematical puzzles, games, stunts, and recreations.
5. Reports on mathematical movies.
6. Bibliographies on topics for club use.

7. Short papers on topics for discussion at meetings.
8. Descriptions of student publications.
9. Material relating to undergraduate contests.
10. Poems relating to mathematical themes.
11. Miscellaneous papers such as hints for study, reading lists, how to present topics, *etc.*

NOTES

1. *Topic* 51. (See this MONTHLY, vol. 47, p. 484.) *Apportionment of Representatives.* Clubs will be interested in obtaining for their files, *Methods of Apportionment in Congress*, A Survey of Methods of Apportionment in Congress, by E. V. Huntington; 76th Congress, 3rd Session, Senate Document No. 304, 41 pages. Send ten cents to Superintendent of Documents, United States Government Printing Office, Washington, D. C.

Professor Huntington calls our attention to *Congressional Apportionment*, by L. F. Schmeckebier, published by Brookings Institution, Washington, D. C., 1941, $2.50.

2. *How To Study Mathematics.* So many requests have been received for copies of the article by this title which appeared in the December, 1940, MONTHLY, (vol. 47, pp. 704–707), that the author has made arrangements to reprint it and make it available to readers at cost. Requests should be addressed to Professor W. C. Arnold, P.O. Box 466, Greencastle, Indiana. Price postpaid for 50 copies, $1.75; for 100 copies, $3.00.

CLUB REPORTS, 1939-40

Mathematics Club, Boston University

Two guest speakers were Professor W. Ransom of Tufts College who spoke on Parsimonious algebra, and Professor E. V. Huntington of Harvard University whose topic was Congressional reapportionment. Student speakers and their topics included: 300 decade facts, by Theodore Ricci; Map making, by Carlton Molineux; Infinite series, by William Murphy; and Mathematics and music, by Evelyn Karol.

Mathematics Club, Wayne University

In addition to the film *Theory of Relativity*, the members heard discussions on Theory and use of log log scales, by Dr. Borgman; Mathematical economics, by Clifford Simms; Scales of notation, by Blanche Wunderlich; and Theory of nomography, by Dr. Southard.

Harvard Mathematical Club

Professor J. L. Coolidge, who retired at the end of this school year, spoke at a special meeting and banquet held in his honor on Unsolved problems on curves. In addition, the following program was presented: Twisting skew curves, by Professor Mac Lane; Generalized mathematical induction, by Dr. Alaoglu; Completeness of formal systems, by Dr. Quine; 1, 2, 6, 42, 1806—integers such that the congruence classes mod q are a homomorphic partition of the positive integers with respect to exponentiation, by John Bennett; Paradoxes and probability, by Irving Kaplansky; A theorem in topology, by Lynn Loomis; Stirling's formula, by Professor Widder; A random series, by Harry Pollard; Tensor fields and Stokes's theorem, by Paul Olum; Transcendence of π, by Michael Norris; Foundations of probability, by David Grey; and Topology of knotted curves, by Ed Hewitt.

Delta X, University of Kansas City

Monthly evening lectures were sponsored by this club to which the public was invited. The program of the year included: Famous problems in mathematics, by Albert Cahn, Jr.; Imaginary branches of a real curve, by Professor Luby; Geometrical chemistry, by Professor Smith; Higher plane curves, by George Milne, actuary; Mathematical recreation, by Dr. Crull of Park College; Elements of non-euclidean geometry, by Rev. William Doyle of Rockhurst College; Mathematical economics, by Henry G. Hilken; and Some fundamental concepts of mathematics, by Donald E. Kibbey.

Euclid's Circle, Mount St. Scholastica College

At the opening meeting, new members were required to "pass" a test involving identification of geometric figures and models, matching of descriptive passages and famous portraits, characterization of upper club members by use of mathematical terms, solution of especially chosen trick problems. The tree at the Christmas party was decorated with multi-colored miniature mathematical models and symbols. A mock broadcast near Valentine's Day presented a mathematical adaptation of the balcony scene of Romeo and Juliet, and a skit showed how Cupid solved his various problems on Valentine's eve by sending mathematically worded telegrams. The play, "It Can't Happen Here" was presented at a spring meeting at which students from local high schools were invited guests. Papers presented at other meetings included: Status of mathematics in America, by Mary Hughes; The place of Euclid in mathematics, by Sister Helen Sullivan; and Mathematics in sports, by Margaret Kennedy. Two students contributed articles to the college magazine: College students evaluate mathematics, by Bobbe Powers; and Arabic numerals, by Bernadette Schirmer. At the close of the year, this group was formally installed as the *Kansas Gamma* chapter of *Kappa Mu Epsilon* by Miss E. Marie Hove, national secretary.

Mathematics-Physics Club, Haverford College

At meetings held every three weeks, the following topics were discussed: Geometric probabilities, by J. W. Wieder, Jr.; Farey series, by R. J. Hunn; Some mechanical curiosities, by Professor Sutton; Representations of non-euclidean geometry in the plane, by Professor Wilson; Movies and slides on the isograph; The Fibonacci series, by R. B. Dickson; Relativity, by Professor Palmer; and The Morley theorem, by R. G. Strohl.

Mathematics Club, University of Cincinnati

At one meeting, Robert Buck presented and discussed the solution and generalization of the following problem: Given three bars, gold, silver, and lead, no two of which are of equal length and no two of which are of equal weight. We are also given the following set of conditions or statements concerning these bars:

1. The heaviest bar is not the longest bar.
2. The silver bar is not the shortest bar.
3. The silver bar is heavier than the lead bar.
4. The lightest bar is not the shortest bar.
5. The lead bar is lighter than the longest bar.
6. The shortest bar is not the one of middle weight.

Identify the relative length and weight of each type of bar.

The president, Adolph Goodman, inquires: "Are there many problems which might give a student an opportunity to use his imagination and yet which do not require too much advanced study to appreciate?" Can any club advisors or readers of this department send us suggestions?

Topics discussed at other meetings were: Personal contact with various European mathematicians, by Professor Moore; An introduction to Fourier series, by A. Wayne McGaughey; A lattice proof of Farey's series, by Immanuel Marx; The isoperimetric problem, by Harry Kieval; Infinite products, by Adolph Goodman; A method of approximating the sum of certain convergent series, by Kenneth Schraut; Maxima and minima without the use of the calculus, by Professor Barnett; Calendar problems and elementary number theory, by William Restemeyer; and Measures of relationship, by Robert Canning.

Kappa Mu Epsilon, Nebraska State Teachers College at Wayne

Mathematical short cuts and oddities was the theme used at one meeting. Each member was expected to present either a short cut or an oddity. At another meeting, a display and discussion on mathematical models was led by Darel Bright. Two other feature meetings included one on mathematical fallacies in which Ellsworth Macklin, Russel Holdenried, and Keith Johnson participated; and another, a mathematical *Information Please* contest.

Echols Mathematics Club, University of Virginia

The following papers were presented: Number systems, by Professor Whyburn; Pythagorean sets of numbers, by W. A. Blankinship; The solvability of equations, by W. R. Callahan; Prime numbers, by J. L. Kelley; Constructions with compasses only, by Professor Hedlund; Non-euclidean geometry, by J. H. Waite; A short method of long division and generalized arithmetic progressions, by P. A. White; Hyperbolic functions, by Professor Linfield; Elementary calculus of variations, by Professor McShane; Mathematics and the student of physics, by Professor Hoxton; The method of least squares, by Dr. Wallace; and A general solution of the magic square, by R. C. Morrow.

Mathematics Club, Ball State Teachers College

A club member, Olive Leskow, wrote a play presented at one meeting entitled, "Sixteenth Century in Review" in which the famous Cardan-Tartaglia controversy on the cubic equation was portrayed. Papers presented at other meetings were: Highway curves, by Kenneth Conkling; Primitive number systems, by Alice Clark; The history of the calculus, by Ben Poer; Applied mathematics in industrial arts, an illustrated lecture presented by Loren Jones; Norman Eilar, a descendant of Leonard Euler, discussed the works of his ancestor basing his talk on family history and published sources; and Interpolation, by Thomas MacOwan. The club sponsored a college convocation at which Professor Reeve of Teachers College, Columbia University, spoke on "The problem of the gifted and dull normal pupil."

Pi Mu Epsilon, University of Pennsylvania

The policy of having student speakers at most of the meetings was followed throughout the year. These were: Stanley Corrsin, on Mathematics applied to some problems in fluid dynamics; Paul Rosenbloom, on Transfinite numbers; Robert Weinstock, on Calculus of variations and the brachistochrone; Verage Tarzian, on The Schrödinger wave equation and the Hermite polynomials; and David Garber, on Fourier series. At the May meeting, students and faculty from surrounding colleges were invited to hear Professor Gilman of Brown University speak on "Probability, or Hope Springs Eternal in the Gambler's Breast."

Pi Mu Epsilon, University of Alabama

At the regular monthly meetings of the chapter, the following papers were given: Mathematics in Persia, by Dr. Kennedy; Vector analysis, by Louise McClanahan; Nomography, by A. DeBon Owen; Games and puzzles in mathematics, by Dr. L. D. Rodabaugh; Envelopes of a system of conics associated with a fixed point and a fixed circle, by Pattillo Burton; Special cases of an inverse problem of the calculus of variations, by Belle Brasley; On a collineation group in 6 variables with certain characteristics of the ternary hessian group, by Mary Jones; On a collineation group of order 1536, by David Bryan; and Configurations associated with a group of order 48 and degree 6, by Dr. Warnock.

Pi Mu Epsilon, Louisiana State University

Topics discussed at meetings were: The cycloid, The brachistochrone problem, Maxima and minima, Homogeneous Diophantine equations, Certain concepts of modern algebra, and Annuities.

Pi Mu Epsilon, University of Arkansas

During the year there were three outstanding speakers who addressed the group as follows: Dr. Kent, on Irrational numbers; Dr. Richardson, on Calendar; and Dr. G. B. Price, on Modern problems in higher mathematics.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 471. *Proposed by L. G. Johnson, Ann Arbor, Michigan.*

A watch attached to a chain is swung around in a circle with the same angular velocity as that of its second hand. Show that the path traced by the tip of the second hand is a limaçon or a circle according as the sense of motion is clockwise or counterclockwise.

E 472. *Proposed by V. Thébault, San Sebastián, Spain.*

Find positive integers x, y, z (less than 100), such that $x^2+y^2=z^2$ and $X^2+Y^2=Z^2$, where X, Y, Z are derived from x, y, z by inserting an extra digit (the same for all) on the left.

E 473. *Proposed by N. A. Court, University of Oklahoma.*

Two variable transversal planes PQR, $P'Q'R'$, reciprocal with respect to a given tetrahedron $DABC$, meet the edges DA, DB, DC in the pairs of points P and P', Q and Q', R and R'. Show that the line of centers of the two spheres $DPQR$, $DP'Q'R'$ passes through a fixed point. (Two transversal planes are said to be *reciprocal* with respect to a tetrahedron if their traces on each edge are equidistant from the midpoint of the edge. See the proposer's *Modern Pure Solid Geometry*, p. 122, art. 354.)

E 474. *Proposed by Roy MacKay, Eastern New Mexico College.*

For $k>1$, define $a_1=\{k(k-1)\}^{1/2}$, $a_n=\{k(k-1)+a_{n-1}\}^{1/2}$, $b_1=k^{1/2}$, and $b_n=(kb_{n-1})^{1/2}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = k.$$

E 475. *Proposed by J. Goodfellow, West Rumney, N. H.*

Let the diameter AB of a circle S meet a perpendicular chord HH' at O. Take points C and D on AB, such that $CO=OB$ and $OD=OH$. Let G be one of the points of intersection of S with the circle on CD as diameter. Show that we have approximately

$$OG^3 = AO\cdot OB^2.$$

How close an approximation does this construction provide for the classical problem of "duplicating the cube"?

Thus

$$HF_1^2 + HF_2^2 + 2K = \frac{\sum (4p^2 + 2AF_1 \cdot AF_2 \cos A) \tan A}{\sum \tan A}$$
$$= 4p^2 + \frac{2\sum AF_1 \cdot AF_2 \sin A}{\sum \tan A} \cdot$$

But $AF_1 \cdot AF_2 \sin A$ or $AB_1 \cdot AF_2 \sin A$, being twice the area of the triangle AB_1F_2, is the area of the deltoid $AB_1F_2C_1$; and three such deltoids make up the hexagon $AB_1CA_1BC_1$, whose area is twice that of the triangle ABC. Hence

$$4p^2 - HF_1^2 - HF_2^2 = 2K - 4\Delta/(\sum \tan A),$$

which is constant.

The above proof is substantially that of E. H. Neville, *Mathematical Gazette*, vol. 12, 1924–25, p. 23, who remarks that the theorem can be divorced from any consideration of conics, since F_1 and F_2 may be any two isogonal conjugate points, and p is the radius of their common pedal circle. The constant can be computed by considering the special case when F_1 and F_2 are the orthocenter H and circumcenter O. Then the pedal circle is the nine-point circle, $2p=R$, and the constant is R^2-HO^2. But

$$HO^2 = 9R^2 - a^2 - b^2 - c^2.$$

Hence, finally,

$$4p^2 - HF_1^2 - HF_2^2 = a^2 + b^2 + c^2 - 8R^2$$
$$= 8R^2 \cos A \cos B \cos C,$$

which is -2ρ in the notation of Neville, *Mathematical Gazette*, vol. 24, 1940, p. 53.

Also solved by the proposer.

E 437 [1940, 569]. *Proposed by V. Thébault, Le Mans, France.*

For what kind of tetrahedron does the Monge point lie on the circumsphere? (The Monge point lies on planes perpendicular to the edges through the mid-points of the respective opposite edges.)

Solution by P. D. Thomas, Norman, Oklahoma.

"The Monge point of a tetrahedron is the symmetric of the circumcenter with respect to the centroid." (N. A. Court, *Modern Pure Solid Geometry*, p. 69, art. 230.) "The circumcenter of a trirectangular tetrahedron is the symmetric of the vertex of the right angle with respect to the centroid." (*Ibid.*, p. 93, art. 288.) Hence (or obviously) a trirectangular tetrahedron has the desired property, the Monge point coinciding with one vertex.

Also solved thus by the proposer.

Editorial Note. A tetrahedron whose Monge point lies on its circumsphere is not necessarily trirectangular. All that we require is that $OG=\frac{1}{2}R$, where O is the circumcenter, G the centroid, and R the circumradius. By applying Lagrange's theorem (quoted above, E 434) to equal masses at the vertices of the tetrahedron, with F at O, we obtain

$$OG^2 = R^2 - \sum a^2/16\,,$$

where $\sum a^2$ refers to the six edges. Hence the necessary and sufficient condition is

$$\sum a^2 = 12R^2.$$

To take an easy example, suppose that all but one of the edges are of length 1, so that $4R^2=(4-a^2)/(3-a^2)$. Then the condition

$$5 + a^2 = 3(4 - a^2)/(3 - a^2)$$

gives $a^2=(\sqrt{13}+1)/2$.

E 441 [1940, 657]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given ED, construct an isosceles triangle ABC, with apex C, so that E lies on the altitude CD, and two perpendicular transversals drawn through E divide the area of the triangle into four equal parts.

Solution by William Douglas, Courtenay, British Columbia.

Imagine the construction completed, with perpendicular transversals FJ, GH in a symmetrical position, with J on BC, G on CA, F and H on AB. Let $ED=1$, so that $FD=DH=1$ and the area $FEH=1$. Denote CD by x, and AD by y, so that the area of the whole triangle is

$$xy = 4.$$

Draw GK perpendicular to CD. Since the area $GCJE$ is $(x-1)GK=1$, we have $KE=GK=1/(x-1)$, $GE=\sqrt{2}/(x-1)$. By dividing the area $AFEG$ into triangles AFG and FEG, we obtain

$$(y - 1)\left(1 + \frac{1}{x - 1}\right) + \sqrt{2}\left(\frac{\sqrt{2}}{x - 1}\right) = 2,$$

whence $3x=xy+4=8$, and

$$x = 8/3, \qquad y = 3/2,$$

and the points A, B, C can immediately be constructed.

Also solved by C. W. Trigg and the proposer, and (with a different interpretation of the problem) by W. B. Clarke.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkle, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3996. *Proposed by Elbert H. Clarke, Hiram College, Ohio.*

Sum the series

$$\sum_{n=1}^{\infty} [(n-1)k]!/(nk)!,$$

where k is any integer greater than unity.

3997. *Proposed by A. Oppenheim, Raffles College, Singapore, S. S.*

Let $P(x, t)$ denote the probability that an irreducible fraction p/q, $1<q\leqq t$, which satisfies the inequality $|x-p/q|<1/q^2$, should be a convergent to the irrational number x. Prove that (1) $P(x, t)\geqq 1/3$; (2) when t tends to infinity, $P(x, t)$ need not tend to a limit, or can tend to any assigned limit between $1/3$ and 1; (3) if $x=[g+(g^2+4)^{1/2}]/2$, then $P(x, t)\to 1$ when $g=1$, $P(x, t)\to 1/2$ when $g=2$, $P(x, t)\to 1/3$ when $g=3, 4, \cdots$.

3998. *Proposed by V. Thébault, Tennie, Sarthe, France.*

A sphere (S) is tangent to the faces of a tetrahedron $ABCD$ at the points A', B', C', D' and the straight lines AA', BB', CC', DD' are concurrent in the point P. The cones (Γ_A), (Γ_B), (Γ_C), (Γ_D) with vertices at A, B, C, D circumscribe (S). The planes through P parallel to the planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ cut the respective cones in four circles which lie on a sphere concentric with (S).

SOLUTIONS

3894 [1938, 631]. *Proposed by Walter Leighton, The Rice Institute.*

Given a polynomial equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0, \tag{1}$$

where the a_i are rational integers, find all roots of the form $(a\pm\sqrt{b})/c$, where a, b, c are integers, $c\neq 0$, and $\sqrt{b}$ is irrational (possibly imaginary).

II. *Solution by J. M. Thomas, Duke University.*

A solution different from the proposer's [1940, 578] follows. Since $f(z)$ has

rational coefficients, we have $f(a+\sqrt{b})=g(a, b)+\sqrt{b}\, h(a, b)$, where g, h are polynomials with rational coefficients. The number $a+\sqrt{b}$, where a, b are rational and $\sqrt{b}$ is not rational, is a root of $f(z)$ if and only if (i) a is a root of the resultant R_a of g, h written as polynomials in b, and (ii) b is a root of the H.C.F. (usually linear) of g, h evaluated for such an a. Consequently, the search for roots $a+\sqrt{b}$ is immediately reduced to finding the rational roots of polynomials with rational coefficients. (The polynomials g, h, R_a are converted into the g, h, R_x of the paper *Resolvents of a polynomial*, this MONTHLY, November, 1940, by the substitution $x=a$, $yi=\sqrt{b}$.) In particular, for a reduced quartic, R_a is the resolvent cubic in $k=2a$; if R_a has a non-zero rational root, the quartic has a root of the form $a+\sqrt{b}$, where a, b are rational (and $\sqrt{b}$ may also be rational).

The condition employed by Glenn (this MONTHLY, vol. 23, 1916, pp. 313–315) is, in effect, that when f and its quadratic factor have their initials equal to unity and their other coefficients integral, the constant term of R_a is divisible by $2a$. Frumveller (this MONTHLY, vol. 24, 1917, pp. 208–212) has also given a method for determining quadratic factors.

3932 [1939, 602]. *Proposed by V. Thébault, Le Mans, France.*

What must be the base of a number system such that a number of the form $aabb$ is the square of a number of the form cc, where c is a multiple of b? Show that b is always a perfect square, and that there exists no number system possessing squares of the form $aabb$, where c is a multiple of a.

Solution by E. P. Starke, Rutgers University.

Let r be the base of the system of notation. Then by hypothesis we have

$$(ar^2 + b)(r + 1) = (k^2b^2)(r + 1)^2, \qquad c = kb < r.$$

Thus $ar^2-a+a+b=k^2b^2(r+1)$ or $ar-a+(a+b)/(r+1)=k^2b^2$, which implies $a+b=r+1$ and $ar-a+1=k^2b^2$. Elimination of a gives

$$r^2 - b(r - 1) = k^2b^2. \tag{1}$$

Let p be a prime factor of b, so that $b=p^n\beta$, $n\geqq 1$, β prime to p. Then (1) becomes $r^2-p^n\beta(r-1)=k^2p^{2n}\beta^2$. Thus p^n divides r^2 and hence p^m divides r, where $m=n/2$ or $(n+1)/2$ according as n is even or odd. Putting $r=p^m\rho$, we have $p^n[p^{2m-n}\rho^2-p^m\beta\rho+\beta-k^2p^n\beta^2]=0$. If n is odd, we have $2m-n=1$ and thus p divides β contrary to the above. Hence n must be even. Since p was any prime factor of b, b is a square.

We may now put $b=s^2$. From (1), s^2 divides r^2. Let us put $r=vs$; then (1) becomes $v^2-vs=k^2s^2-1$ or

$$(2v - s + 2ks)(2v - s - 2ks) = s^2 - 4. \tag{2}$$

An easy set of solutions is obtained by putting $s=2$; then $v=2k+1$, $b=4$, $c=4k$, $r=4k+2$, $a=4k-1$.

For the general solution of (2), put $g=2v-s+2ks$, $h=2v-s-2ks$, so that (2)

becomes $gh=s^2-4$. Since $g=h+4ks$, (2) can be put in the form

$$s^2 - 4 = h^2 + 4ksh, \quad \text{or} \quad x^2 - (4k^2+1)h^2 = 4, \quad \text{where} \quad x^2 = (s-2kh)^2.$$

All solutions of this equation are given by the iteration formulas

$$\begin{aligned} x_{j+1} &= (8k^2+1)x_j + 4k(4k^2+1)h_j, & x_1 &= 16k^2+2,\\ h_{j+1} &= 4kx_j + (8k^2+1)h_j, & h_1 &= 8k. \end{aligned}$$

For the single value $k=1$, there exist three sequences of solutions, using the same iteration formulas and starting with $(x_1, h_1)=(3, 1)$, $(7, 3)$, and $(18, 8)$. For other values of k only the sequence cited above exists. From these results the values of a, b, c, and r are easily obtained.

If we start with the hypothesis

$$(ar^2+b)(r+1) = k^2a^2(r+1)^2,$$

and follow the same method as in the former case, we obtain

$$ar - a + (a+b)/(r+1) = k^2a^2,$$

whence $a+b=r+1$ and $ar-a+1=k^2a^2$. Thus $r=k^2a+1-1/a$ which is not an integer (unless $a=1$; but then $b=r$, which is impossible).

Solved also by the proposer.

3934 [1939, 656]. *Proposed by V. Thébault, Le Mans, France.*

Let G be the centroid of the tetrahedron $ABCD$. Through A, B, C, D planes are drawn perpendicular to GA, GB, GC, GD, respectively, forming the antipedal tetrahedron of G, with respect to $ABCD$, of volume V_g. Similarly, V_a, V_b, V_c, V_d are the volumes of the antipedal tetrahedrons of A, B, C, D, with respect to the tetrahedrons $GBCD$, $GCDA$, $GDAB$, $GABC$. Show that $V_g=4V_a=4V_b=4V_c=4V_d$.

Solution by the Proposer.

This problem is a particular case of the following problem by J. Neuberg, *Mathesis*, 1915, p. 191, which remained without a solution until our own proof in the same journal, 1940, p. 72:

Given in space any five points A_i, $(i=1, 2, 3, 4, 5)$; the four planes perpendicular respectively at A_i to A_iA_5 determine the antipedal tetrahedron of A_5 with respect to $A_1A_2A_3A_4$. Denote by U_5, V_5 the volumes of the respective tetrahedrons, and, similarly, by V_1 the volume of $A_2A_3A_4A_5$ and by U_1 the volume of the antipedal tetrahedron of A_1 with respect to it. Show that $U_1/V_1 = U_2/V_2 = \cdots = U_5/V_5$.

Let O_1 and $R_1, \cdots, O_5$ and R_5 denote the centers and radii of the spheres circumscribing $T_1 \equiv A_2A_3A_4A_5, \cdots, T_5 \equiv A_1A_2A_3A_4$; $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and d_1, d_2, d_3, d_4 the absolute barycentric and absolute normal coördinates of A_5 with respect to T_5; and S_1, S_2, S_3, S_4 the areas of the corresponding faces of T_5. Then T_5 and T_1 have in common the face $A_2A_3A_4$, whose plane is the radical plane of (O_5) and (O_1), and thus O_1O_5 is perpendicular to this face. Using the theorem in regard

to the difference of the powers of a point A_5 with respect to two spheres (O_5) and (O_1) and noting that A_5 is on (O_1), we have $R_5^2-(A_5O_5)^2=2d_1\cdot O_5O_1$, or

$$\alpha_1\cdot(O_5O_1) = \frac{R_5^2 - (A_5O_5)^2}{6V_5} S_1.$$

Thus the vectors $\alpha_i\cdot(O_5O_i)$, $(i=1, 2, 3, 4)$, are perpendicular to the corresponding faces of T_5 and their lengths are proportional to the areas of the same faces. Hence their sum is zero, and the barycentric coördinates α_i of A_5 with respect to T_5 are also the barycentric coördinates of O_5 with respect to $O_1O_2O_3O_4$, and we have $u_1/V_1=u_2/V_2=\cdots=u_5/V_5$, where the u_i's, $(i=1, 2, 3, 4)$, are the algebraic volumes of the tetrahedrons having the common vertex O_5 and the faces of $O_1O_2O_3O_4$ as bases. But the antipedal tetrahedron of A_5 with respect to T_5 is the transform by similitude of $O_1O_2O_3O_4$ with the pole A_5 and ratio 2:1; and hence $U_5=8u_5$. We now have $U_1/V_1=U_2/V_2=\cdots=U_5/V_5$.

If $A_5\equiv G$, the centroid of T_5, we have $V_5=4V_1=4V_2=\cdots$; and hence, in the notation of 3934, we have $V_g=4V_a=4V_b=\cdots$.

A similar proposition for the plane has been given by Bateman, *Educational Times*, 1907, p. 456. In particular, for triangle ABC with centroid G, we have $S_g=3S_a=3S_b=3S_c$, where S_g, S_a, S_b, S_c are the areas of the antipedal triangles of G, A, B, C with respect to ABC, GBC, GCA, GAB.

Editorial Note. The proof uses the theorem that $\sum n_iS_i=0$, $(i=1, 2, 3, 4)$, where n_i is a unit vector normal to the plane of S_i directed toward the interior of T_5, or toward the exterior. Also, if O_5 is taken as the origin of vectors $o_i=O_5O_i$ and m_i are masses, positive or negative, placed at the vertices O_i, and if $\sum m_io_i=0$, then the centroid of this system of four masses is at the origin O_5, and the barycentric coördinates of O_5 with respect to $O_1O_2O_3O_4$ are proportional to m_1, m_2, m_3, m_4. It may be necessary to provide the barycentric coördinates of A_5 with the proper plus or minus signs in order to obtain those for O_5; but for the present theorem, which considers only the absolute values of the volumes in question, these signs are not needed for the proof. This last remark as to signs applies to other parts of the proof; and it will be convenient to assume that no four of the five given points are in a plane.

3935 [1939, 656]. *Proposed by V. Thébault, Le Mans, France.*

In what system of numeration is it true that a number formed from four identical figures $aaaa$ is a perfect square provided that a is a square? The solution is unique. What happens if a is not a square?

Consider the same questions for a number formed from five identical figures $aaaaa$.

Solution by the Proposer.

The two parts of the problem lead to the equations

$$a(B^3 + B^2 + B + 1) = x^2, \tag{1}$$

$$a(B^4 + B^3 + B^2 + B + 1) = y^2, \tag{2}$$

where B is the required base. If a is a perfect square, so are also the expressions in parentheses. For the resulting equations the only solutions are (1) $B=7$, and (2) $B=3$ (Gérono, *Nouvelles Annales*, 1875, pp. 288 $\cdots$). The only solutions are therefore

$$(1) \qquad 1111 = (26)^2, \qquad 4444 = (55)^2, \quad \text{base } 7,$$

$$(2) \qquad 11111 = (102)^2, \quad \text{base } 3.$$

If a is not a perfect square, we have $N=aaaa=a(B+1)(B^2+1)$; and there are an infinite number of solutions, among which we note $B=7$, 41, 239, 1393, 8119, $\cdots$. For (2) we proceed similarly, setting

$$N = aaaaa = a[B(B+1)(B^2+1)+1] = y^2.$$

3936 [1939, 656]. *Proposed by N. A. Court, University of Oklahoma.*

If of the four circles determined by four coplanar points taken three at a time two circles are orthogonal, the remaining two circles are orthogonal. (*Mathesis*, 1929, p. 130, Q. 2515).

If of the five spheres determined by five points in space taken four at a time three spheres are mutually orthogonal, the remaining two spheres are orthogonal to each other. Prove, or disprove.

Solution by B. A. Hausmann, University of Detroit.

Part I. Let p_i, $(i=1, 2, 3, 4)$, be the four coplanar points. Let C_i be the circle on all the points except p_i. Let C_3 and C_4 be orthogonal. Invert the circles C_i with respect to the point p_1. Let the inverse curve of C_i be L_i. Then L_2, L_3, L_4 are straight lines, and L_3 and L_4 are perpendicular. Now L_1 is a circle on the three distinct points of intersection of L_2, L_3, L_4. Since L_3 and L_4 are perpendicular, L_2 passes through the center of the circle L_1, and hence L_1 and L_2 are orthogonal. Therefore C_1 and C_2 are orthogonal.

Part II. Let p_i, $(i=1, 2, 3, 4, 5)$, be the five points in three-space. Let S_i be the sphere on all points except p_i. Let S_3, S_4, and S_5 be mutually orthogonal. Invert the spheres with respect to p_1 and let I_i be the inverse surface of S_i. Then I_3, I_4, I_5 are mutually perpendicular planes. Now I_2 is a plane intersecting these three planes. The four points of intersection of these four planes taken three at a time determine the sphere I_1, which is the circumsphere of the trirectangular tetrahedron determined by the planes I_2, I_3, I_4, I_5. By Theorem 287, p. 92 in *Modern Pure Solid Geometry* by N. A. Court (Macmillan, 1935), "The median of a trirectangular tetrahedron issued from the right angle passes through the circumcenter of the tetrahedron and is equal to two-thirds of the circumradius of the tetrahedron." Hence the circumcenter is not in the base which is the plane I_2. Hence I_1 and I_2 are not orthogonal. Therefore S_1 and S_2 are not orthogonal.

Solved also by J. W. Clawson and the proposer in a similar manner.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The American Standards Association published in March 1941 *Abbreviations for scientific and engineering terms;* this can be obtained for thirty-five cents from the American Society of Mechanical Engineers, 29 West 39th Street, New York, N. Y. The Mathematical Association has been associated with this series of "American Standards for Abbreviations, Symbols, and Charts" beginning with the report on mathematical symbols published in 1928.

Brown University has announced a special program for the Summer Session, June 23–September 13, 1941 (also for the semesters October 1, 1941–January 31, 1942 and February 11–June 6, 1942) devoted to advanced instruction and research in mechanics. It is approved as a part of the Engineering Defense Training Program of the U. S. Office of Education. Because of subventions, there will be no tuition charges in any of the courses of this program in the summer or in the academic year. Fellowships ranging in size up to $600 are available for the academic year 1941–42. For further information, write to the Dean of the Graduate School, Brown University, Providence, R. I.

New York University announces the following courses in applied mathematics for the Summer Session: Advanced methods in applied mathematics, by Professor Courant; Fluid dynamics and applications, by Professor Friedrichs; Bending and buckling of elastic plates, by Professor Stoker; Transient analysis of electrical networks, by Professor E. Weber; Seminar on vibrations, by Professors Courant, Friedrichs, and Stoker.

EXAMINATION QUESTIONS FOR THE FOURTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, MARCH 1, 1941

MORNING SESSION: 9:00 A.M. to 12:00 NOON. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books may be used.*)

1. Prove that the polynomial

$$(a-x)^6-3a(a-x)^5+\tfrac{5}{2}a^2(a-x)^4-\tfrac{1}{2}a^4(a-x)^2$$

takes only negative values for $0<x<a$.

2. Find the nth derivative with respect to x of

$$\int_0^x\left[1+\frac{(x-t)}{1!}+\frac{(x-t)^2}{2!}+\cdots+\frac{(x-t)^{n-1}}{(n-1)!}\right]e^{nt}dt.$$

3. A circle of *radius* a rolls in its plane along the x-axis. Show that the envelope of a diameter is a cycloid, coinciding with the cycloid traced out by a point on the circumference of a circle of *diameter* a, likewise rolling in its plane along the x-axis.

4. Let the roots a, b, c of

$$f(x) \equiv x^3 + px^2 + qx + r = 0$$

be real, and let $a \leqq b \leqq c$. Prove that, if the interval (b, c) is divided into *six* equal parts, a root of $f'(x) = 0$ will lie in the *fourth* part counting from the end b. What will be the form of $f(x)$ if the root in question of $f'(x) = 0$ falls at either end of the *fourth* part?

5. Show that the line which moves parallel to the plane $y = z$ and which intersects the two parabolas $y^2 = 2x$, $z = 0$ and $z^2 = 3x$, $y = 0$ sweeps out the surface

$$x = (y - z)\left(\frac{y}{2} - \frac{z}{3}\right).$$

6. If the x-coördinate $\bar{x}$ of the center of mass of the area lying between the x-axis and the curve $y = f(x)$, $(f(x) > 0)$, and between the lines $x = 0$ and $x = a$ is given by $\bar{x} = g(a)$, show that

$$f(x) = A \frac{g'(x)}{[x - g(x)]^2} e^{\int dx/[x - g(x)]},$$

where A is a positive constant.

7. Take either (a) or (b). (a) Prove that

$$\begin{vmatrix} 1 + a^2 - b^2 - c^2 & 2(ab + c) & 2(ca - b) \\ 2(ab - c) & 1 + b^2 - c^2 - a^2 & 2(bc + a) \\ 2(ca + b) & 2(bc - a) & 1 + c^2 - a^2 - b^2 \end{vmatrix} = (1 + a^2 + b^2 + c^2)^3.$$

(b) A semi-ellipsoid of revolution is formed by revolving about the x-axis the area lying within the first quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Show that this semi-ellipsoid will balance in stable equilibrium, with its vertex resting on a horizontal plane, when and only when $b\sqrt{8} \geqq a\sqrt{5}$.

AFTERNOON SESSION: 2:00 P.M. to 5:00 P.M. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books may be used.*)

8. A particle (x, y) moves so that its angular velocities about $(1, 0)$ and $(-1, 0)$ are equal in magnitude but opposite in sign. Prove that

$$y(x^2 + y^2 + 1)dx = x(x^2 + y^2 - 1)dy,$$

and verify that this is the differential equation of the family of rectangular hyperbolas passing through $(1, 0)$ and $(-1, 0)$ and having the origin as center.

9. Evaluate the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right);$$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right);$$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right).$$

10. Find the differential equation satisfied by the product z of any two linearly independent integrals of the equation

$$y''+y'P(x)+y\,Q(x)=0.$$

11. Two perpendicular diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are given, and the two diameters conjugate to them are constructed. Show that the rectangular hyperbola passing through the ends of these conjugate diameters passes through the foci of the ellipse.

12. A car is being driven so that its wheels, all of radius a feet, have an angular velocity of ω radians per second. A particle is thrown off from the tire of one of these wheels, where it is supposed that $a\omega^2>g$. Neglecting the resistance of the air, show that the maximum height above the roadway which the particle can reach is $(a\omega+g\omega^{-1})^2/2g$.

13. Assuming that $f(x)$ is continuous in the interval $(0, 1)$, prove that

$$\int_{x=0}^{x=1}\int_{y=x}^{y=1}\int_{z=x}^{z=y} f(x)f(y)f(z)\,dx\,dy\,dz=\frac{1}{3!}\left(\int_{t=0}^{t=1} f(t)\,dt\right)^3.$$

14. Take either (a) or (b). (a) Show that any solution $f(t)$ of the functional equation

$$f(x+y)f(x-y)=f(x)f(x)+f(y)f(y)-1, \qquad (x, y \text{ real}),$$

is such that

$$f''(t)=\pm m^2 f(t), \qquad (m \text{ constant and } \geqq 0),$$

assuming the existence and continuity of the second derivative. Deduce that $f(t)$ is one of the functions $\pm\cos mt$, $\pm\cosh mt$.

(b) With n constant values $a_1, a_2, \cdots, a_n$, supposed all different, let n constant values $b_1, b_2, \cdots, b_n$ be associated, and let a polynomial $P(x)$ be defined by the identity in x

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & P(x) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & b_1 \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & b_n \end{vmatrix} \equiv 0.$$

Given a polynomial $\phi(t)$, let a polynomial $Q(x)$ be defined by the identity in x obtained on replacing $P(x)$, b_1, b_2, $\cdots$, b_n of the identity above by $Q(x)$, $\phi(b_1)$, $\phi(b_2)$, $\cdots$, $\phi(b_n)$. Prove that the remainder obtained on dividing $\phi(P(x))$ by $(x-a_1)(x-a_2)\cdots(x-a_n)$ is $Q(x)$.

NOTE. Chairmen of mathematics departments may obtain copies of the examination questions for the Putnam Competition for 1938, for 1939, for 1940, and for 1941 by writing for them to Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The following results of the fourth annual William Lowell Putnam Mathematical Competition held March 1, 1941, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the readers only by number up to the time of this announcement.

The first prize, five hundred dollars, is awarded to the department of mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Richard Bellman, Peter Chiarulli, Hyman Zimmerberg; to each of these is awarded a prize of fifty dollars.

The second prize, three hundred dollars, is awarded to the department of mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania. The members of the team were S. I. Askovitz, Hyman Kamel, P. C. Rosenbloom; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the department of mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were J. R. R. Baumberger, Eugene Calabi, W. S. Loud; to each of these is awarded a prize of twenty dollars.

The five persons ranking highest in the examination, named in alphabetical order, were: R. F. Arens, University of California at Los Angeles; S. I. Askovitz, University of Pennsylvania; A. M. Gleason, Yale University; E. L. Kaplan, Carnegie Institute of Technology; P. C. Rosenbloom, University of Pennsylvania. Each of these will receive a prize of fifty dollars. The order of the names in this list has no relation to their rank in the examination.

The following teams won honorable mention: Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, the members of the team being R. E. Beatty, E. L. Kaplan, N. H. Painter; Department of Mathematics, Cooper Union Institute of Technology, New York, the members of the team being Murray Klamkin, Benjamin Lax, Samuel Manson; Department of Mathematics, Yale University, New Haven, the members of the team being A. M. Gleason, G. R. MacLane, D. M. Merrill.

Six individuals are given honorable mention, the names listed in alphabetical order: Richard Bellman, Brooklyn College; Harvey Cohn, College of the City of New York; W. S. Loud, Massachusetts Institute of Technology; G. R. MacLane, Yale University; Samuel Manson, Cooper Union Institute of Technology; Hyman Zimmerberg, Brooklyn College.

The order of the names in both lists for honorable mention has no relation to their rank in the examination.

The following is a list of all colleges and universities which entered teams in the Competition, some of these entering also individual contestants. (This list is arranged alphabetically, and the order of the names here has no relation to the rank of the teams in the examination.) Brooklyn College, University of California, University of California at Los Angeles, Carleton College, Carnegie Institute of Technology, University of Cincinnati, Columbia University, Cooper Union Institute of Technology, Duke University, Harvard University, Kansas State College, Massachusetts Institute of Technology, McGill University, McMaster University, University of Nebraska, College of the City of New York, New York University, Northwestern University, Pennsylvania State College, University of Pennsylvania, Queens College, The Rice Institute, Rutgers University, College of St. Thomas, University of Saskatchewan, Swarthmore College, Wayne University, Yale University.

The following additional colleges and universities entered individual contestants only: Brown University, Bryn Mawr College, Case School of Applied Science, College of Charleston, Haverford College, Johns Hopkins University, Kenyon College, Millsaps College, Mount Holyoke College, North Texas State Teachers College, University of Notre Dame, Oberlin College, University of Texas, Tulane University, Wesleyan University, University of Wyoming.

A total of one hundred forty-six undergraduate students representing forty-four institutions took part in the Competition.

W. D. CAIRNS, *Secretary-Treasurer*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., May 3; Washington, Pa., October 25.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 10.
MICHIGAN, Ann Arbor, March 15.
MINNESOTA, St. Joseph, May 10
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, April 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, April 4–5.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit, May 3.

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I hereby give[1] to the Board of Governors of the Mathematical Association of America, Incorporated, the sum of .... Dollars to be used for[2]

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[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association can make good use of funds for scientific publications and for the promotion of scientific activities.

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PART I

CONTENTS

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COLLEGE MATHEMATICS—A STATEMENT FOR THE UNDERGRADUATE

What is mathematics? It is difficult to give an entirely satisfactory answer to this question. Roughly speaking, the purpose of mathematics is to discover facts concerning numbers and geometrical forms, and to organize these facts into a logical system. By stating important facts as theorems or formulas, mathematicians make them readily available for wide application, and in mathematics courses some of these applications are given consideration. According as the applications receive much or little attention, a course may be referred to as one in applied or in pure mathematics.

In undergraduate courses in mathematics, the student not only gains information of a particular type but also has an experience in logical and consecutive thinking. In some of the higher courses, the strictly logical aspect of the subject receives main attention, and mathematics and logic become indistinguishable.

Regarded from the arithmetical side, mathematics has sometimes been called "the art of computing." This computational aspect of the subject is important, but the modern trend is more aptly described as "the art of computing as little as possible." Through the use of theorems and formulas, and of calculating instruments which have been devised, the mathematician tries to reach conclusions briefly and with certainty, and without the use of long computations or vague arguments. For example, by use of a formula he will solve the problem of finding the sum of the first hundred integers by multiplying 50×101; or he will find the pressure on a dam of given dimensions by a short method known as integration instead of by costly experimentation and measurement.

Mathematics is a rapidly growing subject, experiencing in its research aspects an almost feverish activity. Its scope may be suggested by the statement that it includes such diverse subjects as the theory of magic squares, the foundations of logic, the theory of relativity, the motions of the moon and stars, the design of airplanes, a theory of economics, and fundamental laws of psychology and heredity. It has been used to predict with startling accuracy eclipses of the sun, the existence of radio waves, the presence of oil at definite places in the earth's interior, the range of heavy artillery, and the proper design of optical instruments. It has been used in developing a satisfactory basis for life insurance, in testing inexpensively the quality of goods produced by mass production methods, in surveying land and locating boundary lines, in navigating the sea and the air, in analyzing scientific experiments, in determining the shape of the earth and the distances to the stars, in designing bridges and laying out telephone networks, and so on, indefinitely. For the pure mathematician, the pursuit of mathematical research is a voyage of discovery, the charting of unknown regions regardless of the ultimate utility of the lands discovered; he is confident of the value of his work, and gets a deep esthetic pleasure from the perfection of his theories.

Why study mathematics? Some answers to this question may be inferred from the answers to the preceding question. We study mathematics to learn facts about numbers and geometrical forms, to experience the logical development of an important branch of knowledge, to gain power in the art of computing as little as possible, to prepare ourselves to handle problems in science and in technology, to earn the satisfaction that comes from overcoming mathematical difficulties, and perhaps to derive the esthetic pleasure that arises from finding an elegant proof of a mathematical proposition.

For students who are curious about the world—its history, its literature, its art, its science—about the physical world and the world of ideas, mathematics may claim attention as one of the greatest products of the human intellect, perhaps surpassing all others for its logical perfection and its universality. It is worthy of study for its own sake.

For students who are mainly concerned with the direct usefulness of their education, it may be more interesting to know that mathematical symbolism and reasoning are constantly penetrating into more and more fields of activity. For example, without a degree of familiarity with higher mathematics, much of the literature of astronomy, physics, engineering, and chemistry is wholly unintelligible. Moreover, such fields as economics, commerce, and medicine are taking on mathematical aspects, so that a person unfamiliar with algebraic symbolism and higher mathematics may find himself completely baffled in attempting to read some of the current literature in these fields. Thus familiarity with mathematics is important for the student's development in very many directions.

And finally, students trained in mathematics have special opportunities for various kinds of employment. For instance, many employers prefer mathematically trained candidates for secretarial and administrative assistants, especially where statistical or accounting work may be required. Life insurance companies and industrial research organizations provide openings for mathematicians in considerable numbers. Many are employed as teachers, and some enter professions such as engineering, law, and medicine, where mathematical training proves highly advantageous. And lastly we note that a mathematical background is deemed of prime importance for prospective officers in various branches of service in the Army and Navy.

E. J. M.

EDITORIAL NOTE

This issue of the MONTHLY contains an important supplement on *Industrial Mathematics* and a timely committee report *On Education for Service*, which amplify the foregoing statements with regard to applications of mathematics to industry and to military service.

ON EDUCATION FOR SERVICE*

WILLIAM L. HART, University of Minnesota

1. Introduction. A report of activities and recommendations was recently presented to Professor Marston Morse, Chairman of the War Preparedness Committee, by its Subcommittee on Education for Service. The following extracts differ from the complete report through the omission of certain recommendations which at the moment are not sufficiently matured to be appropriate for publication. All curricular recommendations are included in the following paragraphs.

The active members of the Subcommittee, who subscribe unanimously to the report, are as follows: R. S. Burington, H. B. Curry, E. C. Goldsworthy, F. L. Griffin, W. L. Hart, M. H. Ingraham, E. J. Moulton.

2. Activities. Members of the Subcommittee and a few other cooperating mathematicians reviewed or inspected an extensive list of books and pamphlets of a mathematical nature which are employed as text material in service schools of the Army and Navy and in the Civil Aeronautics Program. Personal contacts were established by the Subcommittee with various regular officers of the Army and Navy, teachers in the Civil Aeronautics Program, representatives of the field of secondary mathematics, and certain individuals of experience and reputation in the field of vocational education. In spite of these efforts to obtain an objective foundation for opinions and recommendations, it must be definitely admitted that the need for speedy action, as well as the intangible nature of certain features of the situation, did not permit the Subcommittee to adopt an entirely objective approach. In evaluating evidence and formulating opinions, the Subcommittee was aided by the fact that certain of its members had had military experience during the First World War.

3. Statement of general view-points. Mathematicians who are interested in the contacts between their field and emergency problems of national defense and industry should guard themselves against attaching too much importance to the most advanced mathematical aspects of the situation and also too little significance to the elementary or intermediate mathematics which is of use in many directions. It should be realized that, in our nation, which in the past has always been geared to a peace-time economy, with only brief intermissions when military affairs were reckoned of importance, there is likely to be a large element of surprise in the public reaction to information that military science, in most of its important branches, is mathematical in nature. However, in such a statement we do not imply that the mathematics involved is necessarily of *advanced* type. Also, we must be alert to recognize the presence of methods or theory in military and naval science which involve a mathematical *background* for intelligent appreciation, even though, superficially, no mathematical techniques are em-

* Progress report by the Subcommittee on Education for Service of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America.

ployed. A similar statement can be made concerning an evaluation of the mathematical needs of industrial workers, below the level of engineers.

In arriving at an estimate of the mathematical background which is *desirable* for workers in government and industry, and for officers and enlisted men in the Army and Navy, we recognize the validity of the following pedagogical view-point. *In order that an individual may be able to use effectively any particular body of technique, his school training should extend a reasonable distance beyond the level of difficulty at which he will apply the technique.* Thus, if we wish to prepare a student so that, later, perhaps after some review, he can use *elementary algebra*, he should be exposed to *advanced algebra*, or to some other mathematical subject with elementary algebra as a prerequisite. This pedagogical view-point is at variance with emergency actions which would attempt to give men the bare minima of mathematical techniques necessary for a *formal* approach to their applications. An emergency justifies *any* remedial action, but our efforts should be directed toward making it unnecessary to use hazy emergency shortcuts to mathematical procedures. With our widespread democratic system of secondary and collegiate education, our nation is justified in demanding that we should always have on hand a relative *surplus* of people with mathematical training through substantial secondary mathematics and also a surplus with elementary college training in the subject.

At this point it is pertinent to remark that, in the remainder of this report, any apparent omission relating to content or training at the graduate level is due to our decision that the omitted matter falls more properly in the spheres of other subcommittees of the War Preparedness Committee.

4. Recommendations concerning mathematics for those engaged in non-military activities. The importance of military aspects of the present national emergency should not cause us to lose sight of the equally important mathematical features of the normal and emergency activities of government, the various learned professions, and industry. We observe an enormous expansion in the aircraft and other munitions industries, a continuous drain on the national supply of skilled workers due to Army and Navy calls for enlisted specialists, and the extensive statistical and accounting work associated with our national economy. It is our opinion that these features of the present situation, as well as general underlying trends independent of the emergency, create a need for an increased supply of young men and women with training in mathematics through various levels beyond the junior high school grades. We believe that skilled workers in mechanical industry should have, in their backgrounds, substantial secondary mathematics through the stage of computational trigonometry, and at least an intuitional and *sketching* acquaintance with the fundamental notions of solid geometry. Also, we recommend that increased numbers of men and women should be trained at least through substantial secondary mathematics, to create a reservoir of suitable candidates for positions demanding mathematical skill and for the professions where advanced mathematical

knowledge is of advantage. In particular, it would be desirable to have numerous women trained through the stage of elementary mathematical statistics, for the use of government, the professions, and industry. We believe that these recommendations for mathematical training, in so far as they relate to secondary or elementary college mathematics, would not be harmful in connection with other educational objectives dissociated from the field of mathematics.

5. Evaluation of the mathematical needs of the Army and Navy. If we ask what the Army and Navy would desire as mathematical training for officers under *ideal* conditions, we obtain a sufficient answer by observing the intensely mathematical and technical nature of the curricula in the academies at West Point and Annapolis. The desirable level of training could be maintained during the present expansion of the Army and Navy only if all the officers, particularly in certain branches, were required to be engineers. We may assume that this ideal obviously is impossible of attainment, if we admit the truth of many authoritative statements that a shortage of engineers exists at present even for the needs of industry alone, apart from the requirements of the Army and Navy. Our evaluation of the mathematical needs of the Army and Navy will be made in the light of the emergency situation, with a subjective view as to the levels of training which are possible of attainment if full advantage is taken of our extensive educational system. Moreover, we shall give no consideration to mathematical features of the curricula at West Point and Annapolis, because the graduates of those schools obviously have fine training for their activities. We are interested in the mathematical backgrounds of all others, officers or enlisted men, outside of the commissioned personnel of the Regular Army and Navy, who are in or who will enter our Army and Navy. We shall omit mention of various branches of the Army whose activities obviously are non-mathematical. In summary, we believe that the following specifications of mathematical training for *officers* give *minimum* levels, if our Army and Navy are to be intelligently led. The training specified for various enlisted men may exceed the true minimum levels but probably are the *desirable* levels, and we believe that they can be attained. However, these training goals for both officers and enlisted men will *not* be attained unless special efforts are made by the high schools, colleges, civilian centers of adult education, existing service schools of the Army and Navy, and directors of education in the Army and Navy outside their existing schools.

5.1. Infantry. Even this supposedly non-technical branch of the Army places demands on mathematics. All enlisted men find use for arithmetic and intuitional geometry and would be benefited by the content presented in modern courses in mathematics for grades eight and nine. The officers, non-commissioned officers, and privates first-class should have familiarity with elementary geometry to permit map reading and construction, appreciation of contour designations, and the use of coördinate systems. They should be able to study intelligently the mechanical drawings associated with the rifles, light anti-air-

craft guns, motorized equipment, and other materiel assigned to them. In brief, these officers and the upper groups among the enlisted men should have as substantial a background in mathematics as we consider desirable for skilled workers in mechanical industry. In addition, the officers would find it useful to have an acquaintance with the notions of probability and probable error as met in elementary statistics so as to appreciate the theory of gunfire as applied to fire by the infantry and either opposing or supporting fire by artillery.

5.2. Coast Artillery Corps. This exceedingly mathematical branch controls all artillery for seacoast defense, high altitude anti-aircraft artillery, and mobile artillery of heavy caliber. The officers of this corps have to perform the duties of surveyors on some occasions, and they deal with very complex optical instruments, motorized equipment, and complicated guns. These men should have very strong training in mathematics; in fact, we hestitate to specify training short of that possessed by graduates of an engineering college. But, as a minimum, these officers should have passed through advanced high school algebra, computational plane trigonometry, enough spherical trigonometry for its typical applications in surveying, and the elements of solid geometry. Also, they should have an acquaintance with the notions of probability and probable error for appreciation of the theory of gunfire. In addition, a substantial number of the enlisted men should be as well qualified mathematically as the officers, so as to provide intelligent personnel for technical groups and to permit the training of enlisted understudies for all the officers as insurance against the effects of casualties. It would be desirable if practically all the enlisted men had the mathematical background which we consider suitable for skilled workers in mechanical industry.

5.3. Field Artillery. We make the same minimum stipulations for mathematical training in this branch as in the Coast Artillery Corps, with omission of mention of spherical trigonometry for the officers, and with less insistence on the need for mathematical backgrounds in the case of the enlisted men.

5.4. Signal Corps. The officers should be electrical engineers and the enlisted men should have the mathematical training suitable for skilled workers in mechanical industry.

5.5. Ordnance Department. It needs various specialists, among both the officers and the enlisted men, with highly mathematical backgrounds such as possessed by engineering graduates or college majors in mathematics and physics.

5.6. Ground Force of the Air Corps, in the Army and Navy. The ground service requires many engineers, men with extensive college mathematics and physics especially for the meteorological section, and a large number of enlisted men and officers with the mathematical backgrounds suitable for skilled mechanical industry.

5.7. Pilots or navigator-gunners in the Air Corps of the Army and Navy. They should be acquainted with plane trigonometry, the elements of solid geometry, and an introduction to spherical trigonometry. They need this content for

the study of navigation, elementary aerodynamics, bombing, meteorology, and various other technical subjects. The importance of space concepts and physical reasoning in three dimensions in such subjects adds to the importance of solid geometry for this group of men. Very substantial manipulative algebra is needed by them and we presume that this is included as a prerequisite for the trigonometry course. Moreover, they should have acquaintance with the notions of probability and probable error to aid in the appreciation of certain aspects of the theory of bomb dropping.

5.8. Officers of the Navy, outside of its Air Corps, and officers of the Merchant Marine. We hesitate, again, to specify training short of a college degree in some field of engineering. As an emergency minimum, however, we recommend that these men should have the mathematical background specified for pilots in the Air Corps, with additional emphasis on algebra and spherical trigonometry. We include reference to officers of the Merchant Marine in our remarks because of its general importance and also its complementary relationship to the Navy in time of war.

5.9. Enlisted men in the Navy. They should have the mathematical background desirable for skilled workers in mechanical industry. In addition, a substantial number of these enlisted men should be as well qualified mathematically as the officers, to provide intelligent personnel to serve in technical groups and as understudies for officers.

6. Conclusions drawn from results of the program of reviews of books of a mathematical nature used by the Army, Navy, and Civil Aeronautics Authority.

6.1. By and large, the mathematical exposition in these books is satisfactory, particularly when we take account of the fact that they are intended for readers with a minimum technical background. The reviews do not justify us in calling for prompt revisions of any of the existing text material, although various criticisms of present expositions could be made.

6.2. We believe that any one of these books can be well appreciated by a reader who has a proper mathematical background, in accordance with our preceding recommendations, and a suitable teacher. This opinion permits the Subcommittee to place its main emphasis, *first*, on the preceding recommendations for mathematical backgrounds, and, *second*, on an analysis of the effects of these recommendations on plans for instruction in the civilian and military educational system.

7. Recommendations concerning the field of secondary mathematics.

7.1. In the secondary field, it would be very undiplomatic and harmful if the national emergency were taken as an excuse for a violent attack on certain curricular trends, even though weaknesses of some of these trends may become apparent when they are analyzed under the search-light of present national necessities. We consider it best to state mathematical objectives without stipulating the pedagogical details to be involved in their attainment.

7.2. The National Council of Teachers of Mathematics and all organized

bodies of teachers of mathematics at the secondary level should advertise the utility of mathematics in industry, government, the professions, and military science.

7.3. In high schools it should be advertised that the Navy R.O.T.C. and the Coast Artillery groups of the Army R.O.T.C. in collegés require trigonometry as a prerequisite and that they should require the elements of solid geometry and spherical trigonometry. Also, it should be emphasized that these subjects can be studied efficiently in high school.

7.4. In the junior and senior high schools, each boy and girl *of sufficient mathematical aptitude* should be urged by his advisers to observe that the study of mathematics through the stage of trigonometry and some solid geometry may serve as a distinctly patriotic action.

7.5. We recommend that, in connection with emphasis on so-called *socialized* aspects of secondary curricula, a liberalized definition of *socialized mathematics* should be adopted for students at all ability levels, in contrast to more narrow definitions which give unique prominence to business applications and consumer interests. In the liberalized definition we would emphasize that content with military, scientific, professional, and industrial uses is of a most socialized nature. Also, from the standpoint of a high school student of intelligence, *classical* mathematical content may be very "*socialized*," in a true sense, even though the content possesses only *delayed* utility, as contrasted with *immediate* utility in the student's experience.

7.6. The military, industrial, and scientific utility of a considerable quantity of space intuitions and at least a little spherical trigonometry, causes us to recommend that the high school work in solid geometry, both intuitional and demonstrative, be given more prominence than in recent years. The classical course in solid geometry might be modified by replacing some of its content with a treatment of the elements of spherical trigonometry. Or, the intuitional and demonstrative plane geometry presented in grades eight through ten might be modified to include sufficient material from solid geometry.

7.7. We *strongly* recommend that a *single* set of courses be used in any high school for students of *appropriate ability* in attaining desired ends relating to industry, military service, or future collegiate education. We recommend this single treatment rather than separate curricula, some designed to fit men for industry or military service and some planned for those who will delve more deeply into mathematics and related fields in college. In the case of superior students, substantial mathematics, fitted to their intelligence, is likely to serve them better, whenever they will use mathematical content, than specifically pointed vocational mathematics or military mathematics. Thus, we argue against a curricular division in mathematics among secondary students which would be based on their present economic status, their *momentary* expectations about attending or not attending college, or *transient* vocational preferences, and we advise instead a curricular division based on the *intelligence* of the students.

7.8. As a temporary measure, we recommend that boys of intelligence who now are in grades 11 and 12 and who have previously omitted substantial mathematics, should be offered an abbreviated treatment of logarithms, plane trigonometry, intuitional solid geometry, and perhaps an introduction to spherical trigonometry, to train them for their practically certain entrance into skilled industry, the Army, or the Navy.

7.9. We advise the evening schools in cities to give new emphasis to courses in advanced high school mathematics through the stage of trigonometry.

7.'10. We advance the opinion that a shortage of engineers and physicists is at hand. This should be brought to the attention of boys of mathematical ability in the high schools; if these boys extend their exposure to secondary mathematics but later fail to become engineers or physicists, their mathematical training will have sufficient general utility to justify our recommendation.

7.'11. At all stages of secondary mathematics we recommend emphasis on applications. However, the teacher and student should not anticipate that all these applications, or even a majority of them, will be of intrinsically natural types. The pedagogical aim in this connection should be to convince the student that mathematics has not only important cultural and theoretical sides but also is intensely *useful* in our civilization; the applications, artificial or real, should give the student experience and confidence in applying general mathematical techniques as auxiliaries in related fields.

8. Curricular recommendations at the college level.

8.1. We wish to re-emphasize an early recommendation of the War Preparedness Committee by suggesting that as many college teachers of mathematics as possible should carry out measures of self-instruction in one or more of the following fields: Hydrodynamics, Aerodynamics, Meteorology, Probability and Statistics, Computation, Industrial Applications of Mathematics, Exterior Ballistics, Navigation, Artillery Fire Control and Orientation, Cryptanalysis. An enlarged background in these fields would be useful to the teacher of college mathematics at any time and, at this moment, would aid him in introducing problems of emergency interest into his routine courses. Also, such self-instruction would prepare the teachers for emergency use in directions where their talents would be of advantage. Those interested in such self-instruction should consult the bibliography published in the *Report of the War Preparedness Committee*, Bulletin of the American Mathematical Society, vol. 46, 1940, page 713.

8.2. We consider it justifiable for the Department of Mathematics in any college to introduce undergraduate courses in hydrodynamics and aerodynamics, and courses in meteorology in conjunction with the Physics Department, in case such courses are not being given by other departments.

8.3. We recommend a course in exterior ballistics, with a prerequisite of at least elementary calculus and a first course in differential equations, for consideration in the organization of work in applied mathematics.

8.4. We suggest that, in the senior college years, a combination field of

major concentration in mathematics and its applications be provided by the Department of Mathematics and the Departments of the Physical Sciences, perhaps in collaboration with the Departments of Engineering if they are present in the college. Such an undergraduate major might involve some sacrifice of pure mathematical content as compared with a narrow major in mathematics. If the curriculum were organized on a five-year basis, through cooperation between Mathematics, Physics, Chemistry, and Engineering, the curriculum would offer training which could be very useful in industry.

8.5. We wish to caution Departments of Mathematics to avoid indiscriminate introduction of elementary courses in *war* mathematics. In particular, we do not consider it desirable to suggest any special course pointed at Army or Navy service which does not have at least plane trigonometry and substantial manipulative algebra as a prerequisite. This opinion is due to our conviction that the classical material just mentioned is more valuable than a preliminary exposure, with a weak foundation, to military and naval applications in advance of their later study while in service in the Army or Navy. Moreover, we observe with satisfaction that the Army and Navy, in certain directions, are giving explicit credit in priority ratings to men with credit in *classical* courses in trigonometry and advanced algebra, and also in more advanced mathematics. Hence, for any given group of students, we recommend that a Department of Mathematics should carefully investigate the relative advantages of *more classical mathematics* as compared to any *emergency* course in war mathematics before it is introduced.

8.6. For certain special groups of students, we recommend consideration of the following emergency courses, with plane trigonometry and college algebra as prerequisites:

8.61. For students who are fairly sure that they will enter the Air Corps, the Navy, or the Merchant Marine: a course in navigation, including necessary spherical trigonometry and solid geometry.

8.62. For students who are fairly sure to enter the Air Corps, the Navy, or the Coast Artillery Corps of the Army: a brief course, to be offered two hours per week for three months, in the elements of solid geometry and spherical trigonometry.

8.63. For students who plan to enter the Army or the Navy in any capacity and who wish a review of probably useful material with moderate additions: a semester course, to be offered three or four hours per week, which will present, *first*, a review of geometry, necessary algebra, logarithms, and trigonometry, with emphasis on its numerical aspects, and, *second*, new material relating to solid geometry, spherical trigonometry, and probability as involved in the theory of gunfire and bombing. We recommend this semi-review course for use in college extension curricula and in evening schools, as well as in daytime college work.

8.7. We suggest only the following moderate modifications in standard courses:

8.71. Considerably increased emphasis on applications and computational techniques; in this connection the teacher could bring to bear his extended background as advised in Section 8.1.

8.72. In trigonometry, the inclusion of a treatment of mil measurement (the Army system of angular measure), emphasis on vector language and applications, and, for certain students, an expansion of the course to include some spherical trigonometry in colleges where that subject is not usually taught.

8.73. In college algebra, expansion of the work in probability to include introduction of the normal probability curve, the general notion of a frequency curve and its probability significance, and use of the language of probable error.

8.8. We recommend that women who are studying college mathematics be given a course in mathematical statistics with at least freshman mathematics as a prerequisite, regardless of the individual educational objectives of these students. In addition to broadening their backgrounds for their peace-time vocations, this work in statistics might prove useful to the women in search for employment in present emergency activities.

8.9. We suggest that each Department of Mathematics canvass the situation of young men and women in its college who have mathematical talent, even though they may not be taking courses in college mathematics. The men should be made acquainted with existing opportunities for preference in the Army and Navy as the result of training in mathematics. The women should be informed of those semi-mathematical fields, for instance, business statistics, accounting, and drafting, where a continued drain on available man power may create openings for women.

8.'10. Regardless of justified attitudes which cause colleges to avoid teaching elementary parts of secondary mathematics, we recommend that each Department of Mathematics should do everything in its power to aid interested college men, or high school graduates not in college, in learning secondary mathematics which will be of use to them in the Army or Navy. Such instruction could be offered through extension or correspondence courses, as well as in daytime classes, and should be arranged so as to avoid duplication with efforts of neighboring high schools.

8.'11. If a Department of Mathematics has taken all appropriate actions in accordance with the preceding Sections 8.2 through 8.'10, with particular attention paid to the cautions of Section 8.5, and in addition desires to provide a general elementary course in war mathematics, the following outline by Professor Griffin may be useful. This outline gives a rounded view of typical applications of mathematics in military and naval science which should interest men and women alike and which would have specific utility for men who will enter the Army or Navy. The content could be taught well only by one who has an extensive background gained either through actual service in the Army or Navy, or by study as suggested in Section 8.1.

WAR MATHEMATICS

(Three hours per week for one year; suggested by F. L. Griffin)

PURE MATHEMATICAL TOPICS

Chapter I. Preliminary Ideas.
II. Trigonometric Functions.
III. Logarithmic Calculations.
IV. Coördinates and Notions from Analytic Geometry.

ARTILLERY AND MACHINE GUN PROBLEMS

Chapter V. Position Calculations. Methods of locating a fixed target: by direct observation; indirect observation involving trigonometry; map location; sound methods.
VI. Ballistic Calculations. Initial firing data; adjustment of fire; probable errors; bracketing; effect of fire; velocity and angle of impact; penetration.
VII. Safety Zones and Dead Areas.
VIII. Barrage Fire.
IX. Theoretical Ballistics. Discussion of the construction of firing tables.

ARMY ENGINEERING PROBLEMS

Chapter X. Graphical Methods; Rates; Maxima; Work; Momentum.
XI. Statics; Bridge Structures; Cranes; Inclined Planes.
XII. Flexure of Beams; Suspension Cables.

AVIATION PROBLEMS

Chapter XIII. Principles of Flight. Stability; Equilibrium.
XIV. Bombing.
XV. Spherical Trigonometry and Navigation.

NOTE. For further information concerning the preceding course, consult Professor F. L. Griffin, Reed College, Portland, Oregon.

MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-nine persons have been elected to membership on applications duly certified:

H. J. BARTEN. Precision machinist, Naval Gun Factory Optical Shop, Washington, D. C.

R. E. BASYE, Ph.D.(Texas) Acting Instr., A. and M. Coll. of Texas, College Station, Texas.

Sister M. MIRABELLA BOEHMER, M.S.(Catholic Univ.) Instr., Alverno Teachers Coll., Milwaukee, Wis.

D. H. BROWNE. Buffalo, N. Y.

H. E. CALCAGNO. Montevideo, Uruguay.

J. V. COOKE, Ph.D.(Peabody) Asst. Prof., North Texas State Teachers Coll., Denton, Texas.

P. C. COX, A.M.(New Mexico) Instr., Alabama Poly. Inst., Auburn, Ala.

JOHN DE CICCO, Ph.D.(Columbia) Instr., Illinois Inst. of Tech., Chicago, Ill.

E. E. ELY. Supt. of Bridges, State Highway Dept., Astoria, Ore.

F. A. FICKEN, Ph.D.(Princeton) Instr., Cornell Univ., Ithaca, N. Y.

RUTH FIKE, A.M.(Duke) Instr., Florida Southern Coll., Lakeland, Fla.

J. V. FINCH, A.M.(Wisconsin) Grad. student, Univ. of Wisconsin, Madison, Wis.

CLARENCE FORD, A.M.(Kentucky) Teacher, Louisville High School, Louisville, Ky.

L. M. GARRISON, A.M.(Missouri), Ed.M.(Peabody) Head of Dept., Acting Dean, Snead Jr. Coll., Boaz, Ala.

GEORGE GROSSMAN, A.M.(Columbia) Teacher, Manhattan High School of Aviation Trades, New York, N. Y.

Sister MARY CHARLOTTE HOLLAND, A.M.(Catholic Univ.) Registrar, Instr., St. Xavier Coll., Chicago, Ill.

H. K. HUMPHREY, M.S.E.E.(Union) Chm. of Board, Winnetka Trust and Savings Bank, Winnetka, Ill.

MORRIS KLINE, Ph.D.(New York Univ.) Instr., New York Univ., New York, N. Y.

ANNE S. MCCARTHY, A.M.(Boston Coll.) Instr., Mount St. Mary Coll., Hooksett, N. H.

W. A. MCLAUGHLIN, Pet.E.(Colo. School of Mines) Asst. Prof., Math. and Physics, Eastern New Mexico Coll., Portales, N. M.

A. G. MAKAROV, A.M.(Pennsylvania) Instr., Univ. of Delaware, Newark, Del.

C. G. MAPLE, A.M.(Cincinnati) Instr., New River State Coll., Montgomery, W. Va.

D. C. MOORE, A.M.(Emory) Instr., Reinhardt Jr. Coll., Waleska, Ga.

W. H. MYERS, Ph.D.(Stanford) Asst. Prof., San José State Coll., San José, Calif.

J. S. PETERSEN, JR., M.S.(St. Louis Univ.) Dir. of Dept., Math. and Physics, Ursuline Coll., New Orleans, La.

P. V. REICHELDERFER, Ph.D.(Ohio State) Instr., Stanford Univ., Stanford University, Calif.

F. E. RILEY, B.S. in Educ.(Arizona) Asst., Univ. of Arizona, Tucson, Ariz.

ROSE ROLL, A.M.(Columbia) Chm. of Dept., Washington Irving High School, New York, N. Y.

LUCILE L. ROREX, A.M.(Brown) Head of Dept., Campbell Coll., Buies Creek, N. C.

E. D. SCHELL, A.B.(Hiram) Jr. Economist, Cost of Living Div., U. S. Bureau of Labor Stat., Washington, D. C.

H. M. SCHWARTZ, Ph.D.(Pennsylvania) Instr., Montana State Coll., Bozeman, Mont.

W. M. SCOTT, Ph.D.(Michigan) Instr., Univ. of Alabama, University, Ala.

Sister MARY LUCINA SKELLY, Ph.D.(Georgetown) Prof., Georgetown Visitation Jr. Coll., Washington, D. C.

A. E. TAYLOR, Ph.D.(California Inst. of Tech.) Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

W. C. TAYLOR, Ph.D.(Wisconsin) Instr., Univ. of Cincinnati, Cincinnati, Ohio.

C. W. TOPP, A.M.(Illinois) Instr., Fenn Coll., Cleveland, Ohio.

V. J. VARINEAU, Ph.D.(Wisconsin) Instr., Univ. of Wyoming, Laramie, Wyo.

WILLIAM WALLIS, A.B.(Texas Tech.) Grad. student, Texas Tech. Coll., Lubbock, Texas.

C. H. WOLFE, JR., A.B.(Michigan) Teacher, Lakeside High School, Lakeside, Ohio.

W. D. CAIRNS, *Secretary-Treasurer*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-first regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Redlands, Redlands, California, on Saturday, March 8, 1941. Professor O. W. Albert, chairman of the Section, presided.

The attendance was seventy, including the following thirty-four members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, H. M. Bacon, Harry Bateman, Clifford Bell, L. T. Black, G. S. Cook, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, J. R. Gorman, H. J. Hamilton, Frances L. C. Hinds, G. H. Hunt, C. G. Jaeger, G. R. Livingston, Ada A. McClellan, G. F. McEwen, I. P. Maizlish, W. E. Mason, A. D. Michal, P. M. Niersbach, W. T. Puckett, Jr., Lena E. Reynolds, G. E. F. Sherwood, C. E. Smith, D. V. Steed, A. E. Taylor, B. P. Taylor, H. C. Van Buskirk, Mabel G. Whiting, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, L. J. Adams, Santa Monica Junior College; Vice-Chairman, Morgan Ward, California Institute of Technology; Program Committee, D. C. Duncan, Chairman, C. K. Alexander, and the Secretary. The next meeting was tentatively scheduled to be held on March 14, 1942, at Occidental College.

The following six papers were read:

1. "The place of mathematics in the curriculum" by Professor G. R. Livingston, San Diego State College.

2. "Derivatives and the study of curves" by Professor A. E. Taylor, University of California at Los Angeles.

3. "Mathematics in the Naval Reserve Training School" by Captain R. M. Fawell, University of Southern California, introduced by Professor Steed.

4. "Methods for solutions of secular equations" by Professor J. H. Wayland, University of Redlands, introduced by Professor Albert.

5. "Points of contact between mathematics and the civilian pilot training program" by Professor W. T. Whitney, Pomona College, introduced by Dr. Hamilton.

6. "The resistance of ships" by Professor Harry Bateman, California Institute of Technology.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Livingston emphasized the importance of mathematics in general education. Mention was made of the wide use of mathematics as a tool and also as a way of thinking. Typical proposals for cutting down the mathematics offered in high school were answered briefly. A greater appreciation of the values of mathematics on the part of teachers of other subjects was mentioned as highly important, if high school students are to receive proper guidance in this matter.

2. Professor Taylor discussed the study of arcs of the type $y=f(x)$ by means

of the derivatives of f. Principal attention was centered on the behavior of the first derivative. The little-used theorem of Darboux, to the effect that if f' exists on the closed interval (a, b) it assumes all values between $f'(a)$ and $f'(b)$, was proved. It was also shown that f' is continuous except at those points where $f'(x)$ fails to approach a limit from at least one side. An elementary discussion of convex functions was given. From this basis the topics of sense of concavity and points of inflection were introduced, and a few tests indicated.

3. Captain Fawell, U. S. Navy (retired), spoke on the organization and administration of the Naval Reserve Officers Training Corps and on the importance of mathematics to prospective members of that Corps. He stressed the importance of a knowledge of logarithms and trigonometry for such prospective members.

4. A critical study of the various methods for solving equations of the form $|A-I\lambda|=0$ shows that in general the direct methods of solution, as developed by Aitken and Duncan and Collar, are not as rapid as expansion into polynomial form and subsequent solution of the polynomial equation. Professor Wayland gave a comparison of the general expressions for the number of operations required for each of the important methods of expansion and showed that the method of Danielewsky is the most economical of effort. The author has routinized this method for use with a calculating machine, and has added a check. He also discussed methods for handling the general determinantal equation $|A+B\lambda+\cdots+E\lambda^n|=0$.

5. Professor Whitney described the scope of the primary and secondary ground school and flight programs as given at Pomona College. Two years of college work is required, but no scientific prerequisites are specified. This results in a somewhat non-homogeneous group of students, for many of whom additional study is required to develop a sufficient background, particularly for the secondary (or advanced) course. The bulletins provided by the C.A.A. as textbooks present their subject with a minimum of mathematical analysis and formal development. Graphical and tabular presentations take the place of mathematical formulas wherever possible.

In the field of aero-dynamics, however, both the fundamentals of physics and mathematics are so involved that the mathematical treatment of these problems becomes more necessary, and we find the basic theorem of Bernoulli developed and applied to the specific conditions of flight.

Similarly, in the field of meteorology, the discussion of the circulation of the earth's atmosphere introduces the Coriolis force resulting from the earth's rotation, while numerous diagrams and graphs carry the burden of presentation of a subject whose analysis is daily becoming more complex.

In the study of air navigation (the determination of the pilot's position), the graphical solution by velocity vector triangles is first mastered by the student, but is later replaced by the use of graphs and computing instruments. Likewise, in the study of celestial navigation, advantage is taken of the recently devised navigation tables and simplified almanac, so that a pilot can determine his posi-

tion without necessarily having acquired any adequate knowledge of the theoretical bases involved.

6. A dimensionless coefficient which enters into the expression for the drag, drift, or lift of a ship is a function of certain physical and geometrical numbers which cannot all be made the same for model and full-sized ship. When there is no helpful theory, one aim of tank research is to find critical values of some of these numbers so that for values close to these the dimensionless quantity is a slowly varying function, while another aim is to find cases in which the quantity varies almost linearly so that the effect of small changes can be predicted. The best value of the block coefficient is known to some extent, and the worst value of the ratio of draft to depth of water may also be regarded as known. The mathematical interpretation of these results by the theory of wave resistance is beset with difficulties but some progress has been made, the work of T. H. Havelock being mathematically the most interesting on account of the occurrence of confluent hypergeometric functions in the analysis. This analysis was amplified to some extent by Professor Bateman and indications were given of some problems that require fuller investigation. These include cases of variable motion in both smooth and rough sea when virtual mass and moments of inertia need to be calculated and the physical and geometrical numbers are augmented by others depending on acceleration, angle of helm, velocity, and height of the waves. The coefficients of the rolling, pitching, and yawing couples also need to be calculated for ships under way, and we need to determine the effects of planing or V-shaped bottoms in altering draft and stability.

P. H. DAUS, *Secretary*

THE FEBRUARY MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association in Tulsa on Friday morning, February 7, 1941. Professor A. H. Diamond, chairman of the Section, presided.

Fifty-four representatives of high schools and colleges attended the meeting, including the following sixteen members of the Association: Joseph Barnett, Jr., J. C. Brixey, N. A. Court, A. H. Diamond, W. V. N. Garretson, H. L. Hall, O. H. Hamilton, J. O. Hassler, Dora McFarland, W. C. Randels, W. T. Short, H. W. Smith, C. E. Springer, E. B. Wedel, B. S. Whitney, J. H. Zant.

At the business session the following officers were elected: Chairman, E. B. Wedel, Holdenville Junior College; Vice-Chairman, C. E. Springer, University of Oklahoma; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following papers:

1. "On exterior ballistics" by Professor C. E. Springer, University of Oklahoma.

2. "Vibrating plates" by Professor H. W. Smith, Oklahoma A. and M. College.

3. "A problem connected with meteor velocities" by B. S. Whitney, University of Oklahoma.

4. "A symposium on the teaching of general mathematics" by Professor J. H. Zant, Oklahoma A. and M. College, and Professor C. E. Abraham, Panhandle A. and M. College, introduced by Professor Zant.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Springer gave an exposition of the method of numerical integration in the computation of trajectories, and discussed some aspects of the theory of differential corrections.

2. Professor Smith in his expository paper reviewed the classical work of Poisson and Kirchoff in obtaining the necessary partial differential equation and boundary conditions. The solution of the equation for the circular plate and its nodal system were discussed.

3. Mr. Whitney discussed the problem: Determine the locus of the symmetric of a fixed line through the vertex of a right circular cone with respect to an element of the cone. The problem arose from the consideration of the use of a rocking mirror for determining meteor velocities. The locus is a cone of the fourth degree.

4. Professors Zant and Abraham discussed the problem of organization of material suitable for junior college general mathematics, and the need for such courses.

J. C. BRIXEY, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The eighteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 15, 1941. Professor K. W. Folley, chairman of the Section, presided at all sessions.

Ninety persons attended the morning session and more than a hundred persons attended the afternoon session. Among the attendants were the following forty-two members of the Association: N. H. Anning, W. L. Ayres, J. W. Baldwin, E. F. Beckenbach, F. A. Beeler, G. D. Birkhoff, W. M. Borgman, Jr., J. W. Bradshaw, I. W. Burr, R. V. Churchill, A. H. Copeland, Max Coral, C. C. Craig, Wayne Dancer, P. S. Dwyer, J. P. Everett, Peter Field, C. H. Fischer, K. W. Folley, H. H. Goldstine, V. G. Grove, T. H. Hildebrandt, J. D. Hill, L. A. Hopkins, E. E. Ingalls, L. G. Johnson, L. S. Johnston, Wilfred Kaplan, Theodore Lindquist, E. D. McCarthy, D. C. Morrow, A. L. Nelson, G. Y. Rainich, E. D. Rainville, Arthur Rosenthal, L. J. Rouse, T. R. Running, E. R. Sleight, B. M. Stewart, G. B. Van Schaack, T. O. Walton, E. T. Welmers.

The following officers were elected for the coming year: Chairman, T. R. Running, University of Michigan; Secretary-Treasurer, R. V. Churchill, Uni-

versity of Michigan. A resolution was passed commending the retiring Secretary for the manner in which he has administered the responsibilities of his office and extending to him a vote of appreciation for the time and energy which he has devoted to the Section. It was decided to have a fall meeting in 1941 at the University of Detroit and to have the annual meeting in 1942 in conjunction with the meetings of the Michigan Academy of Science. It was further decided to continue to encourage and publish the program of student papers, even though these papers are not now featured at the Section meetings.

The Section had as its guest Professor G. D. Birkhoff of Harvard University, who was the invited speaker at the afternoon session. A large number of the members of the Association also attended the lecture on "Aesthetic measure" given by Professor Birkhoff on Friday afternoon, March 14, by invitation of the Michigan Academy of Science.

The following papers were presented:

1. "On skew symmetry" by D. K. Kazarinoff, University of Michigan.

2. "Some theorems on sub-series" by Professor J. D. Hill, Michigan State College.

3. "*The Whetstone of Witte* by Robert Record" by Professor E. R. Sleight, Albion College.

4. "Miquel polygons" by Dr. B. M. Stewart, Michigan State College.

5. "Solution of total differential equations by means of successive approximations" by S. Kaner, Wayne University, introduced by the Chairman.

6. "A famous geometrical problem that originated in a false hunch" by Professor J. W. Bradshaw, University of Michigan.

7. Three minute talks.

8. "Uniform rectilinear drawing" by Professor G. D. Birkhoff, Harvard University.

Abstracts of the papers follow, in the order numbered above:

1. The method of skew symmetry was illustrated to prove that a necessary and sufficient condition for two planes to cut from a non-ruled central quadric two segments of the same volume V is that each plane should divide its conjugate semi-diameter in the same ratio k. Mr. Kazarinoff proved that the formula $V=\frac{1}{4}(k^3-3k+2)V_1$, where $V_1=\frac{4}{3}\pi abc$, is valid for the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ if $k\leqq 1$, and valid for the hyperboloid of two sheets $x^2/a^2-y^2/b^2-z^2/c^2=1$ if $k>1$. Reference was made to the treatise on the conoids and spheroids by Archimedes.

2. The arithmetical average of all sub-sums (including the void sum) of a given finite sum $s_n\equiv u_1+u_2+\cdots+u_n$ is observed to be $\frac{1}{2}s_n$. Professor Hill indicated that this suggests the problem of finding a mean value for the sums of all infinite sub-series of a given absolutely convergent infinite series $\sum u_k=s$. A solution is obtained by means of the function $\phi(\xi)$ defined on the interval $0\leqq\xi\leqq 1$ as follows: $\phi(0)\equiv 0$, and $\phi(\xi)\equiv\sum_{k=1}^{\infty}\alpha_k u_k$ for $0<\xi\leqq 1$, where $0.\alpha_1\alpha_2\alpha_3\cdots\alpha_k\cdots$ is the non-terminating binary representation of ξ. The set of all functional values $\phi(\xi)$ for $0<\xi\leqq 1$ coincides with the set of the sums of all infinite sub-series of

$\sum u_k$. By employing the functions $\phi_n(\xi)\equiv\sum_{k=1}^{\infty}\alpha_k u_k$, $(n=1, 2, 3, \cdots)$, which are seen to be step-functions, it is shown that the integral $\int_0^1\phi(\xi)d\xi$ exists in the sense of Riemann and has the value $\frac{1}{2}s$. This integral mean value furnishes a generalization of the above fact concerning finite sums. If $\sum u_k$ converges, but not absolutely, the sub-series $\sum\alpha_k u_k$ will in general diverge. More exactly, it is shown that for all values of ξ on $0<\xi\leqq 1$ except those in a set of the first category, the partial sums $\phi_n(\xi)$ satisfy the relations $\liminf_n \phi_n(\xi)=-\infty$ and $\limsup_n \phi_n(\xi)=+\infty$. A precisely analogous result is obtained when the fundamental interval $0<\xi\leqq 1$ is replaced by an abstract metric space D in which a point x is a strictly increasing infinite sequence of positive integers.

3. Professor Sleight gave a brief account of the life and works of Robert Record, with special emphasis on his second arithmetic, which he called *The Whetstone of Witte*. In this text the author discussed certain types of numbers such as diametral numbers, but the purpose of this second arithmetic is best stated in the author's own words in that "it contains extraction of roots, the cossike practice with the rules of equations and the works of surd numbers." Here is found for the first time the processes of extracting square and cube roots which are now in use; also, the equality sign and many other signs of operation which are now in use. The dialogue method is used throughout.

4. Dr. Stewart reported on a number of examples illustrating his paper *Cyclic properties of Miquel polygons*, published in the August-September, 1940, number of the MONTHLY.

5. Mr. Kaner established the existence of a solution of the total differential equation $dz=Adx+Bdy$ through a given initial point, using the Picard method of successive approximations. The hypotheses included the usual one of complete integrability.

6. Professor Bradshaw first discussed the problem commonly known as Malfatti's problem: Within a triangle, to construct three circles each of which shall be tangent to the other two and to two sides of the triangle. Malfatti in 1803 gave a solution of this problem, but it is not the problem which he originally proposed, namely: Given a right prism with a triangular base, of any material, such as marble, to cut from it three right circular cylinders of the same altitude as the prism, whose combined cross-section shall have a maximum area, or in other words, so that the waste shall be a minimum. Malfatti apparently thought that the two problems were equivalent, but they are not. Lob and Richmond in 1929 have called attention to the fact that in the case of the equilateral triangle, the inscribed circle and two little circles that can be squeezed into the corners have a combined area that exceeds that of the Malfatti circles. Professor Bradshaw considered these two sets of circles as limits of a sequence of three variable circles. He also treated two analogous problems in space, involving spheres and the regular tetrahedron. Models were used to illustrate the relations.

7. (a) Professor Anning found a determinant of the second order for the double of the area of a plane quadrilateral in a form which showed it to be equal to the vector product of the diagonals.

(b) Professor Running wrote the cubic in the form $y=(x^2-2ax+a^2+b^2)(x-k)=0$, and used the fact that $y'=b^2$ when $x=a$ and the relations involving the coefficients of the x^2 and the x^0 terms. Thus, in the problem $y=x^3+17x^2-46x+29=0$, we have at once, $3a^2+34a-46=b^2$; $2a+k=-17$; $(a^2+b^2)k=-29$. It is then possible to eliminate the k and the b with a resulting cubic in a. The roots of the original cubic are then $a\pm bi$, and k.

(c) Dr. Paxson showed that in a Banach space the pair of postulates $2(x+y)=2x+2y$, $-(x+y)=-x-y$, replaces and implies the postulate $\lambda(x+y)=\lambda x+\lambda y$, (all real λ).

8. Professor Birkhoff developed the general theory of rectilinear drawing in the plane on the assumption that the straight lines involved were uniform and indefinitely extended. As these lines were taken more and more numerous and of smaller and smaller breadth, a limiting problem arose which yielded an integral equation of the form

$$f(r, \theta) = \int_0^{2\pi} F(r \sin(\phi - \theta), \phi) d\phi.$$

Here $f(r, \theta)$ is the density (of the lead in a pencil drawing), while $F(s, \phi)$ measures the frequency of the lines at distance δ from the origin of polar coördinates (r, θ), making an angle ϕ with the initial line $\theta=0$. The material presented has been partly published in an article *On drawings composed of uniform straight lines* in Liouville's Journal de Mathématiques Pures et Appliquées, and the remainder is contained in a lecture on "Rectilinear drawing," about to appear in the *Rice Institute Pamphlet.*

P. S. DWYER, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-sixth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Thursday April 3, 1941, with an afternoon session, dinner, and evening session. Professor J. R. Musselman, chairman of the Section, presided at these sessions.

Seventy-one persons registered attendance, including the following forty-seven members of the Association: W. E. Anderson, Max Astrachan, H. M. Beatty, H. A. Bender, Henry Blumberg, M. G. Boyce, Louis Brand, J. B. Brandeberry, Foster Brooks, R. S. Burington, V. B. Caris, F. E. Carr, E. H. Clarke, Rufus Crane, Carl Denbow, T. M. Focke, B. C. Glover, R. C. Hildner, F. C. Jonah, E. M. Justin, H. W. Kuhn, A. C. Ladner, Lincoln La Paz, H. M. MacNeille, Florentina Mathias, C. C. Morris, Max Morris, J. R. Musselman, L. F. Ollmann, Jesse Pierce, H. S. Pollard, Tibor Radó, S. E. Rasor, Maxwell Reade, C. E. Rhodes, Hortense Rickard, R. F. Rinehart, S. A. Rowland, K. C. Schraut, S. A. Singer, E. R. Stabler, G. W. Starcher, H. E. Stelson, C. F. Thomas, J. H. Weaver, Fern Welker, C. O. Williamson.

The following officers were elected for the coming year: Chairman, Louis Brand, University of Cincinnati; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, Lincoln La Paz, Ohio State University; Member of Program Committee, R. C. Hildner, Mt. Union College. It is expected that the next meeting will be held on Thursday, April 2, 1942, at the Ohio State University.

The following nine papers were presented:

1. "Aerial photogrammetry" by Professor J. R. Musselman, Western Reserve University.

2. "The first twenty-five years of the Mathematical Association" by Professor C. C. Morris, Ohio State University.

3. "Aerodynamics and airplane performance" by Major Bradley Jones, University of Cincinnati, introduced by Professor Bender.

4. "Stress analysis in airplanes" by Professor H. W. Sibert, University of Cincinnati, introduced by Professor Bender.

5. "An exposition of ciphering systems and deciphering methods" by Dr. R. F. Rinehart, Case School of Applied Science.

6. "Finite permutational vector envelopes" by Fenton Stancliff, Cuyahoga Falls, Ohio, introduced by Professor Bender.

7. "The method of moment distribution employed for exact results" by Professor Louis Brand, University of Cincinnati.

8. "Adjoint systems" by Professor Lincoln La Paz, Ohio State University.

9. "Mathematical statements in the history of mathematics" by Professor Emeritus G. A. Miller, University of Illinois.

Owing to illness Professor Miller was unable to be present, but sent his paper to Professor Bender, chairman of the Program Committee.

Abstracts of these papers follow:

1. After a brief historical sketch of aerial photogrammetry, together with a description of a modern camera used for such purposes, Professor Musselman discussed the radial line method of constructing a map from vertical aerial photographs. The errors due to parallax, tip, and tilt, and their effect on the map were outlined. The type of map was a polyconic projection upon which was superimposed a rectangular grid. Contour lines may be added either by a plane table in the field or by utilizing a stereoscopic model seen in a machine like the Talley Stereocomparator. Canadian, English, and German machines were mentioned briefly.

2. Professor Morris expressed appreciation of what the Association and the MONTHLY have done in the field of classical mathematics, but voiced the belief that adequate attention has not been given to the needs of those who use mathematics as a tool. He cited the fact that the federal government has found it necessary to establish a graduate school for the purpose of providing mathematical training for men in governmental work. He claimed that the problems of the engineer, the tester of materials, the biologist, the economist, and the agriculturist have not been emphasized. It is his opinion that problems involving

mathematics as a tool are just as important as classical problems, and he expressed the hope that those who direct the activities of the Association through the second twenty-five-year period will pay more attention to this side of mathematics.

3. Major Jones showed and explained certain apparatus suitable for demonstrating to a class lift forces on an airfoil, drag forces on objects, Magnus effect, and autorotation. Then he deduced the fundamental formulas for airplane performance.

4. Professor Sibert demonstrated by models some of the unsolved mathematical problems involved in the stress analysis of metal airplanes, such as shear lag, elastic center, tension-field beams, and elastic instability phenomena, and pointed out the inadequacy of present theories concerning these problems.

5. Dr. Rinehart discussed the elementary fundamental types of ciphers, namely, simple substitution, double substitution, and transposition, and described the cryptanalytic methods for breaking down ciphers of each of these types. Cryptographic variations of these elemental themes were also explained.

6. Mr. Stancliff showed methods of representing any permutation geometrically and uniquely by points on a circle, and in other ways. When these points are permuted by power residues of primes having certain primitive roots and connected by directed lines, envelopes are produced whose nature has not in all cases been identified. Parabolas and circles appear, all with direction. Simple interpretations are found possible for such concepts in group theory as transitivity, powers, and multiplications. Curves of permutations of degree 400 were shown geometrically. Algorismic design, a system of practical design of interest to artists, is implicated.

7. Professor Brand showed how, when side-sway is neglected, the method of moment distribution for the analysis of indeterminate structures produces successive contributions to the terminal moments which form one or more geometric series. These series may be summed to give the terminal moments which satisfy the slope-deflection equations. This method is feasible in many of the simpler indeterminate structures, such as those treated in Bulletin No. 108 of the University of Illinois Engineering Experiment Station, and leads to formulas which show the dependence of the terminal moments upon the loads and the structure constants. This method may be adapted to cases in which side-sway must be taken into account. Continuous beams are often readily dealt with by the above method. In fact, a very simple proof of the Theorem of Three Moments may be given by applying the method of moment distribution to two successive spans of a continuous beam.

8. The theory of adjoint systems of differential expressions due to Frobenius has been the point of departure for most of the investigations of inverse variation problems of the Hirsch and Darboux type. In 1927, D. R. Davis, in his Chicago dissertation, solved the Darboux inverse problem in three-space for three special four-parameter families of curves (straight lines, semicircles, catenaries). In 1928, Professor La Paz, in his Chicago dissertation, showed that if

F and G are of class C^{iv}, a pair of differential equations $y''=F(x, y, z, y', z')$, $z''=G(x, y, z, y', z')$ cannot define a system of extremals unless the determinant of a certain square matrix M vanishes. Necessary and sufficient conditions for such a pair of equations to define an extremal system in the general case, that in which the rank of M is 2, were obtained as integrability conditions of a certain differential system, S_0. The more special cases in which the rank of M was less than 2 did not prove amenable to this line of attack. Recently, Jesse Douglas, restricting the functions F, G to be analytic, has attacked this inverse problem from a different direction and has given a complete solution of it in terms of the integrability conditions of a differential system S. The purpose of the present paper of Professor La Paz was to show that the system S is identical with the system S_0, and hence that, if we are willing to make the hypothesis of analyticity, a complete solution of the Darboux inverse problem in three-space is possible in terms of adjoint theory alone.

9. Professor Miller emphasized the fact that a multisensual historical statement does not necessarily imply that its author was ignorant of the exact situation, but should be noted in order to avoid misunderstanding on the part of the student. In mathematics, as well as in its history, serious difficulties to the beginner are frequently overlooked by those who are widely acquainted with the subjects concerned, because the language employed is often construed by the latter in the light of additional knowledge instead of being confined to the actual meaning of the language used. He illustrated this remark by noting various statements contained in some of the most widely used recent publications relating to the history of our subject. He deplored the fact that we still do not have a good modern mathematical dictionary in the English language, and the fact that recent mathematical publications do not always lay enough stress on accuracy in their historical statements.

RUFUS CRANE, *Secretary*

GENERATING FUNCTIONS IN THE THEORY OF STATISTICS

J. H. CURTISS, Cornell University

1. Introduction. This paper consists of an elementary expository treatment of the theory and applications of moment and semi-invariant generating functions in theoretical statistics. The aim in presenting the exposition is to indicate a way in which a prerequisite of only elementary calculus can be extensively and advantageously utilized in an up-to-date introductory course in probability and statistics. Features of the program for such a course developed here are that methods of proof are standardized, notable abbreviations are made possible in certain demonstrations, and convincing partial proofs are provided for some essential propositions which require advanced calculus for full proof. In addition to the material of the usual introductory course in differential and integral calculus, we shall use a few formulas such as Leibniz's rule for differentiating a product, and the fact that

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi};$$

but we shall not assume any knowledge of multiple integrals except possibly in the proof of Theorem 2.2 immediately below.

2. Definitions and fundamental relations. To shorten the preliminary portion of our discussion as much as possible we shall use the language of probability theory.* Let X be a chance variable in space of one dimension and let $E[f(X)]$ denote the expected or mean value of the function $f(X)$. We shall denote the probability that $X=k$ by $P(k)$, and denote the probability that $k_1<X<k_2$ by $P(k_1<X<k_2)$. The following theorems will be needed:

THEOREM 2.1. *If c is any constant, then* $E[cf(X)]=cE[f(X)]$.

THEOREM 2.2. *Let $f_1(X_1)$, $f_2(X_2)$, $\cdots$, $f_N(X_N)$ be functions of the N independent chance variables X_1, X_2, $\cdots$, X_N, and let $W=\prod_{j=1}^N f(X_j)$. Then $E(W)=\prod_{j=1}^N E[f(X_j)]$.*

Theorem 2.1 is trivial. An elementary proof of Theorem 2.2 for two variables with a finite number of values is easily given [**14**, pp. 170–172], and extensions to the infinite discrete and continuous cases may then be worked out in appropriate detail.

We now define the moment generating function of X with respect to the arbitrary point a by the equation

$$G_X(\alpha, a) = G(\alpha, a) = E[e^{\alpha(X-a)}], \tag{2.1}$$

provided that the right side exists for all real values of α in some interval con-

* See [**14**, chapters IX, XII, and XIII]. For a more sophisticated treatment, see [**4**, chapters I–III].

taining the origin as an interior point.* We shall restrict ourselves in this paper to chance variables for which this is true. Certain of the arguments in the later sections will depend upon this assumption:

The moment generating function of a distribution uniquely determines the distribution.

It is well known that this statement admits rigorous mathematical proof [**14**, pp. 271–274; **4**, pp. 28–29], but not at the level of this exposition. However, we can justify an omission of proof here on possibly better grounds than those of mere expediency. It is traditional in applied statistics to suppose that a distribution is characterized chiefly by its moments, and since many students in an introductory course in theoretical statistics will have some background in the practical applications, it would not seem wholly inappropriate to adopt the above assumption as an axiom at this stage of instruction.

We here define the moments of X with respect to a by

$$\mu_{r:X}(a) = \mu_r(a) = G^{(r)}(0, a), \qquad (r = 1, 2, \cdots), \tag{2.2}$$

where the superscript (r) in the right member indicates the rth derivative of G with respect to α. We write $\mu_1(0) = \mu$, $\mu_r(\mu) = \mu_r$, $\mu_2 = \sigma^2$. It is customary to call μ the mean and σ the standard deviation of the distribution of X. If the range of X consists of only a finite number of values, we see at once by carrying out the differentiation in (2.1) that this definition of moment reduces to the more usual one,

$$\mu_r(a) = E[(X - a)^r], \qquad (r = 1, 2, \cdots). \tag{2.3}$$

This relation remains true in the case of an infinite discrete range, or a continuous range, but the proof involves termwise differentiation of an infinite series, or differentiation under an integral sign. It is perhaps of interest to remark that we shall never use (2.3) in the sequel, and that for most practical purposes it seems to be sufficient to establish (2.3) as a consequence of (2.2) only for a discrete variable with a finite number of values.†

Referring to (2.1) and Theorem 2.1, we observe that

$$G(\alpha, a) = E(e^{\alpha X} e^{-\alpha a}) = e^{-\alpha a} G(\alpha, 0). \tag{2.4}$$

We obtain the formula for change of origin in a system of moments by applying Leibniz's rule for the differentiation of a product to the last member of (2.4),

* This function was employed extensively by Laplace in his classical Théorie Analytique des Probabilités. If α is taken as a pure imaginary (as would be possible throughout this paper without any formal changes), the moment generating function becomes a Fourier transform, and is called the characteristic function of the distribution. Questions of existence then disappear in (2.1), but reappear in (2.2) and (2.3) below.

† For example, explanations of the rôle of the moments in determining the shape of a distribution of infinite range are perhaps most easily and accurately given in terms of an approximating finite discrete distribution. A similar remark applies to discussions of the rationale of curve fitting by the method of moments.

and setting $\alpha=0$ in the result. We have

$$\mu_r(a) = \mu_r(0) + {}_rC_1\mu_{r-1}(0)(-a) + {}_rC_2\mu_{r-2}(0)(-a)^2 + \cdots + (-a)^r, \quad (r = 1, 2, \cdots). \tag{2.5}$$

By letting $r=1$ and $a=\mu$, we find that $\mu_1(\mu)=0$; by letting $r=2$, we find that $\sigma^2=\mu_2(0)-\mu^2$. The fact that $\mu_2(a)$ is a minimum if $a=\mu$ is also easily obtained from (2.5).

Again, if c is any constant,

$$G_{cX}(\alpha, 0) = E(e^{\alpha cX}) = G_X(c\alpha, 0). \tag{2.6}$$

Repeated differentiation yields

$$\mu_{r:cX}(0) = c^r\mu_{r:X}(0), \quad (r = 1, 2, \cdots). \tag{2.7}$$

If $X_1, X_2, \cdots, X_N$ are N mutually independent chance variables, and if $L=c_1X_1+c_2X_2+\cdots+c_NX_N$, where the c_j's are arbitrary constants, then Theorem 2.2 enables us to write the important formula

$$G_L(\alpha, 0) = E(e^{\alpha L}) = E\left[\prod_{j=1}^{N} e^{\alpha c_j X_j}\right] = \prod_{j=1}^{N} E(e^{\alpha c_j X_j}) = \prod_{j=1}^{N} G_j(c_j\alpha, 0), \tag{2.8}$$

where $G_j(\alpha, 0)$ is the moment generating function of X_j.

Equations (2.4) and (2.8) suggest that for some purposes the logarithm of $G(\alpha, a)$ might be more useful than G itself. This was first observed by Thiele [13], who gave the name semi-invariants* to the quantities obtained when G is replaced by its logarithm in (2.2). Thus we define the semi-invariant generating function of X by

$$H_X(\alpha, a) = H(\alpha, a) = \log G(\alpha, a),\dagger \tag{2.9}$$

and the rth semi-invariant with respect to a by

$$\lambda_{r:X}(a) = \lambda_r(a) = H^{(r)}(0, a), \quad (r = 1, 2, \cdots).$$

The superscript (r) again indicates differentiation with respect to α. From (2.4), (2.6), and (2.8) we at once obtain the relations

$$\begin{aligned} H(\alpha, a) &= -a\alpha + H(\alpha, 0), \\ H_{cX}(\alpha, 0) &= H_X(c\alpha, 0), \\ H_L(\alpha, 0) &= \sum_{j=1}^{N} H_j(c_j\alpha, 0), \end{aligned} \tag{2.10}$$

where $H_j(\alpha, 0)$ is the semi-invariant generating function of X_j. Repeated dif-

* The semi-invariants of Thiele are a sub-class of the seminvariants which appear in algebraic invariant theory, although Thiele makes no mention of this fact in his book. See [5], especially pp. 31–36; also [6].

† All logarithms in this paper are to the base e.

ferentiation of these equations yields

$$(2.11)\qquad \lambda_1(a) = -a + \lambda_1(0), \quad \lambda_r(a) = \lambda_r(0), \qquad (r = 2, 3, \cdots),$$

$$(2.12)\qquad \lambda_{r:cX}(0) = c^r\lambda_r(0), \qquad (r = 1, 2, \cdots),$$

$$(2.13)\qquad \lambda_{r:L}(0) = \sum_{j=1}^{N} c_j^r\lambda_{r:j}(0), \qquad (r = 1, 2, \cdots),$$

where $\lambda_{r:j}(0)$ means the rth semi-invariant of X_j.

From (2.11) it is seen that all the semi-invariants after the first one are independent of the origin. We shall therefore on occasion write $\lambda_r(a)=\lambda_r$, $r>1$. The relation (2.13) is especially useful in sampling theory, and its existence is one of the chief reasons for introducing semi-invariants into our discussion.* Another reason is that the semi-invariants of certain special distributions are much simpler and easier to find than the moments.

Although the semi-invariants can easily be expressed in terms of the moments, and *vice versa*, by repeated differentiation of (2.9) or of the equation $G=e^H$, a more convenient derivation of such identities is made possible by a set of linear equations which we obtain as follows. Since $H^{(1)}=G^{(1)}/G$, we have $G^{(1)}=H^{(1)}G$. Applying Leibniz's rule to the second member of this formula, and setting $\alpha=0$ in the result, we obtain

$$(2.14)\qquad \begin{cases} \mu_1(a) = \lambda_1(a), \\ \mu_{r+1}(a) = \lambda_1(a)\mu_r(a) + {}_rC_1\lambda_2\mu_{r-1}(a) \\ \qquad\qquad + {}_rC_2\lambda_3\mu_{r-2}(a) + \cdots + \lambda_{r+1}, \qquad (r = 1, 2, \cdots). \end{cases}$$

From the equations obtained by writing out (2.14) for $r=1, 2, 3, \cdots$,† we rapidly find that

$$(2.15)\qquad \begin{cases} \mu_1(a) = \lambda_1(a), \\ \mu_2(a) = \lambda_2 + (\lambda_1(a))^2, \\ \mu_3(a) = \lambda_3 + 3\lambda_1(a)\lambda_2 + (\lambda_1(a))^3, \\ \mu_4(a) = \lambda_4 + 4\lambda_3\lambda_1(a) + 6(\lambda_1(a))^2\lambda_2 + 3\lambda_2^2 + (\lambda_1(a))^4. \end{cases}$$

It is to be observed that $\lambda_1(0)=\mu$, $\lambda_2=\mu_2$, $\lambda_3=\mu_3$.

3. Semi-invariants and moments of certain distributions. We shall now study four important special distributions by the methods of §2.

(a) *Binomial distribution.* The chance variable X is said to have a binomial distribution with $n+1$ values and parameter p, if its values are the integers

* The importance of semi-invariants was apparently overlooked by the English statisticians for over twenty-five years after Thiele's work. Comprehensive treatments of semi-invariants in sampling theory were given in 1928 simultaneously by Craig [**2**] and Fisher [**7**]. Fisher and his school call semi-invariants "cumulants."

† Craig in [**2**, pp. 10–13], sets up the determinant of the linear equations (2.14) and lists the first twelve of the relations (2.15) with $a=\mu$.

0, 1, 2, $\cdots$, n, and if $P(X=k)={}_nC_k q^{n-k}p^k$, $(k=0, 1, 2, \cdots, n)$, where $q=1-p$. The variable X may be concretely interpreted as the number of white balls obtained in a set of n drawings from an urn containing white and black balls, when the probability of drawing a white ball is equal to p in each drawing. For many purposes it is convenient to express X in this manner:

$$X = z_1 + z_2 + \cdots + z_n, \tag{3.1}$$

where the z_j's are identical independent chance variables, each with the two values 0 and 1 assumed with respective probabilities q and p. In the concrete interpretation, z_j stands for the result of the jth drawing.

Letting z stand for any one of the z_j's, we have $G_z(\alpha, 0)=q+pe^\alpha$, $H_z(\alpha, 0)=\log(q+pe^\alpha)$, by (2.1) and (2.9). Then using (2.8) and (2.10), we have

$$\begin{aligned} G_X(\alpha, 0) &= (q + pe^\alpha)^n, \\ H_X(\alpha, 0) &= n \log (q + pe^\alpha). \end{aligned} \tag{3.2}$$

By differentiating these expressions and setting $\alpha=0$ in the various derivatives, we can obtain in terms of n and p as many of the moments or semi-invariants as desired. For example, from (3.2) we obtain

$$\begin{aligned} &\lambda_1(0) = \mu_1(0) = np, \qquad \lambda_2 = \mu_2 = npq, \\ &\lambda_3 = \mu_3 = npq(q-p), \ \lambda_4 = npq[(p-q)^2 - 2pq]. \end{aligned}$$

The formal work becomes tedious after the third differentiation. However, a semi-recursion formula is immediately available, for after one differentiation of (3.2), we have

$$(q + pe^{\alpha})H_X^{(1)}(\alpha, 0) = npe^{\alpha};$$

then applying Leibniz's rule to the left member of this equation and letting $\alpha=0$ in the result, we obtain at once

$$\lambda_{r+1} + {}_rC_1 p\lambda_r + {}_rC_2 p\lambda_{r-1} + \cdots + {}_rC_{r-1}p\lambda_2 + p\lambda_1(0) = np.$$

By successive substitutions in this formula, the semi-invariants can be rapidly written down in unsimplified form.*

The variable $X'=X/n$, where X has a binomial distribution, is frequently encountered in the applications. According to (2.7) and (2.12), the rth moment or semi-invariant of X' can be obtained from the corresponding moment or semi-invariant of X simply by multiplying the latter by $1/n^r$.

(b) *Poisson distribution.*† The chance variable X is said to have a Poisson distribution with parameter m if its values are the positive integers and zero,

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* Romanovsky [12] gave a recursion formula for the binomial moments involving $d\mu_r/dq$. See also [1]. For the analogous formula for the semi-invariants, and for a list of the first twelve semi-invariants, see [9]. Kirkham [11] discusses a semi-recursion formula for the moments and lists the first eight moments about the mean.

† An especially complete exposition of the applications of this distribution is given in [8].

and if $P(X=k)=m^k e^{-m}/k!$, $(k=0, 1, 2, \cdots)$. We have

$$(3.3)\qquad G(\alpha, 0) = \sum_{k=0}^{\infty} \frac{e^{\alpha k} m^k e^{-m}}{k!} = e^{-m} \sum_{k=0}^{\infty} \frac{(e^{\alpha} m)^k}{k!} = e^{-m} e^{m e^{\alpha}} = e^{m(e^{\alpha}-1)},$$

$$H(\alpha, 0) = m(e^{\alpha} - 1).$$

It is immediately obvious that

$$\lambda_r(0) = m, \qquad (r = 1, 2, \cdots).$$

The first four moments may now be written down in terms of m by using (2.15). It is to be noticed that $\mu_1(0)=\mu_2=\mu_3=m$.

(c) *Normal distribution.* The continuous chance variable X is said to have a normal distribution if it has the frequency function

$$\phi(x) = Ke^{-h^2(x-b)^2}.$$

Here h and b are parameters, and K is a function of h and b such that $G(0, b) = \int_{-\infty}^{+\infty}\phi(x)dx = 1$. We have

$$G(\alpha, b) = \int_{-\infty}^{+\infty} Ke^{\alpha(x-b)-h^2(x-b)^2}dx = \frac{Ke^{\alpha^2/4h^2}}{h}\int_{-\infty}^{+\infty} e^{-u^2}du = \frac{Ke^{\alpha^2/4h^2}}{h}\sqrt{\pi}.$$

(The transition from the second member to the third is effected by the substitution $u=h(x-b)-\alpha/2h$.) Setting $G(0, b)=1$, we find that $K=h/\sqrt{\pi}$. Since obviously $G^{(1)}(0, b)=0$, it follows that $\mu_1(b)=0$ and $b=\mu$. Also, $\sigma^2=G^{(2)}(0, b) =1/2h^2$, so $h=1/\sigma\sqrt{2}$. We now rewrite ϕ and G as follows:

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},$$

$$(3.4)\qquad G(\alpha, \mu) = e^{\alpha^2\sigma^2/2}.$$

Also,

$$H(\alpha, \mu) = \tfrac{1}{2}\alpha^2\sigma^2.$$

From the last equation we find that $\lambda_r=0$, $r>2$. Of course, $\lambda_1(\mu)=0$, $\lambda_2=\sigma^2$. Substituting these values into (2.14), we obtain the recursion formula

$$\mu_{r+1} = r\sigma^2\mu_{r-1}, \qquad (r = 2, 3, \cdots).$$

We are now able to write down a complete set of semi-invariants and moments in terms of μ and σ. We have

$$\lambda_1(0) = \mu, \quad \lambda_2 = \sigma^2, \quad \lambda_r = 0, \qquad (r = 3, 4, \cdots),$$

$$\mu_{2\nu-1} = 0, \quad \mu_{2\nu} = \sigma^{2\nu}\frac{(2\nu - 1)!}{2^{\nu-1}(\nu - 1)!}, \qquad (\nu = 1, 2, \cdots).$$

(d) *Pearson type III distribution.* The continuous chance variable X is said to have a Pearson type III distribution with parameters b and h, $(b>0, h>0)$, if

it has the frequency function*

$$\phi(x) = \begin{cases} 0, & x < 0, \\ Kx^{b-1}e^{-hx}, & x \geqq 0. \end{cases}$$

Here K is again a function of b and h. The discussion is similar to that of the normal curve, so we shall omit the details. Employing a simple substitution in the integral defining $G(\alpha, 0)$, we find that

$$G(\alpha, 0) = \left(1 - \frac{\alpha}{h}\right)^{-b}, \tag{3.5}$$

$$H(\alpha, 0) = b \log h - b \log (h - \alpha).$$

It then develops that $b = \mu^2/\sigma^2$, $h = \mu/\sigma^2$, and

$$\lambda_r(0) = (r-1)!bh^{-r}, \qquad (r = 1, 2, \cdots),$$

$$\mu_r(0) = b(b+1)(b+2)\cdots(b+r-1)h^{-r}, \qquad (r = 1, 2, \cdots).$$

4. **Addition theorems.** It is an important fact that each of the distributions studied in § 3 satisfies a reproductive law. We can derive these laws rapidly by using the uniqueness assumption of § 2 concerning moment generating functions.

THEOREM 4.1. *Let the independent variables $X_1, X_2, \cdots, X_N$ have binomial distributions with the common parameter p and with $n_1+1, n_2+1, \cdots, n_N+1$ values respectively. Then the variable $S = \sum_{j=1}^N X_j$ has a binomial distribution with parameter p and $1+\sum_{j=1}^N n_j$ values.*

THEOREM 4.2. *Let the independent variables $X_1, X_2, \cdots, X_N$ have Poisson distributions with respective parameters $m_1, m_2, \cdots, m_N$. Then the variable $S = \sum_{j=1}^N X_j$ has a Poisson distribution with parameter $\sum_{j=1}^N m_j$.*

THEOREM 4.3. *Let the independent variables $X_1, X_2, \cdots, X_N$ have Pearson type III distributions with respective parameters $(b_1, h), (b_2, h), \cdots, (b_N, h)$. Then the variable $S = \sum_{j=1}^N X_j$ has a Pearson type III distribution with parameters $b = \sum_{j=1}^N b_j$, and h.*

The proofs are so similar that we give only that of Theorem 4.2. Here, according to (3.3),

$$G_j(\alpha, 0) = e^{m_j(e^\alpha - 1)},$$

where G_j is the moment generating function of the variable X_j. Applying (2.8) with $c_j = 1$, $(j = 1, \cdots, N)$, we find that

$$G_S(\alpha, 0) = \prod_{j=1}^N e^{m_j(e^\alpha - 1)} = e^{(e^\alpha - 1)M}, \quad M = \sum_{j=1}^N m_j,$$

and this is the moment generating function of a Poisson distribution with

* See [3]. The case $b = N/2$, $h = 1/2$ is often called a χ^2 distribution with N degrees of freedom.

parameter $\sum_{j=1}^{N} m_j$.

Our theorem concerning the normal distribution is a little more elaborate.

THEOREM 4.4. *Let the independent variables $X_1, X_2, \cdots, X_N$ have normal distributions with common mean $\mu = 0$ and respective standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_N$. Then the variable $L = \sum_{j=1}^{N} c_j X_j$, where the c_j's are arbitrary constants not all zero, has a normal distribution with mean $\mu = 0$ and with standard deviation $\sigma = [\sum_{j=1}^{N} c_j^2 \sigma_j^2]^{1/2}$.*

The result is an immediate consequence of (3.4), (2.8), and the uniqueness assumption.

5. Sampling theory and the central limit theorem. We define a sample of N observations to be N independent determinations of a chance variable X, whose distribution we call the parent distribution. Before the values of these observations have been determined, the observations may be thought of as N independent chance variables $X_1, X_2, \cdots, X_N$, each having the parent distribution. A statistic is any function of the observations, and from the *a priori* point of view (which we adopt henceforth) is in general itself a chance variable [4, p. 13].

The moments of the sample with respect to the fixed point (or chance variable) Y, are statistics defined by

$$m_r(Y) = \frac{\sum_{j=1}^{N} (X_j - Y)^r}{N}, \qquad (r = 1, 2, \cdots).$$

We write $\bar{x} = m_1(0)$, $s^2 = m_2(\bar{x})$, and call these variables respectively the mean and variance of the sample.

Two of the most important problems in sampling theory are as follows:

(1) Given the form of the parent distribution, deduce the forms of the distributions of various statistics, such as the sample moments.

(2) Given an arbitrary parent distribution of unknown form, find the approximate forms of the distributions of certain statistics for large values of N.

Many of the classical results in connection with problem (1) can be easily derived by the methods of this paper. We illustrate this by first obtaining some formulas for the semi-invariants of a sample moment, and then by studying the special case in which the parent distribution is assumed to be normal.

By letting $c_j = 1/N$, $(j = 1, 2, \cdots, N)$, in (2.13), we obtain the following relation between the semi-invariants of the sample mean and the semi-invariants of the parent distribution:

$$\lambda_{r:\bar{x}}(0) = N^{1-r}\lambda_{r:X}(0), \qquad (r = 1, 2, \cdots). \tag{5.1}$$

The corresponding relation for the moments of the sample mean are less simple; they may be found by successive substitution in (2.14). Of course $\mu_{2:\bar{x}}$ and $\mu_{3:\bar{x}}$ are given at once in terms of $\mu_{2:X}$ and $\mu_{3:X}$ by (5.1.).

Again, letting $z = X^k$, we find that

$$\lambda_{r:k}(0) = N^{1-r}\lambda_{r:z}(0), \qquad (r = 1, 2, \cdots), \tag{5.2}$$

where $\lambda_{r:k}$ stands for the rth semi-invariant of $m_k(0) = \sum_{j=1}^{N} X_j^k/N$. Since $\mu_{r:z}(0) = E[(X^k)^r] = \mu_{rk:X}(0)$, the moments of $m_k(0)$ are readily expressed in terms of the parent moments by use of (5.2). For example, letting $r = 2$ and using (2.15), we have for the standard deviation of $m_k(0)$,

$$\sigma_k = \sqrt{\frac{1}{N}\sigma_z^2} = \sqrt{\frac{1}{N}[\mu_{2:z}(0) - \mu_z^2]} = \sqrt{\frac{1}{N}[\mu_{2k:X}(0) - (\mu_{k:X}(0))^2]}. \tag{5.3}$$

Similar results may be obtained for the moments of $m_k(\bar{x})$; they are less simple because the variables $(X_j - \bar{x})^k$, $(j = 1, \cdots, N)$, are not independent [**2**, chapter III].

We now assume that the parent distribution is a normal distribution with mean zero and standard deviation σ. Letting $c_j = 1/N$, $(j = 1, \cdots, N)$, in Theorem 4.4, we have the following:

THEOREM 5.1. *The sample mean $\bar{x}$ has a normal distribution with mean zero and standard deviation $\sigma/\sqrt{N}$.*

Let $\bar{y}$ denote the mean of another sample from the same population, this time with M observations in the sample.

THEOREM 5.2. *The statistic $\bar{x} - \bar{y}$ has a normal distribution with mean zero and standard deviation $\sigma\sqrt{(1/N) + (1/M)}$.*

Consider now the statistic $m_2(0)$. By a simple change of variables in the frequency function for X_j, it may be shown that X_j^2 has a Pearson type III distribution with $b = 1/2$, $h = 1/2\sigma^2$. Thus, according to (2.8) with $c_j = 1/N$, and (3.5), we have

$$G_2(\alpha, 0) = \left(1 - \frac{2\sigma^2\alpha}{N}\right)^{-N/2}, \tag{5.4}$$

where G_2 denotes the moment generating function of $m_2(0)$. Hence, using the uniqueness assumption, we deduce the following:

THEOREM 5.3. *The statistic $m_2(0)$ has a Pearson type III distribution with parameters $b = N/2$, $h = N/2\sigma^2$.*

If we are now willing to make the assumption that the statistics s^2 and $\bar{x}$ in samples from a normal parent distribution are independent chance variables, we can go very much farther. Unfortunately, a simple proof of this independence does not seem to be available, and it is suggested that an empirical demonstration might be useful in the classroom at this point.

Using this assumption and the identity $s^2 + \bar{x}^2 = m_2(0)$, we have, by (2.8),

$$G_2(\alpha, 0) = G_{s^2}(\alpha, 0) \cdot G_{\bar{x}}^2(\alpha, 0),$$

in the notation of (5.4). Now by the change of variables used in the proof of

Theorem 5.3, we know that $\bar{x}^2$ has a Pearson type III distribution with $b=1/2$, $h=N/2\sigma^2$, since $\sigma_{\bar{x}}^2=\sigma^2/N$ by Theorem 5.1. Thus, using (5.4), we have

$$\left(1-\frac{2\sigma^2\alpha}{N}\right)^{-N/2} = G_{s^2}(\alpha, 0)\left(1-\frac{2\sigma^2\alpha}{N}\right),$$

$$G_{s^2}(\alpha, 0) = \left(1-\frac{2\sigma^2\alpha}{N}\right)^{-(N-1)/2}.$$

The uniqueness assumption then yields the following:

THEOREM 5.4. *The sample variance s^2 has a Pearson type III distribution with parameters $b=(N-1)/2$, $h=N/2\sigma^2$.*

The way is now open for a fairly complete discussion, within the mathematical limits of this paper, of the distribution of s, and also of Fisher's z, and of "Student's" t.*

We now turn to problem (2). The most fruitful approach to this aspect of sampling theory lies in the direction of the famous Central Limit Theorem of the theory of probability, first stated by Laplace, and the subject of many investigations since his time. A complete proof of even the restricted form of the theorem used in sampling theory involves a certain amount of heavy analysis. However, we can easily "prove" this restricted case of the theorem if we are willing to extend our uniqueness assumption to include the assertion that the limiting form of a moment generating function determines the limiting form of its distribution.† The special case of the theorem which we shall use may be stated as follows:

THEOREM 5.5. *Let the independent chance variables $T_1, T_2, \cdots, T_N$ have identical distributions with means zero and unit standard deviations. Let the moment generating function of T_j exist for α in some neighborhood of $\alpha=0$. Let*

$$\tau = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} T_j.$$

Then

$$\lim_{N\to\infty} P(h_1 < \tau < h_2) = \frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}dx \tag{5.5}$$

uniformly for all h_1 and h_2.

We give two demonstrations. The first uses the Taylor series with remainder for the semi-invariant generating function of T_j, which we denote by $H(\alpha, 0)$. By (2.10) with $c_j=1/\sqrt{N}$, $(j=1, \cdots, N)$, we have

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* A satisfactory treatment can be developed by using material from [**10**], chapter VII, §§7, 11, 12, and 9 in this order.

† For a precise statement and proof, see [**4**, pp. 29–31]; also [**14**, pp. 284–289].

$$H_\tau(\alpha, 0) = NH\left(\frac{\alpha}{\sqrt{N}}, 0\right) = N\left[H(0, 0) + H^{(1)}(0, 0)\frac{\alpha}{\sqrt{N}}\right.$$
$$\left. + \frac{H^{(2)}(0, 0)}{2!}\left(\frac{\alpha}{\sqrt{N}}\right)^2 + \frac{H^{(3)}(\xi, 0)}{3!}\left(\frac{\alpha}{\sqrt{N}}\right)^3\right], \qquad 0 < \xi < \frac{\alpha}{\sqrt{N}}\cdot$$

But $H(0,0) = 0$, and by hypothesis, $H^{(1)}(0,0) = \lambda_1(0) = \mu = 0$, $H^{(2)}(0,0) = \lambda_2 = \sigma^2 = 1$. Substituting these values, we find at once that

$$\lim_{N\to\infty} H_\tau(\alpha, 0) = \tfrac{1}{2}\alpha^2,$$

or

$$\lim_{N\to\infty} G_\tau(\alpha, 0) = e^{\alpha\ /2}. \tag{5.6}$$

Comparing the right member of (5.6) with (3.4), we see that it is the moment generating function of a normal distribution with mean zero and unit standard deviation. By our extended assumption, the limiting form of the distribution of τ is then normal, which is the conclusion stated more precisely in (5.5).

Our second approach avoids Taylor's theorem, and is perhaps more in the spirit of applied statistics. As above, we omit the second subscripts on the moments and semi-invariant of T_j. By (2.13), we have

$$\lambda_{1:\tau}(0) = N\cdot\frac{\lambda_1(0)}{\sqrt{N}} = 0, \quad \lambda_{2:\tau} = N\cdot\frac{\lambda_2}{(\sqrt{N})^2} = 1,$$

$$\lim_{N\to\infty} \lambda_{r:\tau} = \lim_{N\to\infty} N\cdot\frac{\lambda_r}{(\sqrt{N})^r} = 0, \qquad (r = 3, 4, \cdots).$$

Thus the semi-invariants of τ approach the semi-invariants of the normal distribution. By using (2.14), we find as in the case of the normal distribution that

$$\lim_{N\to\infty} \mu_{r+1:\tau} = \lim_{N\to\infty} r\mu_{r-1:\tau}, \qquad (r = 2, 3, \cdots).$$

Successive substitutions in this equation yield

$$\lim_{N\to\infty} \mu_{2\nu-1:\tau} = 0, \quad \lim_{N\to\infty} \mu_{2\nu:\tau} = \frac{(2\nu-1)!}{2^{\nu-1}(\nu-1)!}, \qquad (\nu = 1, 2, \cdots).$$

That is, the moments of τ approach the moments of a normal distribution of the type involved in (5.5), which can be considered strong circumstantial evidence for the truth of Theorem 5.5. The Tchebycheff-Markoff theorem [**14**, pp. 384–387] is needed to complete a rigorous demonstration of (5.5).

Several applications of Theorem 5.5 of great importance in the theory of statistics lie immediately at hand. Consider first the mean $\bar{x}$ of a sample of N observations from an arbitrary parent distribution possessing a moment generating function. From (5.1) we know that $\mu_{\bar{x}} = \mu$, $\sigma_{\bar{x}} = \sigma/\sqrt{N}$, where μ and σ denote

the mean and standard deviation of the parent distribution. We write the identity

$$\frac{\bar{x}-\mu}{\sigma}\sqrt{N} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\left(\frac{X_j-\mu}{\sigma}\right),$$

and verify readily that the variables $(X_j-\mu)/\sigma$ satisfy the conditions imposed on the T_j's in Theorem 5.6. Hence we have

$$\lim_{N\to\infty} P\left(h_1 < \frac{\bar{x}-\mu}{\sigma}\sqrt{N} < h_2\right) = \frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}dx \tag{5.7}$$

uniformly in h_1 and h_2; that is, *the mean of a large sample is approximately normally distributed no matter what the parent distribution may be.* If we examine the right member of the following identity, which is a consequence of (5.2) and (5.3),

$$M_k(0) = \frac{\sqrt{N}\left[m_k(0) - \mu_k(0)\right]}{\sqrt{\mu_{2k}(0) - (\mu_k(0))^2}} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\frac{z_j-\mu_z}{\sigma_z},$$

we see that (5.7) may be generalized as follows:

$$\lim_{N\to\infty} P(h_1 < M_k(0) < h_2) = \frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}dx$$

uniformly in h_1 and h_2.

We conclude with an application of Theorem 5.5 of a somewhat different type, but closely related to sampling theory. If the variable X has a binomial distribution as in § 3(a), then by (3.1) we can write

$$\frac{X-np}{\sqrt{npq}} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{z_j-p}{\sqrt{pq}}.$$

Obviously the variables $(z_j-p)/\sqrt{pq}$ satisfy the conditions imposed on the T_j's in Theorem 5.5. Thus we find that

$$\lim_{n\to\infty} P\left(h_1 < \frac{X-np}{\sqrt{npq}} < h_2\right) = \frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}dx$$

uniformly in h_1 and h_2. This is the important DeMoivre-Laplace theorem.*

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FOCAL POINTS AND FOCAL LOCI

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1. Foci of conics. All conics have foci. The ellipse has two; the hyperbola has two; the parabola has one; and the circle may be described as an ellipse whose axes are equal and whose two foci coincide at the center. A very remarkable property of foci of conics was proved by Plücker in 1833. He showed that if tangents are drawn to any conic from the circular points at infinity, the real points of intersection of these tangents are the foci of the conic. This result is proved in Salmon's *Conic Sections* [**1**, pp. 238–239], and in other more advanced texts on conics.

This discovery of Plücker's has had far-reaching consequences in the study of the geometry of conics and of higher plane curves. The characteristic property of foci (that they are the points of intersection of the tangents from the circular points) has been taken by Charlotte Scott [**2**], Hilton [**3**], Graustein [**4**], and others as the definition of foci, not only for conics, but for cubics, quartics, and curves of higher degree.

2. Focal points. We shall use Plücker's characteristic property as a definition, and by a generalization of it we shall locate points, not on the axes of the conic, which have some of the properties of the points commonly called foci. We shall call these new points "focal points" to distinguish them from the points ordinarily called foci.

The equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

represent an ellipse and a hyperbola, respectively, when the coördinate axes are oblique as well as when they are rectangular. In the second case the coördinates

are referred to the axes of the conic; in the first, to a pair of conjugate diameters [1]. To avoid ambiguity we shall use a and b to denote the semi-axes when the coördinates are rectangular, and a' and b' to denote the semi-conjugate diameters when the coördinates are oblique.

We have defined foci as the points of intersection of the tangents from the circular points to the conic. There are four foci (two real and two imaginary) on the axes of any central conic which is not a circle. The coördinates of these foci are:*

$$\textit{for the ellipse} \qquad (\pm\sqrt{a^2-b^2},\ 0), \qquad (0,\ \pm i\sqrt{a^2-b^2});$$
$$\textit{for the hyperbola} \qquad (\pm\sqrt{a^2+b^2},\ 0), \qquad (0,\ \pm i\sqrt{a^2+b^2}).$$

The algebra which is involved in their derivation, when applied to the equations

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1, \qquad \frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}=1,$$

which are with reference to a pair of conjugate diameters as oblique axes, will give as the intersections of tangents from the points $(1, i, 0)$ and $(1, -i, 0)$† the following points on the conjugate diameters:

$$\textit{for the ellipse} \qquad (\pm\sqrt{a'^2-b'^2},\ 0), \qquad (0,\ \pm i\sqrt{a'^2-b'^2});$$
$$\textit{for the hyperbola} \qquad (\pm\sqrt{a'^2+b'^2},\ 0), \qquad (0,\ \pm i\sqrt{a'^2+b'^2}).$$

Since these points arise from a generalization of the definition of foci, we shall call them "focal points." For the ellipse the points $(\pm\sqrt{a'^2-b'^2},\ 0)$ are real if $a'>b'$. That is, the real focal points lie on the longer of the two conjugate diameters. If the conjugate diameters are equal (*i.e.*, if $a'=b'$), the four focal points coincide at the center of the ellipse. For the hyperbola the real focal points are $(\pm\sqrt{a'^2+b'^2},\ 0)$, which lie on that one of the two diameters which should be called transverse, since it meets the hyperbola in real points.

We shall now seek the locus of these focal points for all pairs of conjugate diameters (including the axes of the conic, upon which the focal points are foci). Since the number of diameters is infinite and there is one in every direction through the center, the locus is presumably a continuous curve. We shall first determine it for the ellipse.

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* For a derivation of these coördinates from the Plücker definition, see Graustein [**4**, pp. 133–136].

† In this case the points $(1, i, 0)$ and $(1, -i, 0)$ are *not* the circular points at infinity, but rather what may be called certain elliptic points at infinity—points in which the line at infinity meets the ellipse $x^2+y^2=1$. In rectangular coördinates this equation represents a circle (an ellipse of zero eccentricity); in oblique coördinates it represents an ellipse whose eccentricity depends upon the angle between the coördinate axes. In terms of that angle ω, the eccentricity is $\sqrt{1-\cot^2\omega/2}$. As ω varies from $\pi/2$ to π, $\omega/2$ goes from $\pi/4$ to $\pi/2$, and $\cot\omega/2$ from 1 to 0. Therefore, by varying ω, the eccentricity may be made to assume all values between zero and one.

3. The focal point locus for the ellipse. In Figure 1 is shown an ellipse with axes whose lengths are $2a$ and $2b$, $(a>b)$. It has the two real foci, F $(c, 0)$ and $F'(-c, 0)$, where $c=\sqrt{a^2-b^2}$, and a pair of conjugate diameters, $M'OM$ and $N'ON$. On the longer of these, $M'OM$, are marked the focal points f and f', whose oblique coördinates with respect to the conjugate diameters as axes are $(c', 0)$ and $(-c', 0)$, where $c'=\sqrt{a'^2-b'^2}$, $(a'>b')$.

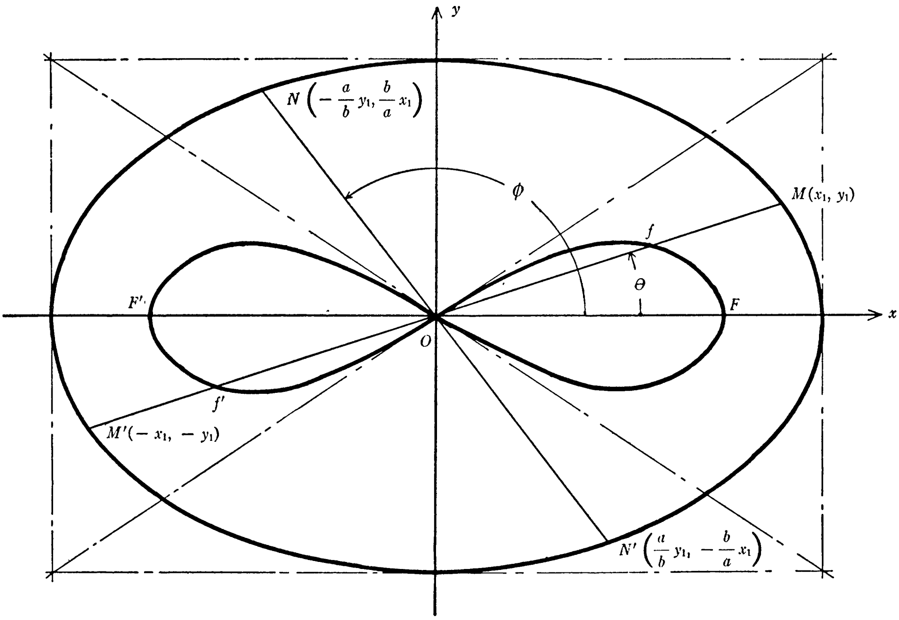

FIG. 1. Ellipse and focal locus.

To obtain the locus of the focal point f on the diameter $M'OM$ as that diameter assumes all possible positions, we denote by (x_1, y_1) the rectangular coördinates of M with respect to the x- and y-axes as indicated in Figure 1. The diameter $N'ON$ conjugate to $M'OM$ is a line through the center O parallel to the tangent to the ellipse at M. The slope of that tangent is $\tan\phi=-b^2x_1/a^2y_1$, and the equation of $N'ON$ is therefore $b^2x_1x+a^2y_1y=0$. This line intersects the ellipse $x^2/a^2+y^2/b^2=1$ at the points $(-ay_1/b, bx_1/a)$, $(ay_1/b, -bx_1/a)$.

The distance r from the center to the focal point f on $M'OM$ is given by the relation $r^2=a'^2-b'^2$. But from the figure we have

$$a'^2 = x_1^2 + y_1^2 \quad \text{and} \quad b'^2 = \frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2},$$

where $a'=OM$ and $b'=ON$. Therefore

$$r^2 = \left(1-\frac{b^2}{a^2}\right)x_1^2 + \left(1-\frac{a^2}{b^2}\right)y_1^2 = (a^2-b^2)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right).$$

To get the equation of the locus of f in polar coördinates (with respect to the x-axis as polar axis), we express x_1 and y_1 in terms of the angle θ which OM makes with the positive x-axis, as follows:

Since $\tan\theta = y_1/x_1$, we have $x_1^2\tan^2\theta - y_1^2 = 0$. Also, $b^2x_1^2 + a^2y_1^2 = a^2b^2$, because (x_1, y_1) is on the ellipse. These two equations are linear in x_1^2 and y_1^2. Their solution is

$$x_1^2 = \frac{a^2b^2}{a^2\tan^2\theta + b^2}, \qquad y_1^2 = \frac{a^2b^2\tan^2\theta}{a^2\tan^2\theta + b^2}.$$

The substitution of these values in the equation for r^2 gives

$$r^2 = (a^2 - b^2)\left(\frac{b^2 - a^2\tan^2\theta}{b^2 + a^2\tan^2\theta}\right),$$

which is the equation in polar coördinates of the focal point locus for the ellipse. Its equation in rectangular coördinates x and y, obtained by means of the equations

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta,$$

turns out to be

$$(x^2 + y^2)(b^2x^2 + a^2y^2) = (a^2 - b^2)(b^2x^2 - a^2y^2).$$

We have derived this equation for the focal locus for the ellipse on the assumption that $a > b$. However, in case $a < b$ the focal locus is given by exactly the same equation. This may be verified by substituting b for a, a for b, y for x, and $-x$ for y in the equation for the focal locus in rectangular coördinates.

This quartic curve is shown in Figure 1. It has a double point at the origin. The two tangents there are $y = \pm(b/a)x$. The origin is therefore a crunode; more than that, it is a bi-flecnode, since for each branch of the curve the origin is a point of inflection. From the polar form of the equation we see that the curve lies wholly in the acute angle between the two tangents at the origin, for if $b^2 - a^2\tan^2\theta < 0$ (*i.e.*, if $|\tan\theta| > b/a$), r is imaginary.

This focal locus is a circular quartic since it passes through the circular points at infinity, but it is not a lemniscate which is a bicircular quartic.

4. The focal locus for the hyperbola. It is possible to find the focal locus for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

directly, by a method which parallels that which we have used for the ellipse. However, it is not necessary to go to that much trouble if we recognize the fact that the result can be obtained quickly by replacing b^2 by $-b^2$. The resulting equations of the focal locus are

$$r^2 = (a^2 + b^2)\left(\frac{b^2 + a^2 \tan^2 \theta}{b^2 - a^2 \tan^2 \theta}\right)$$

in polar coördinates, and

$$(x^2 + y^2)(b^2x^2 - a^2y^2) = (a^2 + b^2)(b^2x^2 + a^2y^2)$$

in cartesian coördinates. This quartic curve is shown in Figure 2. As in the case of the focal locus for the ellipse, it is a circular quartic of species II and has the same Plücker numbers. However, although it is of the same general type, in appearance it in no way resembles the other curve. In fact, the double point at the origin in this case is an acnode instead of a crunode, since the tangents $y = \pm(ib/a)x$ at that point (the origin) are imaginary.

The focal locus has the same asymptotes $y = \pm(b/a)x$ as the hyperbola itself. If $\tan^2\theta > b^2/a^2$, the radius vector r is imaginary and the curve has no real trace. The real part lies in the region between the lines $y = \pm(b/a)x$ which contains the x-axis.

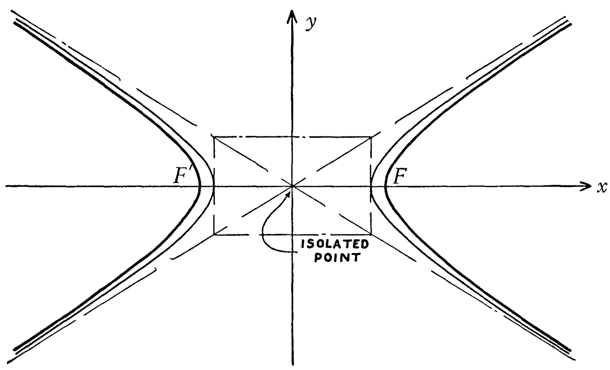

FIG. 2. Hyperbola and focal locus.

5. The focal locus for the equilateral hyperbola. If $a = b$, the hyperbola becomes equilateral. The equations of the focal locus are

$$r^2 = 2a^2\left(\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta}\right)$$

in polar coördinates, and

$$(x^2 + y^2)(x^2 - y^2) = 2a^2(x^2 + y^2)$$

in rectangular coördinates. This last quartic is equivalent to the two second degree equations

$$x^2 + y^2 = 0 \quad \text{and} \quad x^2 - y^2 = 2a^2.$$

The focal locus of an equilateral hyperbola $x^2 - y^2 = a^2$ is therefore another hyperbola $x^2 - y^2 = 2a^2$, and a circle of zero radius $x^2 + y^2 = 0$. The vertices of

the second hyperbola are at the foci of the first, and both have the same asymptotes.

6. The focal locus for the parabola. A separate treatment is required to find the focal locus for the parabola. The equation $y^2=px$ represents a parabola when the coördinate axes are oblique as well as when they are rectangular. In the second case the x-axis is the axis of the parabola, the y-axis is the tangent at the vertex, and p (usually called the principal parameter of the parabola) is the length of the latus rectum (the double ordinate through the focus). In the first case the x-axis is a diameter, the y-axis is the tangent at the end of that diameter, and p is that which is called the parameter for the particular diameter involved [**1**, p. 197]. For the parabola whose equation in rectangular coördinates is $y^2=px$, the Plücker definition of foci leads to just one focus, the real point $(p/4, 0)$ [**4**, p. 134]. The algebra which is involved in its derivation, when applied to the equation $y'^2=p'x'$, which is with respect to a diameter and the tangent at the end of the diameter as oblique axes, will give $(p'/4, 0)$ as the intersection for the tangents from the points $(1, i, 0)$ and $(1, -i, 0)$. The point $(p'/4, 0)$ is a real point on the diameter which is the axis of abscissas. As we did in the case of central conics, we shall call this point a focal point. Every diameter of the parabola has such a point; the focal point of the axis is the focus. We shall now find the locus of these focal points. Since the number of diameters is infinite (there being one, parallel to the axis, through each point of the parabola), the locus is presumably a continuous curve.

Figure 3 shows the parabola whose equation is $y^2=px$ with respect to rectangular axes. Point F is the focus; its coördinates are $(p/4, 0)$. With respect to the oblique axes $Y'MX'$, the equation of this parabola is $y'^2=p'x'$, and the focal point f has coördinates $(p'/4, 0)$.

To obtain the locus of this focal point f on the diameter MX' as that diameter assumes all possible positions, we denote by (x_1, y_1) the rectangular coördinates of M with respect to the x- and y-axes, as indicated in the figure. The coördinates of the focal point f referred to these rectangular axes are $x=x_1+(p'/4)$ and $y=y_1$. But [**1**, p. 202] since $p'=p+4x_1$, we have $x=2x_1+(p/4)$ and $y=y_1$. Since $y_1^2=px_1$, we obtain

$$y^2 = \frac{p}{2}\left(x - \frac{p}{4}\right)$$

as the equation of the focal point locus. It is another parabola whose vertex is at the focus of the given parabola and whose parameter is $p/2$, half that of the given parabola.

The curve which we have derived is shown in Figure 3. It is rather surprising that it should be a conic instead of a quartic, but it may be shown that it is a limiting form of either of the quartics which we have derived as the focal loci of the ellipse and the hyperbola—a limiting form as one vertex and corresponding focus are kept fixed while the eccentricity varies and approaches unity. The

work necessary to prove this statement can be done after the manner of Salmon's proof [1, p. 200] that the parabola is a limiting form of an ellipse.

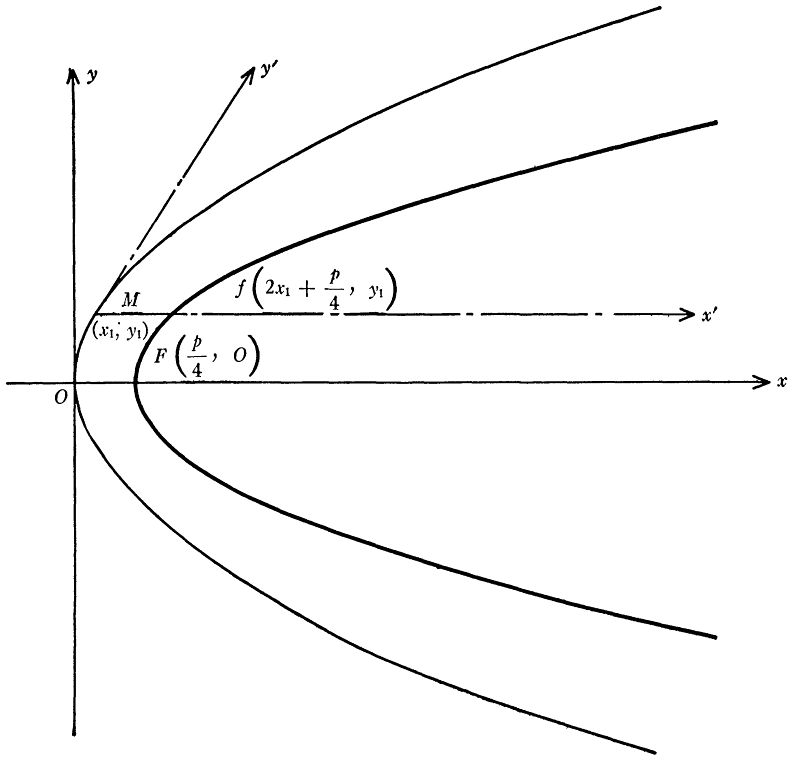

FIG. 3. Parabola and focal locus.

7. Another characteristic property of focal points. We have defined focal points of conics by generalizing a certain characteristic property of foci. (See paragraphs 1 and 2.) There is another characteristic property of foci which might be made the basis for a generalization which would lead to the same focal points and focal loci. According to Salmon [1, p. 241] the focus of any conic may be considered as an infinitely small circle having double contact with the conic in two imaginary points situated on the directrix. Here the word directrix has the meaning usually associated with it in the theory of conics. For our purpose, a directrix of a conic may be thought of as the polar of a focus; an ellipse has two, a hyperbola has two, and a parabola has one. Their equations, in terms of the usual standard forms, are as follows:

	Conic	Directrices
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,$	$x = \pm \dfrac{a^2}{c},\qquad c = \sqrt{a^2 - b^2};$

Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$ $\quad x = \pm \dfrac{a^2}{c},$ $\quad c = \sqrt{a^2 + b^2};$

Parabola $\quad y^2 = px,$ $\quad x = -p/4.$

To verify this characteristic property for the ellipse, we solve simultaneously the equations of the ellipse and the circle of zero radius whose center is at the focus $(c, 0)$. These equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (x - c)^2 + y^2 = 0.$$

For the abscissas of the common points, we find that $c^2x^2 - 2a^2cx + a^4 = 0$. Since this may be written as $(cx - a^2)^2 = 0$, it indicates that each of these points is on the directrix $x = a^2/c$, and that each one counts as two points of intersection of the ellipse and the circle of zero radius. It is just as easy to make this verification for the other focus of the ellipse, and for the hyperbola and parabola. We omit the details.

If we now suppose the equations for the ellipse, hyperbola, and parabola to be referred to oblique axes, we may define a directral line as the polar of a focal point and get the following equations for directral lines in terms of the standard equations which we have used for conics referred to oblique axes:

	Conic	Directral lines	
Ellipse	$\dfrac{x'^2}{a'^2} + \dfrac{y'^2}{b'^2} = 1,$	$x' = \pm \dfrac{a'^2}{c'},$	$c' = \sqrt{a'^2 - b'^2};$
Hyperbola	$\dfrac{x'^2}{a'^2} - \dfrac{y'^2}{b'^2} = 1,$	$x' = \pm \dfrac{a'^2}{c'},$	$c' = \sqrt{a'^2 + b'^2};$
Parabola	$y'^2 = p'x',$	$x' = -p'/4.$	

The algebra which we used and indicated in the preceding paragraph for the corresponding equations without the primes, when applied to these equations with the primes, will show that the focal point $(c', 0)$ of the ellipse $(x'^2/a'^2) + (y'^2/b'^2) = 1$ may be thought of as an infinitely small conic $(x' - c')^2 + y'^2 = 0$, which has double contact with the ellipse in two imaginary points of the directral line $x' = a'^2/c'$. In this case the infinitely small conic, whose only real point is the focal point $(c', 0)$, is not a circle. Since the coördinates are oblique, it must be considered as an infinitely small ellipse having the same eccentricity as the ellipse $x'^2 + y'^2 = 1$ which we used, in paragraph 2, to determine elliptic points at infinity.

8. Complementary focal loci. We have found the equation of the focal locus for the ellipse to be

$$r^2 = (a^2 - b^2)\left(\frac{b^2 - a^2 \tan^2 \theta}{b^2 + a^2 \tan^2 \theta}\right).$$

The equation

$$r^2 = -(a^2 - b^2)\left(\frac{b^2 - a^2 \tan^2 \theta}{b^2 + a^2 \tan^2 \theta}\right)$$

represents a quartic which we shall call the "complementary focal locus." It has real points for values of the angle θ for which the focal locus has imaginary points. The focal locus is in the acute angles between the lines $y = \pm(b/a)x$ and meets the x-axis in the points $(c, 0)$ and $(-c, 0)$; the complementary focal locus is in the obtuse angle and meets the y-axis in the points $(0, c)$ and $(0, -c)$. This curve, which is shown in Figure 4, is also a bi-flecnodal quartic of species II and has the same Plücker numbers as has the focal locus.

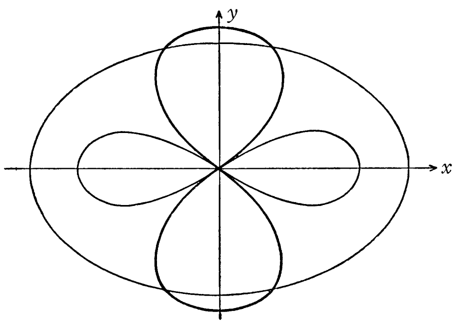

FIG. 4. Ellipse with focal locus and complementary focal locus.

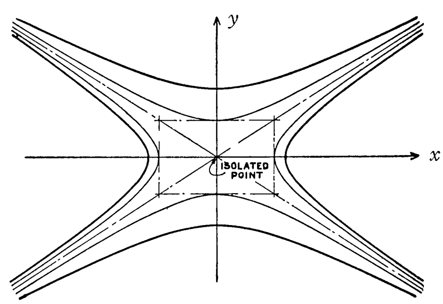

FIG. 5. Hyperbola and conjugate hyperbola, focal locus and complementary focal locus.

In a similar way we define the complementary focal locus for the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ to be

$$r^2 = -(a^2 + b^2)\left(\frac{b^2 + a^2 \tan^2 \theta}{b^2 - a^2 \tan^2 \theta}\right).$$

This turns out to be the focal locus for the conjugate hyperbola $(y^2/b^2) - (x^2/a^2) = 1$. It is shown in Figure 5.

In paragraph 2 we defined imaginary as well as real focal points for the ellipse and the hyperbola. The quartics which we have derived as the loci of the real focal points are satisfied by the coördinates of the imaginary focal points also and may therefore be regarded as complete focal loci. The complementary focal loci which we have defined may be thought of as loci of real points whose distances from the origin are the absolute values of the imaginary distances from the origin to the imaginary focal points.

9. Inverses of the focal loci. Two points P and P' are said to be one the inverse of the other with respect to a circle whose center is at O and whose radius is k, if OPP' is a straight line and $OP \cdot OP' = k^2$. A plane figure is called the inverse of another plane figure if each point of the first is the inverse of a point on the second, and conversely. This is the ordinary definition of inverse curves.

The constant k^2 is thought of as real and positive since it represents the square of the length of the radius of a real circle. Some authors interpret inversion for negative values of k^2 (*i.e.*, for pure imaginary values of k). For example, Hilda P. Hudson (in her *Ruler and Compasses*, p. 92) says, "If the radius k is pure imaginary, then $OP\cdot OP'$ is negative, and P, P' are on opposite sides of the center O." Roger Johnson (in his *Modern Geometry*, p. 45) says, "Some geometers find it desirable to define as an inversion the transformation determined by the equation $OP\cdot OP' = -c^2$, the circle of inversion being imaginary, with radius $c\sqrt{-1}$." In discussing inverses of focal loci we shall use a more general k^2, namely k^2 a pure imaginary, and show that it leads to real inverse curves.

We have found that the focal locus of the ellipse $(x^2/a^2)+(y^2/b^2)=1$ (for $a>b$ or $a<b$) is, in polar coördinates,

$$r^2 = (a^2 - b^2)\left(\frac{b^2 - a^2\tan^2\theta}{b^2 + a^2\tan^2\theta}\right).$$

If we choose $k^2=\sqrt{a^4-b^4}$, the inverse curve will have for its equation

$$r^2 = (a^2 + b^2)\left(\frac{b^2 + a^2\tan^2\theta}{b^2 - a^2\tan^2\theta}\right),$$

which is exactly the focal locus for the hyperbola $(x^2/a^2)-(y^2/b^2)=1$.

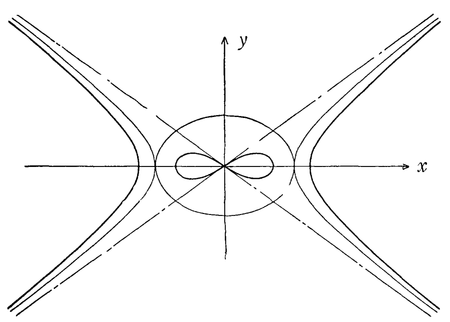

FIG. 6. Ellipse, hyperbola, and inverse focal loci, $a>b$.

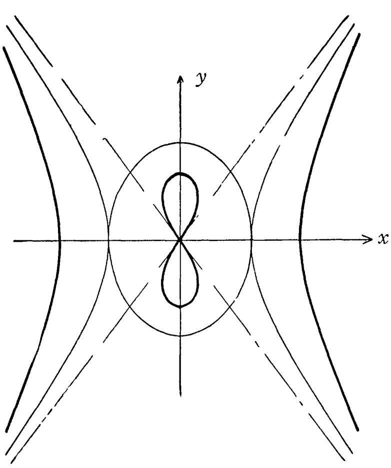

FIG. 7. Ellipse, hyperbola, and inverse focal loci, $a<b$.

Figures 6 and 7 show the ellipse, the hyperbola, and the inverse focal loci for $a>b$ and $a<b$, respectively. In the one case $(a>b)$, the constant $k^2=\sqrt{a^4-b^4}$ is real and the two inverse focal loci lie wholly in the acute angles between the lines $y=\pm(b/a)x$; in the other case $(a<b)$, the constant $k^2=\sqrt{a^4-b^4}$ is pure imaginary and the focal locus of the ellipse lies in the acute angles while its

inverse, the focal locus of the hyperbola, lies wholly in the obtuse angles. In this case a line through the origin which meets one of the focal loci in real points meets the other in imaginary points, a result of the pure imaginary nature of k^2.

10. M. Jacob's parameter curves. The quartic curves which we have derived as focal loci were discovered and described many years ago, from a different point of view, by M. Jacob, a captain of the French artillery. His account appeared in the Nouvelles Annales de Mathématiques, vol. II, 1843, under the title *Parameter curves of conics*. Apparently he had no knowledge of the characteristic property of foci which Plücker had discovered just a few years before; or, if he did, he attached no significance to it. He developed the curves in terms of the Apollonius "parameter" [**5**, Prop. 22, 23]. His paper contains the equations for the focal loci of the ellipse and the hyperbola, but not for the parabola, although he indicates how it could be found. Gino Loria in his *Spezielle Algebraische und Transzendente Ebene Kurven*, vol. I, pp. 233, 234, has a reference to this paper of Jacob's, but he mentions the "parameter curve" for the ellipse only. Jacob found his "parameter curve" as the locus of the points of intersection of diameters and the parameter chords associated with them. To use the language of Loria, it was a "quartic curve derived from a conic."

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

PLOTTING CURVES IN POLAR COÖRDINATES

P. C. HAMMER, Oregon State College

To many calculus students the plotting of curves in polar coördinates is a process which is both tedious and uncertain. I have found that the elementary mapping approach given below is appreciated by most calculus students. This "solution" of the problem presumes with some justification that students are more familiar with cartesian than with polar coördinates.

Let $f(\rho, \theta)=0$ be the equation of a curve to be plotted in the polar coördinate system. Taking ρ as the ordinate and θ (in radians, say) as the abscissa, plot the graph of this function in cartesian coördinates and then map this curve into the polar coördinate system in the obvious manner—*i.e.*, interpret the abscissas as angles and the ordinates as distances from the origin. Using compasses, many points can be plotted in the polar coördinate system with relative ease.

This method is not a panacea for all the ills of plotting curves in polar coördinates. In many cases the customary device of reverting to a cartesian coördinate system superimposed upon the polar coördinate system is better, and in other cases direct plotting is superior. This mapping method has, nonetheless, many advantages in certain types of problems. The graph of any equation of the form $\rho=a+b\cos(c\theta+d)$ is made very simple by mapping, in that the symmetries and required range of θ are immediately clear from the "cartesian" curve and the general shape of the curve is evident at a glance. Thus all the ordinary n-leaf roses, the cardioids, and the limaçons are easily plotted. Other common curves which are easily done are the lemniscates, the conchoid of Nicomedes, and the customary spirals. This method has the advantage of introducing in a natural way a continuous transformation which is not one-to-one. For the student who likes to play around with oddities, the mapping of other plane curves from the cartesian to the polar system will provide some sport. Letting c be an irrational number in the above equation, the range of θ must be infinite and the student is introduced to curves rarely mentioned in the texts.

AN ELEMENTARY PROPERTY OF THE TRAJECTORY OF A PROJECTILE

GUIDO FUBINI, Institute for Advanced Study

It is well known that *the angle of fall of a projectile is greater than its quadrant elevation.* I will give here some generalizations of this theorem.

As usual, the origin O is at the muzzle of the gun, the y-axis is vertical, directed upwards, and the x-axis is the horizontal projection of the trajectory. The *inclination* of the trajectory, or of one of its tangent lines t, is the angle $\theta=xt$;

the initial value θ_0 of θ is the *quadrant elevation*. If P is a point of the trajectory, the line $l=OP$ is the *line of site*, and $\alpha=xl$ is the *site* or *angle of site*. Let us write $Y=y-x\tan\alpha$; this function is equal to zero at the points O, P; let m be the greatest value that it takes on the points of the arc OP, and let M be the corresponding point of this arc. From the law of the mean we deduce that the value θ_M of the inclination at the point M is equal to α. The point M divides the arc OP into two partial arcs, the arc OM and the arc MP. Let v be the velocity of the projectile and $u=v\cos\theta$ its horizontal velocity. From the fundamental equations of exterior ballistics (see *Note* at the end of this paper), we deduce that

$$dY = -\frac{u^2}{g}\frac{\sin(\theta-\alpha)d\theta}{\cos^3\theta\cos\alpha}, \tag{1}$$

or

$$g\frac{dY}{u^2} = -\tan\theta\, d\tan\theta + \tan\alpha\, d\tan\theta.$$

We integrate now:

1) along the arc OM (by supposing $\theta_0 \geqq \theta \geqq \alpha$);

2) along the arc PM (by supposing $\theta_P \leqq \theta \leqq \alpha$).

We get

$$\tfrac{1}{2}\tan^2\alpha - \tan\alpha\tan\theta_0 + \tfrac{1}{2}\tan^2\theta_0 = g\int_0^m \frac{dY}{u^2}, \qquad \text{(along } OM\text{)},$$

$$\tfrac{1}{2}\tan^2\alpha - \tan\alpha\tan\theta_P + \tfrac{1}{2}\tan^2\theta_P = g\int_0^m \frac{dY}{u^2}, \qquad \text{(along } PM\text{)}.$$

But u is *decreasing* along the trajectory; therefore, the latter integral is greater than the former, and consequently

$$\tfrac{1}{2}\tan^2\theta_P - \tan\alpha\tan\theta_P > \tfrac{1}{2}\tan^2\theta_0 - \tan\alpha\tan\theta_0. \tag{2}$$

Since θ is decreasing along the trajectory, $\tan\theta_0 - \tan\theta_P > 0$. From the preceding inequality we deduce

$$\tfrac{1}{2}(\tan^2\theta_0 - \tan^2\theta_P) < \tan\alpha(\tan\theta_0 - \tan\theta_P)$$

or, dividing by $\tan\theta_0 - \tan\theta_P > 0$,

$$\tan\theta_0 - \tan\alpha < \tan\alpha - \tan\theta_P. \tag{3}$$

If P is the level point, $\alpha=0$, $\omega=-\theta_P>0$ is the angle of fall; and we obtain the quoted theorem for the *angle of fall*. Besides the generalization (3) of this theorem, we can obtain also another generalization by supposing for instance that P is on the *ascending* branch of the trajectory ($\theta_0>\alpha>\theta_P>0$). From (1) we deduce that

$$\frac{g\cos\theta}{v^2}\cos\alpha\, dY = -\sin(\theta-\alpha)\, d\theta.$$

But $\theta > 0$ and $v > 0$ are decreasing functions along the ascending branch of the trajectory, and therefore $\cos \theta / v^2$ is increasing. By integrating along the arcs OM and PM, we obtain by a similar method,

$$1 - \cos(\theta_0 - \alpha) < 1 - \cos(\theta_P - \alpha) = 1 - \cos(\alpha - \theta_P),$$

or

$$\theta_0 - \alpha < \alpha - \theta_P, \qquad (\theta_0 > \alpha > \theta_P > 0). \tag{4}$$

Then (4) is the new generalization; but we cannot now suppose $\alpha = 0$, and deduce the theorem concerning the angle of fall, because we have supposed $\theta_P > 0$ (in the ascending branch of the trajectory) whereas $\theta = -\omega < 0$ at the level point. In the general case we must make use of the preceding inequality (3).

Note. If g is the acceleration of gravity, and $F(v)$ the retardation due to the air, these equations are:

$$dx = v \cos \theta \, dt, \qquad dy = v \sin \theta \, dt,$$
$$d(v \cos \theta) = -F(v) \cos \theta \, dt, \qquad d(v \sin \theta) = -[F(v) \sin \theta + g] dt.$$

By eliminating $F(v)$ we deduce

$$dt = -\frac{v}{g} \frac{d\theta}{\cos \theta},$$

and consequently

$$dx = -\frac{v^2}{g} d\theta, \qquad dy = -\frac{v^2}{g} \tan \theta \, d\theta.$$

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Elementary Mathematical Concepts from the Historical and Logical Point of View. By J. H. Zant and A. H. Diamond. Minneapolis, Burgess Publishing Co., 1941. 125 pages. $1.50.

A Treatise on Algebra. By G. Peacock. Vol. I, Arithmetical Algebra. Reprinted from the 1842 edition. 16+399 pages. Vol. II, On Symbolic Algebra and Its Applications to the Geometry of Position. Reprinted from the 1845 edition. 10+455 pages. New York, Scripta Mathematica, Yeshiva College, 1940. $6.50 for the set.

Introduction to the Theory of Equations. By N. B. Conkwright. Boston, Ginn and Co., 1941. 8+214 pages. $2.00.

The Stereographic Projection. By F. W. Sohon. Brooklyn, Chemical Publishing Co., Inc., 1941. 9+210 pages. $4.00.

Trigonometry. Revised edition. By N. J. Lennes and A. S. Merrill. With tables. New York, Harper and Brothers, 1939. 12+243+92 pages. $2.20.

Six-Place Tables. Sixth edition. By E. S. Allen. New York, McGraw-Hill Book Co., 1941. 23+181 pages. $1.50.

REVIEWS

College Algebra. By H. T. Davis. New York, Prentice-Hall, Inc., 1940. 13+423 pages. $2.50.

This book differs from the usual college algebra text both in the number of topics covered and in the emphasis which is placed on the history of the subject.

The first seven chapters (pp. 1–134) are devoted mainly to a review of elementary algebra, although the work on exponents and logarithms is combined in a single chapter and the section on progressions precedes the consideration of linear and quadratic equations. On the whole the material presented is sufficient for a thorough review, but the reviewer feels that in the sections on linear and quadratic equations there should be a greater number of applied problems and some consideration of equations involving radicals and equations in quadratic form. An unusual feature is the introduction in these chapters of the ideas of slope, fitting a straight line to data, and least squares.

The next nine chapters (pp. 135–285) deal with more advanced topics, including probability, theory of equations, determinants, infinite series, inequalities, and theory of investment. Mathematical induction is treated briefly in the chapter on the binomial theorem, and the trigonometric functions are introduced in the chapter on proportion, which immediately precedes that on complex numbers.

Of the topics covered in the last four chapters (pp. 286–383), practically the only ones common to this and the usual text are continued and partial fractions. Eighteen pages are devoted to an introductory chapter on statistics, and many of the topics in the remaining three chapters should prove of special interest and value to the student who is preparing to teach mathematics. The connection between the binomial expansion and the process of extracting square and cube roots is made clear, as is also the basis for the criteria for divisibility by 3, 9, *etc.* Topics treated briefly in the chapter on mathematical recreations include scales of notation, prime and perfect numbers, magic squares, and some of the famous problems of antiquity. The final chapter contains brief biographies of nineteen outstanding mathematicians of the past.

A comparison of the uses of the word "term" on pp. 13, 29, 33, and 127 would be rather confusing to the student. The reviewer objects particularly to this sentence, which appears on page 33: "First we factor the numerator and denominator and then cancel the common terms." It should be made clear in each of the examples on p. 17 that a is equal to b, and the restriction of the discussion to polynomial equations should precede the definition of degree on p. 215. Formula I, p. 201, should read $(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i$, and there are minor misprints on pages 67, 159, and 353.

The material is very attractively arranged and printed.

ETHEL I. MOODY

Exterior Ballistics. A reprint of Chapter X, Exterior Ballistics, and Chapter XII, Bombing from Airplanes, from *Elements of Ordnance.* Prepared under the direction of Lieutenant Colonel T. J. Hayes. New York, John Wiley and Sons, 1938. 98 pages. $1.00.

The purpose of this little book is to give an introduction to that branch of mechanics which deals with the motion of projectiles through the air and their behaviour during flight. The differential equations of the trajectories are derived and the various force terms occurring in them are discussed in their dependence on the structure of the atmosphere, the velocity, and the angle of yaw of the projectile. The mechanism of the gyroscopic action under the influence of initial spin is explained, and the main results of an analysis of this problem are stated. Various methods for the integration of the ballistic equations are mentioned and the method of numerical integration is discussed in sufficient detail to apply it to the solution of problems. Exterior ballistic tables are introduced and their application is explained by means of a discussion of the characteristics of trajectories. In the separate chapter on bombing from airplanes, the problem of hitting a given target from the cruising or diving plane is treated in a clear manner, and in this connection there is given an account of the tasks to be accomplished by bomb sights.

It is the reviewer's opinion that this book can be recommended as a competent first introduction to the fundamentals of exterior ballistics.

ERIC REISSNER

The Foundations of Geometry. (Mathematical Expositions, No. 1.) By G. de B. Robinson. Toronto, University of Toronto Press, 1940. 11+167 pages. $2.00.

Mathematical Expositions is a new series of books, published under the auspices of the University of Toronto, and with an editorial board consisting of S. Beatty, R. Brauer, H. S. M. Coxeter, L. Infeld, G. de B. Robinson, and J. L. Synge. This series is designed to meet the need for books in English which emphasize fundamental principles while presenting the material less elaborately than do those texts which treat a subject exhaustively. *The Foundations of Geometry* achieves this aim to a very considerable degree.

If this book had a sub-title, it might well have been "The Theorems of Desargues and Pappus," for these two theorems play a central rôle throughout the text; their dependence on and independence of various axioms is continually studied. Nowhere before, in English, has the importance of these two theorems been so carefully demonstrated.

In the first half of the book the author develops projective geometry axiomatically, he discusses affine and euclidean geometry from the point of view of the Erlanger Programm, and then he reconsiders euclidean geometry by developing it from Veblen's order axioms and Hilbert's axioms on congruence and continuity.

The second half of the book is concerned with the foundations of analytic projective geometry. Here the concept of number is studied; Von Staudt's "algebra of points" links the algebraic theory with the geometry, and serves as a basis for the introduction of coördinates. Order and continuity are taken up in chapter 8, in part from the point of view of Vailati. In the last chapter the author returns to the study of projective transformations, but now he presents Von Staudt's conception of imaginary elements and he closes with a peep into complex projective geometry.

The make-up of the book is quite satisfactory except for some of the figures which seem crowded. There are a few errors: on p. 15 the author would seem to suggest that Desargues's triangle theorem holds for all special positions of the two triangles; and on p. 40 the involution theorem is stated incorrectly. Nevertheless, this is a most excellent book, and it will inspire many students of mathematics.

HARRY LEVY

Facsimiles of Two Papers by Bayes. With Commentaries. Prepared under the direction of W. E. Deming. Washington, D. C., The Graduate School of The Department of Agriculture, 1940. 16+52 pages. $1.00.

This little volume contains facsimile reprints of two papers by the Rev. Thomas Bayes, reproduced from the pages of the Philosophical Transactions of the Royal Society, vol. 53, 1763, pp. 370–418 and pp. 269–271. The first paper is Bayes's famous essay on inverse probability entitled *An Essay Toward Solving a Problem in the Doctrine of Chances*, with the Rev. Richard Price's introduction

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

FILMS

The Multi-Sensory Aids Committee of the National Council of Teachers of Mathematics has arranged for a temporary loan of films produced in Great Britain relating to mathematics. Clubs or departments interested in using these films should make reservations with the editor of this department at as early a date as possible. Transportation charges must be paid both ways, and a charge of twenty-five cents per reel will be made to defray handling costs. Some of these films may also be purchased at the prices given, by writing directly to the producers in England. All films are 16 mm. silent and non-flammable. Those available are as follows:

1. *The Equation* $\ddot{X}+X=0$. 120 feet. $6.00. Apply B. G. D. Salt, 4 Polayn Garth, Welwyn, Garden City, Herts, England. This film deals with the differential equation for free vibrations.

2. *The Equation* $\ddot{X}+X=A \sin Nt$. 300 feet. $16.00. Apply B. G. D. Salt. Deals with the differential equation for forced vibrations.

3. *A Hypocyclic Motion*. 300 feet. $24.00. Apply B. G. D. Salt. Demonstrates some of the theorems of elementary kinematics.

4. *The Generation of Involute Gear Teeth*. 100 feet. $7.00. Apply B. G. D. Salt. Shows (a) the generation of involute gear teeth as the loci of points attached to a crossbelt connecting two pulleys; (b) the generation of teeth as the envelope on the disc of a zig-zag cutter moving relatively to it.

5. *Rate of Change*. 230 feet. £6. 10s. per reel, plus spool, container, and freight charges. Apply Visual Education Ltd., 31 St. Martin's Lane, London, W.C.2. This is a diagram film presenting an introduction to the differential calculus.

6. *Simple Harmonic Motion: Resultant Circle and Straight Line, and Resultant Ellipses*. 50 feet. 22s. 6d. Apply Educational and General Services Ltd., 37 Golden Square, London, W.1. Shows the composition of simple harmonic motions.

7. *Hypocycloid Gear Ratio 2:1*. 50 feet. 15s. Apply Educational and General Services Ltd. A rolling circle, half the diameter of a fixed circle, traces a straight line—the diameter of the larger circle.

8. *The Theorem of Pythagoras*. 100 feet. Not for sale. Produced by B. G. D. Salt.

9. *Angle Sum of a Triangle*. 100 feet. Not for sale. Produced by B. G. D. Salt.

10. *Levers*. 212 feet. £3. 5s. Apply Educational and General Services Ltd., 37 Golden Square, London, W.1. An elementary introduction to levers—suitable for high school use.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 476. *Proposed by A. H. Stone, Graduate College, Princeton.*

Show that it is possible to fit together six isosceles right triangles, all of different sizes, so as to make a single isosceles right triangle.

E 477. *Proposed by V. Thébault, San Sebastián, Spain.*

Consider four spheres (S_1), (S_2), (S_3), (S_4), whose centers are the vertices of a tetrahedron $S_1S_2S_3S_4$. Let (G_1) be the sphere whose center is the centroid of the face $S_2S_3S_4$ and which passes through the points of intersection of spheres (S_2), (S_3), (S_4). Defining (G_2) and (G_3) similarly, prove that the three spheres (G_1), (G_2), (G_3) intersect on the radical axis of (S_1), (S_2), (S_3). (A similar problem for three circles was discussed in *Mathesis*, 1891, p. 238.)

E 478. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the successive differences of ath powers, in the notation $\Delta r^a=(r+1)^a-r^a$, satisfy the relation $\sum_{n=0}^{a}(-1)^n\Delta^n 1^a=0$.

E 479. *Proposed by Daniel Arany, Budapest, Hungary.*

In the plane of a given triangle ABC, find the locus of a point from which the sides BC and CA subtend equal angles.

E 480. *Proposed by D. E. Lynch, Jr., Brooklyn, N. Y.*

Construct a pentagon whose sides and diagonals are all commensurable. (For definiteness, suppose there are four equal sides, and three equal diagonals.)

SOLUTIONS

E 438 [1940, 569]. *Proposed by J. S. Frame, Brown University.*

If p is any odd prime, show that the decimal expansion of the fraction $1/p$ will repeat in $(p-1)/2$ digits or some factor thereof if and only if $p\equiv \pm 3^k$ (mod 40).

Solution by E. P. Starke, Rutgers University.

The repeating of the decimal expansion in $(p-1)/2$ digits, or some factor thereof, will occur if and only if $10^{(p-1)/2}\equiv 1$ (mod p). But this is Euler's criterion that 10 be a quadratic residue (mod p). Now it is known that 5 is a quadratic

residue of every prime of the form $5n\pm 1$, while 2 is a quadratic residue of every prime of the form $8n\pm 1$. Also, 10 is a quadratic residue (mod p) if both 5 and 2 are residues or non-residues. Thus p must be of both the stated forms, or of neither; *i.e.*, $p=40n\pm r$, where $r=1$, 3, 9, or 27. Since $3^4=81\equiv 1 \pmod{40}$, this is the desired result. (It has been assumed that $p\neq 5$.)

Also solved by the proposer.

E 439 [1940, 569]. *Proposed by J. H. M. Wedderburn, Princeton University.*

ABC is a triangle; lines are drawn external to it, parallel to AC and BC at distances which bear a fixed ratio to the lengths of AC and BC, respectively, making a parallelogram of which CD is one diagonal. If the length of CD is kept constant, show that the locus of C is obtained as follows. Draw two equal circles with centers A and B, and let a line, equal in length to the diameter of the circles, slide with its ends on the two circles; then C is on the locus of the mid-point of this line. (The radius of the circles is determined by any one point on the locus.)

Solution by the Proposer.

Let the parallelogram formed by the sides of the triangle and the two parallels be $CFDE$, the point F being on AC produced; then, from the given construction,

$$CF \sin C = kBC, \qquad CE \sin C = kAC.$$

Let DC produced meet AB in G, and set $\angle ECD=\alpha=\angle BCG$, $\angle FCD=\beta=\angle ACG$; draw CH perpendicular to GC, and let the perpendicular bisectors of BC and AC meet CH in L and M, respectively. Then $2CL=BC/\sin\alpha$, $2CM=AC/\sin\beta$; but $BC/AC=CF/CE=\sin\alpha/\sin\beta$, and hence $CL=CM$. Also, $BC=CF\sin C/k=CD\sin\alpha/k$; hence, $2CL=CD/k$, which is constant. Thus L and M lie on circles with radius $CD/2k$ and centers A and B.

The problem arose in surveying. If the base angles are both increased by the same small amount $\delta\theta$, then, to quantities of the first order, the new rays are parallel to the original ones and give

$$CF = a\delta\theta/\sin C, \qquad CE = b\delta\theta/\sin C.$$

The displacement of the station is easily shown to be

$$CD = 2r\delta\theta/\sin C,$$

where r is the length of the median through C.

It may be of interest to note that $AG/GB=b^2/a^2$.

E 440 [1940, 569]. *Proposed by H. S. M. Coxeter, University of Toronto.*

Prove that, for every integer $n>2$, there are from ten to thirteen n-digit numbers whose digits are the same as the last n digits of their cubes, and that for $n=6$ the thirteen numbers are $5\cdot 10^5\pm 1$, 10^6-1, 5^8, 10^6-5^8, $5\cdot 10^5\pm 5^8$, $5\cdot 10^5\pm(5^8-1)$, $2\cdot 5^8-1$, $10^6-2\cdot 5^8+1$, $2\cdot 5^8-5\cdot 10^5-1$, $15\cdot 10^5-2\cdot 5^8+1$. (G. Fistie, in *Sphinx*, 1935, p. 4, found only nine such numbers.)

Solution by E. P. Starke, Rutgers University.

By hypothesis, our desired x is a number such that

$$x^3 \equiv x \pmod{10^n}. \tag{1}$$

If $n>2$, there are fourteen positive integers, less than 10^n, which satisfy (1); they may be found as follows. We observe that (1) is equivalent to the simultaneous sets of congruences

$$x \equiv 0,\ 1,\ -1 \pmod{5^n}, \tag{2}$$

$$x \equiv 0,\ 1,\ -1,\ 2^{n-1}+1,\ 2^{n-1}-1 \pmod{2^n}. \tag{3}$$

For each choice of a solution of (2) and a solution of (3) there is a solution of (1), and conversely; hence, there are $3\cdot 5$ solutions of (1), including 0 which must be discarded. Of the remaining fourteen, two are even and twelve odd.

The restriction $x \geqq 10^{n-1}$ invalidates the solution $x=1$ and perhaps other solutions. Now, it is easily verified that, if x_0 is an odd solution of (1), then 10^n-x_0 and $\left|\frac{1}{2}10^n \pm x_0\right|$ also are solutions; of this set of four, just one will be less than $10^n/4$. If x_0 is an even solution, so is 10^n-x_0; just one of these will be less than $10^n/2$. Thus the fourteen solutions of (1) fall into three sets of four odd solutions and one set of two even solutions. For a particular n, one member of each of the four sets might conceivably be less than $10^n/10$, and so have less than n digits. It is thus seen that the number of valid solutions is not less than $14-4$, nor more than $14-1$.

For $n=6$, (2) requires $x=5^6k$ or $5^6k\pm 1$. Since $5^6\equiv 9 \pmod{2^6}$, (3) reduces to a set of linear congruences which are easily solved for k. The required values of x are now determined in the form

$$\begin{aligned} 5^6k:&\quad k = 7,\ 25,\ 39,\ 57;\\ 5^6k+1:&\quad k = 7,\ 14,\ 32,\ 46;\\ 5^6k-1:&\quad k = 18,\ 32,\ 50,\ 57,\ 64. \end{aligned}$$

These are seen to be the same as the proposed numbers.

Editorial Note. It is well known (see, for instance, W. W. Rouse Ball, *Mathematical Recreations and Essays*, 11th edition, 1939, p. 60) that there are in general just two "automorphic" numbers, whose n digits are the same as the last n digits of their squares. The digits are easily computed successively, from right to left. According to Fistie, the last 37 digits of these numbers are

$$6\ 109004\ 106619\ 977392\ 256259\ 918212\ 890625$$

and

$$3\ 890995\ 893380\ 022607\ 743740\ 081787\ 109376.$$

There is nothing surprising in the fact that corresponding digits (except the unit digits) add up to 9; for, if N satisfies the congruence

$$x^2 \equiv x \pmod{10^n},$$

so does $10^n - N + 1$.

Starke's four sets of solutions of (1) can be derived by giving x_0 the values 1, $N-1$, N, $2N-1$ in turn, where N is the smaller n-digit automorphic number (possibly with leading digit 0). For example, for $n=4$ we take $x_0=1$, 624, 625, 1249, and deduce the sets

$$\begin{array}{rrrr} 1, & 4999, & 5001, & 9999; \\ 624, & & & 9376; \\ 625, & 4375, & 5625, & 9375; \\ 1249, & 3751, & 6249, & 8751. \end{array}$$

Thus there are eleven 4-digit solutions.

E 442 [1940, 657]. *Proposed by V. Thébault, Le Mans, France.*

In the scale of B there is a perfect square of sixteen digits of the form *abcdefghabcdefgh*. What is the smallest possible value of B?

Partial Solution by E. P. Starke, Rutgers University.

We desire $N^2 = abcdefgh(1+B^8)$. Since $abcdefgh < B^8$, an evident necessary condition is that $1+B^8$ have a square factor greater than 1. If B is odd, B^8-1 is divisible by 2^5; hence B^8+1 is never divisible by 2^2. By Fermat's theorem it is easy to show that, if an odd prime p is a divisor of B^8+1, then p must be of the form 2^4n+1. Thus p is among the primes 17, 97, 113, 193, 241, 257, $\cdots$ (written in the denary scale). Now 17^2 is a factor of $1+B^8$ for any root of the congruence

$$B^8 \equiv -1 \pmod{17^2}.$$

By familiar manipulation we find eight values of B (mod 17^2), of which the smallest is 40. This gives

$$(1 + 40^8) = 17^2K,$$

where $K = 22676816609$ or, in the scale of 40,

$$K = 5\ 21\ 18\ 5\ 10\ 15\ 9.$$

Since K and $4K$ have only seven digits, while $9K$ has a repeated digit, we use instead, $16K = 2\ 8\ 23\ 10\ 4\ 6\ 3\ 24$, and obtain

$$\begin{aligned} 2\ 8\ 23\ 10\ 4\ 6\ 3\ 24\ 2\ 8\ 23\ 10\ 4\ 6\ 3\ 24 &= 16K(1 + 40^8) = (4\cdot 17K)^2 \\ &= (9\ 16\ 18\ 32\ 37\ 25\ 35\ 12)^2. \end{aligned}$$

Thus $B=40$ is a radix of the kind desired. But it has not been shown that for a smaller value of B, B^8+1 might not be divisible by the square of a larger prime than 17.

Also solved (to the same extent) by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3999. *Proposed by G. B. Van Schaack, Michigan State College.*

Let $f(x)$ be a polynomial of degree n with n distinct real roots x_i, $(i=1, 2, \cdots, n)$. Let λ_i be the reciprocal of the slope of the curve $y=f(x)$ at $x=x_i$. Let ρ_j, $(j=1, 2, \cdots, n-1)$, be the algebraic radius of curvature of the curve at the critical point of the curve which lies between x_j and x_{j+1}. (a) Show that if $n>1$, then $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. (b) Show that if $n>2$, then $\rho_1+\rho_2+\cdots+\rho_{n-1}=0$.

4000. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Given the functions $y=f(x)$, $Y=F(X)$, deduce from the transformation formulas

$$\frac{y'}{aX+bY+d}=\frac{-1}{bX+cY+e}=\frac{y-xy'}{dX+eY+f},$$

the following:

$$\frac{Y'}{ax+by+d}=\frac{-1}{bx+cy+e}=\frac{Y-XY'}{dx+ey+f},$$

where a, b, c, d, e, f are arbitrary constants. Give a geometric interpretation; and deduce from it the Legendre transformation.

4001. *Proposed by V. Thébault, Le Mans, France.*

The spheres (O_1'), (O_2'), (O_3'), (O_4'), symmetric to the circumsphere (O) of the tetrahedron $A_1A_2A_3A_4$ with respect to its faces, intersect in sets of three in the points A_1', A_2', A_3', A_4' distinct from the vertices; and the spheres described on the circumcircles (O_1), (O_2), (O_3), (O_4) of the triangles of the faces as great circles intersect in sets of three in A_1'', A_2'', A_3'', A_4''. Show that: (1) the tetrahedrons $A_1'A_2'A_3'A_4'$ and $A_1''A_2''A_3''A_4''$ are homothetic; (2) the centers of the circumspheres of the tetrahedrons $A_1A_2A_3A_4$, $O_1O_2O_3O_4$, $O_1'O_2'O_3'O_4'$, $A_1'A_2'A_3'A_4'$, $A_1''A_2''A_3''A_4''$ are collinear.

SOLUTIONS

3937 [1940, 53]. *Proposed by V. Thébault, Le Mans, France.*

A given circle has the fixed chord BC and a variable point A on its circumference; the midpoints of CA, AB are B_1, C_1; the centers of the inscribed and escribed circles for the angle A of triangle ABC are I and I_a; the parallels to AB through I, I_a meet AC in M, M'; and the parallels to AC through I, I_a meet AB in N, N'. Prove that: (1) the altitudes of triangle AB_1C_1 from B_1 and C_1 pass each through a fixed point; (2) the circles tangent to the sides of angle A with centers at the orthocenters of triangles IMN, $I_aM'N'$ envelop a fixed circle; (3) the locus of the midpoints of MN and $M'N'$ is a limaçon of Pascal.

Editorial Note. The proposer stated the following: (1) The altitudes of triangle AB_1C_1, from B_1 and C_1, cut respectively the perpendiculars to BC through B and C in a fixed point for each. (2) The circle tangent to AB, AC at M and N and the circle tangent to the same lines at M' and N' envelop the circumcircle of triangle BOC, which is fixed, where O is the circumcenter of triangle ABC. (3) The midpoints of MN, $M'N'$ describe a limaçon of Pascal with its double-point at the midpoint P of the arc BC of (O) which is opposite to A, and the director circle has the diameter OP.

We shall give proofs of the above statements. The circle (O), with center O and radius R, has the fixed chord BC and the variable point A; the midpoints of BC, CA, AB are A_1, B_1, C_1. The circles (O_c) and (O_b) on OB and OC as diameters intersect again in A_1 and have radii equal to $R/2$; let E and D be the other extremities of the diameters through A_1 of these two circles, so that $BCDE$ is a rectangle with O as the midpoint of DE. Thus (O_b) and (O_c) are the respective loci of B_1 and C_1. Then C_1E is perpendicular to A_1C_1 and to its parallel AC; and, similarly, B_1D is perpendicular to A_1B_1 and AB; and this proves (1).

Let A_1' be the extremity of the diameter of (O) through A_1, where A_1' and A are on opposite sides of BC. Then $A_1'A$ is the internal bisector of angle A of triangle ABC and contains I and I_a; and it is easily seen by a comparison of angles that $A_1'I$, I_aA_1', $A_1'B$, $A_1'C$ have equal lengths. Let U and U' be the respective midpoints of IA and I_aA; therefore they are centers of the isosceles parallelograms $IMAN$ and $I_aM'AN'$. Then

$$A_1'A = A_1'I + IA = A_1'I + 2UA;$$

and

$$\begin{aligned} A_1'A &= A_1'U' + U'A = A_1'U' + I_aU' = A_1'U' + I_aA_1' + A_1'U' \\ &= A_1'I + 2A_1'U'. \end{aligned}$$

Hence, $A_1'U' = UA$. Also,

$$\begin{aligned} U'U &= U'A + AU = I_aU' + AU = I_aA_1' + A_1'U' + AU \\ &= I_aA_1' = A_1'I = A_1'B = 2R \sin A/2. \end{aligned}$$

From the above we have $A_1'U'=\frac{1}{2}(A_1'A-U'U)$, and

$$A_1'U = A_1'A - UA = A_1'A - \tfrac{1}{2}(A_1'A - A_1'I) = \tfrac{1}{2}(A_1'A + U'U).$$

The locus of the midpoint of $A_1'A$ is a circle (OA_1') on OA_1' as diameter, and $\frac{1}{2}U'U=\frac{1}{2}A_1'B=R\sin A/2$. We now describe the locus of U' and U. Let $A_1'Q$ be a variable secant of $(A_1'O)$ cutting it again in Q; on this secant lay off a segment of length $A_1'B$ with its midpoint at Q. The extremities of this segment are U' and U; and, as the secant varies, U' and U describe part of a limaçon L' with double-point at A_1', consisting of a heart-shaped oval through B and C enclosing an internal oval. Let $A_1'O$ cut (O) again in A_1''; then, as A describes the arc $CA_1''B$ of (O), the points U' and U describe that part of L' obtained by omitting the dented part of the heart oval from C to A_1' and A_1' to B. For the rest of the locus, let us suppose that chord BC lies between A_1' and O. Then, with $(A_1''O)$ as the base circle and with the segment of fixed length $A_1''B$, we obtain in a similar manner the limaçon L''. As A describes the arc $BA_1'C$ of (O), the points U' and U describe that part of L'' which is left after the dented part of the outside oval through C, A_1'', B is removed. Thus the complete locus consists of the two inside ovals of L' and L'' and an oval through B and C consisting of the two undented parts of the outside ovals united at the ends of BC.

In what follows we shall consider only the case where A lies on the arc $CA_1''B$ of (O), since the consideration of A on the rest of (O) is similar, replacing A by $\pi-A$. Set $\theta=\angle OA_1'A$, and let the perpendiculars to AB at N and N' cut $A_1'A$ at the points H_n and $H_{n'}$. Then, from the above we easily find that

$$\begin{aligned} U'A &= R(\cos\theta+\sin A/2), \qquad UA = R(\cos\theta-\sin A/2),\\ A_1'H_n &= R(\cos A\cos\theta+\sin A/2)/\cos^2 A/2,\\ NH_n &= [R(\cos\theta-\sin A/2)\sin A/2]/\cos^2 A/2. \end{aligned}$$

The expressions for $A_1'H_{n'}$ and $N'H_{n'}$ are obtained from the above by replacing the $+$ and $-$ by $-$ and $+$. The circle (O') circumscribing OBC has the radius $r'=R/2\cos A$, and $A_1'O'=R(2\cos A-1)/2\cos A$. Then it may be shown that $O'H_n=r'+NH_n$ by developing $(O'H_n)^2$ by the law of cosines, and reducing the result with the aid of the following identities:

$$\begin{aligned} (1-\tfrac{1}{2}\sec A)^2+\tan^2 A/2\sec^2 A/2 &= (\tfrac{1}{2}\sec A-\tan^2 A/2)^2,\\ \cos^2 A\sec^2 A/2-2\cos A+1 &= \tan^2 A/2,\\ 2\cos A-2+\sec A &= 2(\tfrac{1}{2}\sec A-\tan^2 A/2). \end{aligned}$$

The above reductions serve also to show that $O'H_{n'}=r'-N'H_{n'}$, after noting the change of signs above. Thus, the circles (H_n) and $(H_{n'})$ with the indicated centers and tangent to both AB and AC are also tangent, respectively, externally and internally to (O'), so that (O') is a common part of their envelopes.

We may consider the rest of the envelopes, say for (H_n), as follows. Let a particular (H_n) touch (O') at T and a neighboring (H_n) touch it at T'; then the

common chord of the two circles (H_n) is perpendicular to the straight line joining their centers. Hence, the limit points of the intersection of these two circles are T and the point where the perpendicular from T to the tangent at H_n to the locus of H_n cuts (H_n) again. The locus of H_n is a limaçon, as is easily seen from the above expression for $A_1'H_n$. The determination of the form of the locus of the second limit point does not appear to be easy.

3938 [1940, 53]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres (A), (B), (C), (D) with non-coplanar centers, the four spheres (A'), (B'), (C'), (D') are constructed belonging, respectively, to the four coaxal nets determined by the triads of spheres (B), (C), (D); (C), (D), (A); (D), (A), (B); (A), (B), (C). Find four spheres coaxal, respectively, with the four pairs of spheres (A) and (A'), (B) and (B'), (C) and (C'), (D) and (D'), and forming a coaxal pencil.

Solution by the Proposer.

If the four required spheres are to form a coaxal pencil, it is necessary that their centers be collinear, *i.e.*, that these centers shall lie on a line m meeting the four given lines AA', BB', CC', DD'.

This condition is also sufficient. For, if P, Q, R, S are the traces of a line m on AA', BB', CC', DD', the spheres (P), (Q), (R), (S) having P, Q, R, S for centers and coaxal, respectively, with the pairs of spheres (A) and (A'), *etc.*, will form a coaxal pencil.

Indeed, the sphere (A') belonging to the coaxal net determined by the spheres (B), (C), (D) is orthogonal to any sphere orthogonal to these three spheres and, in particular, to the orthogonal sphere (U) of the four given spheres (A), (B), (C), (D). Furthermore, since the two spheres (A), (A') are orthogonal to (U), any sphere coaxal with them, and in particular the sphere (P), is orthogonal to (U); similarly for the spheres (Q), (R), (S). Consequently, (P), (Q), (R), (S) are coaxal (see Court's *Modern Pure Solid Geometry*, p. 180).

The four lines AA', BB', CC', DD' are met, in general, by two lines m, m' which may or may not be real; hence, the problem may have two, one, or no solutions.

If the four lines considered form a hyperbolic group, the problem will have an infinite number of solutions. If these lines have a point M in common, then the four spheres having this point for center and coaxal, respectively, with the indicated pairs of spheres coincide with the sphere (M) having M for center and orthogonal to the sphere (U).

If the sphere (U) is imaginary, the coaxal pencil determined by (A), (A') is of the intersecting type, and the sphere (P) is necessarily real; the same holds for (Q), (R), (S). If (U) is real, the four spheres (P), (Q), (R), (S) may all be real, or some of them may be imaginary, or they all may be imaginary.

3939 [1940, 53]. *Proposed by J. H. Curtiss, Cornell University.*

Define the function $f(x)$ by the relations

$$f(x) = x \sin(1/x), \qquad x > 0,$$
$$= 0, \qquad x = 0.$$

Show that $|f(x_1)-f(x_0)|/|x_1-x_0|^\alpha$ is bounded for $0 \leqq x_0 \leqq 1$, $0 \leqq x_1 \leqq 1$, if and only if $\alpha \leqq 1/2$.

I. *Solution by E. J. McShane, University of Virginia.*

We may suppose $0 \leqq x_0 < x_1 \leqq 1$. Let x_2 be the greatest number in $[0, x_1]$ for which $f(x_2) = f(x_0)$. It is easily seen that $x_2^{-1} \leqq x_1^{-1} + 2\pi$; for, as x ranges over the interval $[(x_1^{-1}+2\pi)^{-1}, x_1]$, the factor $\sin x^{-1}$ goes through a period of values. If $\alpha \leqq 1/2$, by the inequality of Schwarz we have

$$\begin{aligned}
| f(x_1) - f(x_0) | &= | f(x_1) - f(x_2) | \\
&\leqq \int_{x_2}^{x_1} | f'(x) | \, dx \\
&\leqq | x_1 - x_2 |^{1/2} \left\{ \int_{x_2}^{x_1} | f'(x) |^2 dx \right\}^{1/2} \\
&\leqq | x_1 - x_0 |^{1/2} \left\{ \int_{x_2}^{x_1} [1 + x^{-1}]^2 dx \right\}^{1/2} \\
&\leqq | x_1 - x_0 |^\alpha \{ x_1 - x_2 + 2 \log (x_1/x_2) + x_2^{-1} - x_1^{-1} \}^{1/2} \\
&\leqq | x_1 - x_0 |^\alpha \{ 1 + 2 \log (1 + 2\pi) + 2\pi \}^{1/2}.
\end{aligned}$$

On the other hand, if $\alpha > 1/2$, choose $x_0 = (n\pi)^{-1}$ and $x_1 = [(n+1/2)\pi]^{-1}$, where n is an arbitrary positive integer. We find that

$$\frac{| f(x_1) - f(x_0) |}{| x_1 - x_0 |^\alpha} = 2^\alpha \pi^{\alpha-1} n^{2\alpha-1} \left(1 + \frac{1}{2n}\right)^{\alpha-1},$$

which is unbounded.

II. *Solution by Chang Shou Lien, Yenching University, Peking, China.*

Write

$$\begin{aligned}
I &= x_1 \sin \frac{1}{x_1} - x_0 \sin \frac{1}{x_0} = (x_1 - x_0) \sin \frac{1}{x_1} + x_0 \left(\sin \frac{1}{x_1} - \sin \frac{1}{x_0} \right) \\
&= (x_1 - x_0) \sin \frac{1}{x_1} + 2x_0 \sin \frac{1}{2} \left(\frac{x_0 - x_1}{x_0 x_1} \right) \cos \frac{1}{2} \left(\frac{x_0 + x_1}{x_0 x_1} \right).
\end{aligned} \tag{1}$$

In what follows, it is no loss of generality to assume $x_1 > x_0$. We shall first show that, for $\alpha \leqq 1/2$, the expression

$$R \equiv |I| / |x_1 - x_0|^\alpha \leqq \left|(x_1 - x_0)^{1-\alpha} \sin \frac{1}{x_1}\right| \tag{2}$$
$$+ \left|2x_0(x_1 - x_0)^{-\alpha} \sin \frac{1}{2}\left(\frac{x_1 - x_0}{x_0 x_1}\right) \cos \frac{1}{2}\left(\frac{x_0 + x_1}{x_0 x_1}\right)\right|$$

is bounded for $0 \leqq x_0 < x_1 \leqq 1$; and next, that for $\alpha > 1/2$, we can always find pairs of values (x_0', x_1') in the given range of x_0 and x_1 such that R is unbounded. We note here that for $\alpha \leqq 1$, the first term in the right member of (2) is uniformly bounded.

(*i*) Suppose $\alpha \leqq 1/2$. If $x_0 \leqq (x_1 - x_0)^\alpha$, then R is evidently bounded; if $x_0 > (x_1 - x_0)^\alpha$, then the second term of the right member of (2) becomes

$$\left|2x_0(x_1 - x_0)^{-\alpha} \frac{\sin \theta}{\theta} \cdot \theta\right| \cdot \left|\cos \frac{1}{2}\left(\frac{x_0 + x_1}{x_0 x_1}\right)\right| \leqq \frac{(x_1 - x_0)^\alpha}{x_1} \cdot (x_1 - x_0)^{1-2\alpha},$$
$$\theta \equiv \frac{1}{2}\left(\frac{x_1 - x_0}{x_0 x_1}\right),$$

and consequently R is bounded; and this completes the proof for this part.

Solved also by D. R. Curtiss and Fritz Herzog.

Editorial Note. In solution II, the second part of the proof has been omitted since it is rather long for this simple part of the theorem (see below, the last part of Herzog's work). The proposer's father, D. R. Curtiss, gave the following neat proof of the first part. Set $\phi(x) = (x \sin x^{-1} - a \sin a^{-1})^2 = \phi(x) - \phi(a)$ $= 2(x - a)(\xi \sin \xi^{-1} - a \sin a^{-1})(\sin \xi^{-1} - \xi^{-1} \cos \xi^{-1})$, where the theorem of mean value is used with $a < \xi < x$. Then

$$\phi(x)/(x - a) = 2[\xi \sin^2 \xi^{-1} - a \sin a^{-1} \sin \xi^{-1}$$
$$- \sin \xi^{-1} \cos \xi^{-1} + a\xi^{-1} \sin a^{-1} \cos \xi^{-1}],$$
$$[\phi(x)/(x - a)]^{1/2} \leqq [2(|x| + 3)]^{1/2}.$$

This proves the first part.

Herzog used the sequence $a_n = 1/(n + \frac{1}{2})\pi$, $(n = 0, 1, 2, \cdots)$; and considered $0 \leqq x_0 < x_1 \leqq a_0 = 2/\pi$. Then there exists a positive integer k such that $a_k < x_1 \leqq a_{k-1}$; for x_0 there are two cases: (1) $x_0 \leqq a_{k+1} < a_k < x_1 \leqq a_{k-1}$, (2) $a_{k+1} < x_0 < x_1 \leqq a_{k-1}$. Using inequality reductions for (1), we find that $|f(x_1) - f(x_0)| / |x_1 - x_0|^\alpha \leqq c_1$ (constant). For (2), the theorem of mean value is used with inequalities to find a similar constant c_2, and this completes the first part. If $\alpha > \frac{1}{2}$, we consider $x_0 = a_k$, $x_1 = a_{k-1}$; and then the given expression on the left is

$$2\pi^{\alpha-1} k^{2\alpha-1}\left(1 - \frac{1}{4k^2}\right)^{\alpha-1},$$

which is unbounded; the proof is then complete.

3941 [1940, 114]. *Proposed by N. A. Court, University of Oklahoma.*

The polar plane of a point common to three given spheres, with non-collinear centers, with respect to a variable sphere tangent externally to the three given spheres, describes a coaxal pencil.

I. *Solution by J. S. Frame, Brown University.*

Place a common point of the three given spheres S_i at the origin, and orient the axes so that the three centers lie at the points (a_i, b_i, c) in the plane $z=c$. Denote the radii by r_i. Let the variable sphere S of radius r have its center at the point (X, Y, Z). Then the conditions for external tangency of S and S_i are

$$(X - a_i)^2 + (Y - b_i)^2 + (Z - c)^2 = (r + r_i)^2, \qquad (i = 1, 2, 3), \tag{1}$$

or

$$2Xa_i + 2Yb_i + 2rr_i = X^2 + Y^2 + Z^2 - 2Zc - r^2. \tag{2}$$

The equation of the polar plane of the origin with respect to S is

$$(x - X)(0 - X) + (y - Y)(0 - Y) + (z - Z)(0 - Z) = r^2, \tag{3}$$

or

$$Xx + Yy + Z(z - 2c) = X^2 + Y^2 + Z^2 - 2Zc - r^2, \tag{4}$$

and this plane intersects the plane $z=2c$ in the line

$$Xx + Yy = X^2 + Y^2 + Z^2 - 2\,Zc - r^2, \qquad z = 2c. \tag{5}$$

If we set the right-hand side of equations (2) equal to $2k$, and eliminate the quantities X, Y, r, and k between these three equations and equation (5), we obtain in determinant form a new equation for the line (5) which is independent of the choice of the sphere S, namely,

$$\begin{vmatrix} x & y & 0 & 2 \\ a_1 & b_1 & r_1 & 1 \\ a_2 & b_2 & r_2 & 1 \\ a_3 & b_3 & r_3 & 1 \end{vmatrix} = 0, \qquad z = 2c. \tag{6}$$

Hence, this line (6) is the axis of a pencil of coaxal polar planes of the origin, with respect to the variable spheres S. It lies in the plane parallel to the plane of centers of the three given spheres, which passes through their second common point.

II. *Solution by the Proposer.*

If S, S', S'' are the three external centers of similitude of the three given spheres (A), (B), (C) taken in pairs, the three corresponding spheres of anti-similitude are coaxal and their common circle (c) passes through the point D

common to (A), (B), (C); (see Court's *Modern Pure Solid Geometry*, arts. 616, 602g).

A sphere (V) tangent externally to (A), (B), (C) belongs to the coaxal net (N) conjugate to the coaxal pencil of spheres determined by the circle (c), (*ibid.*, p. 224, art. 690). Thus the polar plane (P) of D for (V) passes through the diametric opposite of D on (c) and is perpendicular to the plane of (c), *i.e.*, (P) passes through the homothetic of the line $SS'S''$ in the homothecy $(D, 2)$ having D for center and 2 for homothetic ratio.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

It is evident that our entire shipment to the British Isles of the MONTHLY for October 1940 (vol. 47, no. 8) was lost at sea. If any American members or subscribers are willing to contribute their copies of this number to replace those lost, the Secretary of the Association will be glad to forward any such copies sent to him for this purpose.

One session of the Association program at Lehigh University on January 1, 1942, will be devoted to contributed papers, briefer than the customary papers on these programs. Abstracts, with a suggestion of the time required, should be in the hands of Professor L. L. Dines, Carnegie Institute of Technology, Pittsburgh, Pa., as early as possible since this is an innovation, and in any case not later than November 1, 1941. These should not be on the one hand purely pedagogical papers or on the other hand papers in advanced research.

Those who have desired copies of R. C. Archibald's *Outline of the History of Mathematics*, which has been out of print for some months, will be glad to know that a new edition is ready, as noted in the advertising pages of this issue of the MONTHLY.

Plane-Strain Distribution of Stress in Elastic Media is the title of a 56-page bulletin (Bulletin 148, 1941) of the Iowa Engineering Experiment Station written by Dr. D. L. Holl. The bulletin presents a method of determining the surface deflections and maximum shearing stresses in a semi-infinite elastic medium, such as earth, induced by various surface loads. A limited number of copies are available for free distribution and may be obtained from the Iowa Engineering Experiment Station, Iowa State College, Ames, Iowa.

The Department of Mathematics at Harvard University has awarded the William Lowell Putnam Memorial Scholarship for 1941 to R. F. Arens of the University of California at Los Angeles. This scholarship is awarded to one of the five ranking highest in the Putnam Competition for the year.

Associate Professor A. A. Albert of the University of Chicago has been promoted to a professorship.

Associate Professor W. L. Ayres of the University of Michigan has been appointed head of the department of mathematics at Purdue University.

President R. W. Brink will represent the Mathematical Association at the celebration of the fiftieth anniversary of the founding of the University of Chicago, September 27–29, 1941.

I. W. Burr of Antioch College has been appointed an assistant professor at Purdue University.

The degree of Sc.D. was conferred upon Professor Emeritus L. E. Dickson of the University of Chicago by Princeton University.

The degree of Sc.D. was conferred upon Professor L. M. Graves of the University of Chicago by Washburn College.

Dr. A. M. Harding, for twenty-two years director of extension service at the University of Arkansas, and formerly professor of mathematics and astronomy, became president of the University July 1, 1941.

H. H. Harman has been appointed chief statistician of the Department of Public Welfare at Springfield, Illinois.

Dr. E. O. Lovett, president of Rice Institute, and formerly professor of mathematics and astronomy at Princeton University, has retired at the age of seventy and will become president emeritus as soon as his successor is chosen.

Professor A. L. O'Toole of Mundelein College has accepted a position with the government in Washington, D. C., and will do statistical work there.

Dr. O. K. Sagen has been appointed chief statistician of the Department of Public Health at Springfield, Illinois.

Assistant Professor I. J. Schoenberg of Colby College has been appointed to an assistant professorship in the Graduate School of the University of Pennsylvania.

Dr. M. S. Webster of Purdue University has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
University of Chicago: Dr. P. V. Reichelderfer
Fenn College: C. W. Topp
Harvard University: Dr. I. E. Segal, L. H. Loomis
University of Michigan: Dr. Sam Perlis
Northwestern University: Dr. E. L. Buell, Dr. J. W. Givens, Dr. S. J. Lawwill

Dr. J. W. Glover, professor emeritus at the University of Michigan, died on July 15, 1941 at the age of seventy-two. He had been connected with the University since 1895 when he became instructor of mathematics. He was a charter member of the Association.

Dr. D. D. Leib, professor of mathematics at Connecticut College since 1918, died suddenly June 15, 1941, at the age of sixty-one. He was a charter member of the Association.

Dr. Ethel I. Moody of Pennsylvania State College was killed in an automobile accident on April 11, 1941. She was a member of the Association since 1931.

Dr. J. E. Trevor, professor emeritus of thermodynamics at Cornell University, died May 4, 1941, at the age of seventy-six. He had been a member of the Cornell faculty for forty-nine years, and a member of the Mathematical Association since 1928.

Miss Alice Winbigler, professor emeritus at Monmouth College, Illinois, died May 27, 1941. She was a charter member of the Mathematical Association.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., May 3; Washington, Pa., October 25.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 10.
MICHIGAN, Ann Arbor, March 15; Detroit, November 15.
MINNESOTA, St. Joseph, May 10.
MISSOURI, Columbia, April 18.
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, April 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, April 4–5.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit, May 3.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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By THORNTON C. FRY

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CONTENTS

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THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Trinity College, Washington, D. C., on Saturday, December 7, 1940, with a morning session, luncheon, and afternoon session. Dean T. McN. Simpson, Jr., chairman of the Section, presided at these sessions.

The attendance was seventy-seven, including the following thirty-four members of the Association: T. E. Berry, Archie Blake, W. E. Bleick, Randolph Church, Abraham Cohen, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, W. C. Flaherty, Michael Goldberg, G. A. Hedlund, F. E. Johnston, L. M. Kells, Solomon Kullback, A. E. Landry, S. B. Littauer, E. J. McShane, Sister Thomas Marie Maloney, W. K. Morrill, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, Harry Siller, T. McN. Simpson, Jr., J. P. Smith, F. W. Sohon, T. H. Taliaferro, J. H. Taylor, C. H. Wheeler, III, G. T. Whyburn, John Williamson.

Lieutenant-Commander P. V. H. Weems, U. S. N. (retired), an authority and author of many books on navigation, gave an address on that subject at the invitation of the Section. A motion was passed expressing the appreciation of the Section to the authorities of Trinity College for their generous hospitality. The next meeting will be held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 10, 1941.

After an address of welcome by Sister Catherine Dorothea, President of Trinity College, the following papers were read:

1. "Curves on ruled surfaces" by Michael Goldberg, Bureau of Ordnance, Navy Department.

2. "On dimension and continuous transformations" by Dr. C. H. Dowker, Johns Hopkins University, introduced by Professor Murnaghan.

3. "The necessary conditions for the problem of Bolza in the calculus of variations" by F. G. Myers, University of Virginia, introduced by Professor Whyburn.

4. "On semi-linear transformations" by Professor Nathan Jacobson, Johns Hopkins University, introduced by Professor Murnaghan.

5. "On the averages of functions" by Dr. R. B. Kershner, Johns Hopkins University, introduced by Professor Murnaghan.

6. "Reducibility conditions for rational quartic and quintic equations" by Professor E. J. Finan, Catholic University of America.

7. "Navigation" by Lieutenant-Commander P. V. H. Weems, U. S. N. (retired), at the invitation of the Section.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Goldberg pointed out that the continuous transformation of plane curves from one form into another can be considered as the varying intersection of a moving plane with a fixed surface. By projecting a plane beam of light

through string models of ruled surfaces, the speaker demonstrated the variation of cubic and quartic curves through their acnodal, cuspidal, crunodal, and degenerate phases. The method was the same as the one demonstrated by Professor Robin Robinson at the twenty-third summer meeting of the Association at Dartmouth.

2. Dr. Dowker pointed out how a metric separable space of dimension n can be mapped essentially on an n-cell but cannot be mapped essentially on a cell of higher dimension. He also stated how a metric separable space R of dimension n can be transformed by identification of all points of a closed set into a space R' which can be mapped essentially on an n-sphere; but if R has dimension less than n this is not possible.

3. Mr. Myers presented the results of several recent papers. The first order necessary conditions for the parametric problem were established without the usual assumption of normality, and the corresponding theorems for the non-parametric problem were obtained by means of a simple transformation. He then established the condition of Jacobi or Mayer by use of "p-normal solutions" of the Jacobi equations (as defined by E. J. McShane). The criteria for conjugate points were developed for both the parametric and non-parametric problems. A brief discussion of focal points and of the accessory boundary value problem was also presented.

4. Professor Jacobson let R be a space of vectors $x = |\xi_1, \xi_2, \cdots, \xi_n|$, where ξ_i take on all complex number values, and defined an anti-linear transformation as one sending x into $x' = (\xi_i')$, where $\xi_i' = \Sigma\alpha_{ij}\bar{\xi}_j$ and the α's are fixed. As a generalization he considered the n-dimensional vector space with respect to any field Φ, and let $\xi \rightarrow \xi^S$ be an automorphism in this field, *i.e.*, $(\xi+\eta)^S = \xi^S + \eta^S$, $(\xi\eta)^S = \xi^S\eta^S$. Then $x \rightarrow x'$, where $\xi' = \Sigma\alpha_{ij}\xi_j^S$ is called a semi-linear transformation with automorphism S. Under a change of coördinates, the matrix $A = (\alpha_{ij})$ is replaced by a matrix $G^{-1}AG^S$. It was pointed out that the theory of semi-linear transformations is equivalent to that of matrices relative to this type of similarity. Professor Jacobson gave a brief outline of this theory, and of the analog in the present set-up of the theory of normal matrices with elements in the field of complex numbers.

5. Dr. Kershner let $x_{r1}, x_{r2}, \cdots, x_{rn_r}$ be a sequence of finite sequences on the interval $[0, 1]$, and also let $N_r(\lambda)$ be defined for a fixed r and for $0 \leqq \lambda \leqq 1$ as the number of elements $x_{rj} \leqq \lambda$. If there is a function $\phi(\lambda)$ such that $\phi(\lambda) = \lim_{r\rightarrow\infty} N_r(\lambda)/n_r$ at every point λ where $\phi(\lambda)$ is continuous, then $\phi(\lambda)$ is called the asymptotic distribution function of $\{x_{rj}\}$. Suppose that $\phi(\lambda)$ has the continuous derivative $\delta(x)$ which does not vanish in $[0, 1]$. It was then shown that

$$\lim_{r\rightarrow 0} \frac{1}{n_r} \sum_{j=1}^{n_r} \frac{f(x_{rj})}{\delta(x_{rj})} = \int_0^1 f(x)dx$$

for every function $f(x)$ which is Riemann integrable on $[0, 1]$.

6. Professor Finan stated that the purpose of his paper was to develop a

practical test by which one may decide whether a given rational quartic or quintic (in one variable) is reducible or not. It was shown that in the case of the quartic the test consisted of examining two sixth degree equations, with rational coefficients, for a rational root. If both equations have a rational root, the quartic is reducible; otherwise, it is not. A similar test was given for the quintic, but in that case one equation of degree 10 is sufficient to determine the reducibility.

7. Commander Weems traced the changes in navigation during the last hundred years. He pointed out the large part the airplane has played in the development of modern navigation, and exhibited a number of charts, tables, and books which are now in use in air-navigation.

C. H. WHEELER, III, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The nineteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of North Carolina, Chapel Hill, N. C., on Friday and Saturday, March 28–29, 1941.

There were in attendance about two hundred persons from forty-seven institutions, including the following fifty-one members of the Association: T. A. Bancroft, D. F. Barrow, Helen Barton, W. S. Beckwith, R. C. Blackwell, J. W. Blincoe, E. T. Browne, Iris Callaway, E. A. Cameron, T. C. Carson, J. B. Coleman, R. W. Cowan, Forrest Cumming, D. C. Dearborn, R. L. Douglass, F. G. Dressel, E. D. Eaves, L. P. Eisenhart, C. H. Frick, L. L. Garner, Leslie J. Gaylord, J. J. Gergen, M. E. Gillis, J. A. Greenwood, C. L. Hair, Archibald Henderson, P. R. Hill, H. M. Hughes, J. A. Hyden, Olive M. Jones, E. S. Kennedy. J. W. Lasley, Jr., E. L. Mackie, W. L. Miser, R. H. Moorman, Sara L. Nelson, W. P. Ott, K. B. Patterson, Annie M. Pegram, W. W. Rankin, Caroline M. Reaves, H. A. Robinson, Lucile L. Rorex, C. L. Seebeck, Jr., R. E. Smith, Ruth W. Stokes, Cora Strong, J. M. Thomas, W. L. Williams, G. F. Woodson, Jr., J. T. C. Wright.

Sessions were held Friday afternoon and evening and Saturday morning. Professor Forrest Cumming, chairman of the Section, presided, except Friday evening and part of Saturday morning when the Section was divided into sub-groups according to the nature of the papers presented. Sub-groups were presided over by Professors J. W. Lasley, Jr., and J. T. C. Wright. On Friday evening a dinner was given in honor of the visiting speaker, Dean L. P. Eisenhart of Princeton University. At this time Professor Archibald Henderson presided.

At the business session on Saturday the following officers were chosen for 1941–42: Chairman, J. W. Lasley, Jr., University of North Carolina; Vice-Chairman, Ruth W. Stokes, Winthrop College; Secretary-Treasurer, H. A. Robinson, Agnes Scott College; Members of Executive Committee: D. H. Ballou, Georgia School of Technology, T. M. Simpson, University of Florida,

F. L. Wren, George Peabody College for Teachers. The next meeting was scheduled for March 1942, at Emory University.

The following twenty-three papers were presented:

1. "Mathematics and the commerce curriculum" by Professor M. A. Hill, Jr., University of North Carolina, introduced by the Secretary.

2. "Sturm's theorem for multiple roots" by Professor J. M. Thomas, Duke University.

3. "A necessary and sufficient condition for bounded Laplace transforms" by Professor N. N. Royall, Winthrop College, introduced by Professor Stokes.

4. "Mathematics and defense" by Professor Forrest Cumming, University of Georgia.

5. "Some interrelations between the philosophy and the mathematics of the last three centuries" by Dr. E. S. Kennedy, University of Alabama.

6. "On isolated essential singularities of functions of two complex variables" by Professor Abe Gelbart, North Carolina State College, introduced by the Secretary.

7. "A new geometrical interpretation of special relativity" by Professor Archibald Henderson, University of North Carolina.

8. "The teaching of mathematics" by Dean L. P. Eisenhart, Princeton University.

9. "Cremona transformations belonging to a family of cubic curves" by Professor Sara L. Nelson, Georgia State College for Women.

10. "The osculating cylinders of the general analytic space curve" by J. A. Pond, University of Georgia, introduced by the Secretary.

11. "On the Euler polynomial coefficients" by H. M. Hughes, University of Tennessee.

12. "The tensor method" by Dean L. P. Eisenhart, Princeton University.

13. "The influence of mathematics on the philosophy of Spinoza" by Dr. R. H. Moorman, Tennessee Polytechnic Institute.

14. "On certain properties of particular solutions of a modified Bessel's equation" by Dr. R. W. Cowan, University of Alabama.

15. "A simple solution of the general quartic" by J. E. Hacke, Jr., University of Georgia, introduced by the Secretary.

16. "On the algebra of pairs" by E. B. Shanks, Vanderbilt University, introduced by Dr. Hyden.

17. "On Boolean algebras" by Dr. Henry Wallman, University of North Carolina, introduced by Professor Henderson.

18. "The concept of generality in secondary mathematics" by Professor W. W. Rankin, Duke University.

19. "Teaching algebraic notation and symbolism" by H. W. Wey, Appalachian High School, Boone, N. C., introduced by Dr. Wright.

20. "Some aspects of the freshman mathematics program at the University of North Carolina" by Professor E. L. Mackie, University of North Carolina.

21. "A demonstration of mathematical models and how they are constructed" by Professor Ruth W. Stokes, Winthrop College.

22. "The rôle of figures in elementary mathematics" by Professor W. L. Miser, Vanderbilt University.

23. "Index theorems for the problem of Bolza in the calculus of variations" by Dr. Katherine E. Hazard, Winthrop College, introduced by Professor Stokes.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Professor Hill gave an account of a new course for commerce students designed by the combined efforts of the mathematics and commerce departments of the University of North Carolina.

2. Professor Thomas proved a form of Sturm's theorem which states the number of roots on an interval closed at the upper end only, and which determines the multiplicity of each root.

3. In terms of the uniform boundedness of a certain related integral, Professor Royall gave a necessary and sufficient condition that a Laplace transform be bounded in a half-plane.

4. In his retiring address, Chairman Cumming discussed certain interesting applications of mathematics used in national defense, and the reports of the War Preparedness Committee.

5. Dr. Kennedy discussed the influence of mathematics on philosophy in the seventeenth century, the lack of any mutual influence in the eighteenth century, and certain interrelationships in the nineteenth century.

6. Professor Gelbart obtained upper and lower bounds for a function of two complex variables which is meromorphic in the neighborhood of an isolated essential singularity.

7. Professor Henderson showed how all the fundamental quantities connected with the Voigt-Lorentz transformation can be exhibited or scaled off on a single diagram constructed by the use of a hyperbola and five circles. This is believed to be the simplest treatment of special relativity ever devised.

8. Whether mathematics holds its rightful place in our school curriculum depends upon the attitude of the teachers and the character of text-books used. Maturity for mathematical thinking is not a matter of the age of the pupil, but the result of experience, and if appropriate experience is withheld, maturity will never develop. Dean Eisenhart believes college teachers are apt to underestimate the quality of performance of which the student is capable.

On its own merits mathematics has a primary place in the curriculum, not when taught merely as a collection of isolated exercises and as a means of making the student work, but when presented in a manner which will sharpen the intellect and quicken the imagination. To learn to state a definition with precision and use it in a proof, to understand the conditions involved in a proposition and see how each condition figures in its proof, to follow through the reasoning resulting in a precise theorem are experiences which are bound to react upon the mind of any student who does his utmost to meet the demands upon his powers.

9. Professor Nelson defined three series of space Cremona transformations by means of a family of cubic curves. Using first a pencil of planes and then a pencil of quadric surfaces she obtained transformations which were, respectively,

symmetric and involutorial. Finally, she considered a transformation obtained by use of a special pencil of planes.

10. Mr. Pond gave seven relations determining the coefficients of an osculating cylinder, and discussed the resolvent sextic and the cases when its roots are all real or all imaginary. Several examples of the osculating cylinders of familiar curves were presented.

11. Mr. Hughes derived several formulas for the general Euler polynomial coefficient and showed that the coefficients could be arranged in a triangle similar to the Pascal triangle with interrelationships paralleling those of the binomial coefficients.

12. In this expository paper, Dean Eisenhart traced the evolution of the tensor method in the works of Descartes, Gauss, Riemann, Ricci, and Einstein.

13. Dr. Moorman pointed out that Spinoza consistently made use of the synthetic geometrical form of reasoning and of mathematical analogies in his treatment of metaphysical problems, natural philosophy, and ethics.

14. Dr. Cowan applied Green's theorem to a particular form of a series solution. The resulting expression for the integrated square was simplified by using several recurrence relations among the functions. Orthogonality conditions over the interval (0, 1) were also obtained.

15. Mr. Hacke's paper appeared in the May, 1941, issue of the MONTHLY.

16. The rational numbers and complex numbers may be defined as ordered pairs (a, b) subject to certain laws of equality and combination. Mr. Shanks showed that, under very light hypotheses, the definitions adopted for equality are unique and to this extent the results serve as a logical basis for these definitions.

17. Dr. Wallman gave a new and simple proof of the theorem that any abstract Boolean algebra is isomorphic to a concrete Boolean algebra.

18. Professor Rankin discussed linear and quadratic functions of algebra from a graphical point of view, concurrency of lines, and generalization of some geometric figures. He pointed out that the nature of mathematics is revealed more through generalization than through application.

19. Mr. Wey traced the history and outlined methods of teaching algebraic symbolism and notation. To the pupil, algebra is a new language. Symbolism should serve as a valuable aid in thinking.

20. Professor Mackie gave some of the results of the system employed at the University of North Carolina since 1938, based upon an attempt to classify the entering freshmen according to individual differences. A report on the success of the examination for advanced standing, with a comparison of the mortality in sections of high, medium, and low ability, was made.

21. Professor Stokes demonstrated a number of models purchased or constructed by her department at Winthrop College to illustrate propositions in solid geometry, analytics, and calculus, and projective geometry.

22. The rôle of figures in elementary mathematics is to supply the student with visual and physical aid just as a modern laboratory does for a student in

science. Professor Miser discussed how to enrich the theory by use of applications.

23. Dr. Hazard considered a problem in the calculus of variations determined by a certain functional subject to a set of end conditions and a set of differential equations.

H. A. ROBINSON, *Secretary*

THE FALL MEETING OF THE KENTUCKY SECTION

A joint meeting of the Kentucky Section of the Mathematical Association of America and the Kentucky Section of the National Council of Teachers of Mathematics was held at the University of Kentucky on Saturday, October 26, 1940. Professor H. A. Wright presided at the morning meeting.

There were fifty-nine in attendance, including the following eighteen members of the Association: N. B. Allison, M. C. Brown, L. W. Cohen, H. H. Downing, Tryphena Howard, W. R. Hutcherson, E. D. Jenkins, Fritz John, C. G. Latimer, F. Elizabeth LeStourgeon, W. L. Moore, R. S. Park, Sallie Pence, D. E. South, Guy Stevenson, Elizabeth C. Strayhorn, S. Helen Taylor, H. A. Wright.

The chairman of the Kentucky Section of the National Council of Teachers of Mathematics, Miss M. Cottell Gregory, presided at the luncheon meeting which followed the morning session.

The following papers were presented:

1. "The training of high school mathematics teachers" by Professor E. D. Jenkins, Eastern Kentucky State Teachers College.

2. "The place of mathematics in secondary education"

a. "From the junior high school view-point" by Russell Garth, Louisville Junior High School, introduced by Professor Wright.

b. "From the senior high school view-point" by Mary E. Clark, Henry Clay High School, Lexington, introduced by Professor Wright.

3. "Mathematical models" by Professor W. L. Moore, University of Louisville.

4. "High school preparation for college mathematics" by Professor W. R. Hutcherson, Berea College.

5. "A pursuit problem" by Professor H. H. Downing, University of Kentucky.

6. "State requirements in mathematics for students and teachers" by Mark Godman, State Department of Education, by invitation.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

1. Professor Jenkins stressed the necessity for a broad, general education of prospective mathematics teachers and described the training program now being used at Eastern Kentucky State Teachers College.

3. The paper by Professor Moore was an exposition of several types of models that are well known and easy to construct. The point of view expressed was that such models stimulate interest in mathematics. A few described were those of surfaces found in certain problems in calculus. These aid the student in visualizing the problem.

4. Professor Hutcherson reviewed the change in the content material of college algebra during the last fifteen years. Berea College has just introduced an intermediate algebra course of three credits for the poorly prepared students, while college algebra is restricted to the better prepared. A college teacher of English, education, agriculture, home economics, as well as of the social and natural sciences, expects the student to be abreast of the fundamentals of arithmetic and the simplest principles of algebra and geometry.

5. Professor Downing discussed in detail his solution of Advanced Problem 3942 of the February, 1940, issue of this MONTHLY. A few remarks were made on the more general types of pursuit problems.

D. E. SOUTH, *Secretary*

THE TWENTY-SEVENTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-seventh annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State College, Manhattan, on Friday and Saturday, April 4–5, 1941, in conjunction with the meetings of the Kansas Academy of Science and the Kansas Association of Teachers of Mathematics. There were three sessions. On Friday evening, Professor E. O. Deere, president of the Academy, presided. Professor G. B. Price, chairman of the Section, introduced Professor Hart. Professor Price presided at the two Saturday sessions. Luncheon for the two mathematics sections was served at the Manhattan Country Club.

The attendance was one hundred and thirty-five, including the following thirty-eight members of the Association: Sister Ann Elizabeth, Sister Mary N. Arnoldy, R. W. Babcock, Wealthy Babcock, E. A. Beito, Lois E. Bell, C. V. Bertsch, Florence L. Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, D. D. Driver, W. H. Garrett, Edison Greer, W. L. Hart, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Thirza A. Mossman, O. J. Peterson, P. S. Pretz, G. B. Price, C. B. Read, C. A. Reagan, B. L. Remick, J. A. G. Shirk, D. T. Sigley, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, J. J. Wheeler, A. E. White, Ferna Wrestler.

The officers elected for the coming year are: Chairman, C. V. Bertsch, Southwestern College; Vice-Chairman, C. F. Lewis, Kansas State College, Manhattan; Secretary, Lucy T. Dougherty, Junior College, Kansas City. The time and place of the next meeting were left to the executive committee.

The following papers and reports were presented, and are numbered in the order of their place on the program:

1. "Mathematics and national preparedness" by Professor W. L. Hart, University of Minnesota.

2. "Kansas resources and the national defense" by Professor L. C. Heckert, Kansas State Teachers College, Pittsburg, introduced by Professor Price.

3. "Mathematics and the junior colleges" by Dean Helen Moore, Kansas State College, Manhattan, introduced by Professor Emma Hyde.

4. "Modified high school algebra" by Professor R. W. Babcock, Kansas State College, Manhattan.

5. "The Council" by Miss Mary A. Potter, President of the National Council of Teachers of Mathematics, introduced by Professor Price.

6. "The rôle of mathematics in airplane design and navigation" by Professor R. G. Smith, Kansas State Teachers College, Pittsburg.

7. "Curricular suggestions related to the national emergency" by Professor W. L. Hart, University of Minnesota.

8. "Applications of statistics in agriculture" by Professor H. C. Fryer, Kansas State College, Manhattan, introduced by Professor Stratton.

9. "Report of the Committee on Placement Test" by
   a. Professor U. G. Mitchell, University of Kansas.
   b. Professor W. T. Stratton, Kansas State College, Manhattan.
   c. Professor W. H. Garrett, Baker University.

10. "Report of the representative on the Board of Governors for Region Number 10, Kansas-Missouri-Nebraska" by Professor O. J. Peterson, Kansas State Teachers College, Emporia.

Abstracts of most of the reports and papers follow, numbered in accordance with their listing above:

1. Professor Hart emphasized the fact that a discussion of the rôle of mathematics in national service or preparedness should not be limited to the period of the present national emergency. He used the general activities of the War Preparedness Committee and the special activities of its Subcommittee on Education for Service, of which he is chairman, as a framework for presenting a picture of certain aspects of mathematics as it enters into industry and military and naval science. The essential parts of Professor Hart's paper were similar to those in *Mathematics in the defense program* by Marston Morse and W. L. Hart, published in the MONTHLY, May, 1941.

2. According to the census, Kansas has lost 82,000 in population in the last ten years. Professor Heckert said that examination shows that this loss is not from the "dust bowl" area, but is scattered over the whole state, and is from the trained, employable, tax-paying, and family-raising groups. One reason is that, although Kansas is rich in the natural resources, coal, oil, gas, lead, zinc, pyrite, salt, gypsum, and a host of other products, including some of the rarer metals, up to the present the processing of these has been done away from this area, giving to other localities the advantage of the jobs (for both scientific and unskilled labor), as well as the financial profits. This work can be done here as economically and certainly more safely than on either coast. Kansas is anxious

to do its part in the defense emergency, but it must be realized that drainin the state of its men, money, and materials, while not setting up reasonable in dustry within the state, will cripple it for years to come. We, as scientists, must do everything in our power to forward the establishment of industries in this area.

4. Dean Babcock said that current instruction in the first unit of secondary algebra does not prepare students for mathematics of college grade. Over one-fourth of the freshmen in Kansas State College in curricula where mathematics is required were subject to scholastic probation or dismissal in January 1941. Mathematics was involved in eighty per cent of these actions. A new course for the Manhattan public schools has been approved, to be effective in September 1941. Those high school freshmen who now plan on subsequent admission to colleges, particularly as engineers, will be segregated into a "modified" course in algebra. This will include constant drill in the principles of formal algebra, with particular emphasis on fundamental operations, factoring, fractions, laws of exponents, equations, graphs, and elementary quadratic equations. It is hoped that such a course will act as a vocational guidance test for those who plan for admission to college.

5. Miss Potter gave in outline the history, aims, organization, and activities of the National Council, as it has grown during its twenty-one years from a group of one hundred and twenty-seven enthusiastic teachers to an association of six thousand members living in every state in the union. Before the outbreak of World War II, there were also two hundred thirty-two foreign members representing every continent and thirty-seven different countries.

6. Professor Smith gave a brief outline of the use of mathematics in the research problems of obtaining high ratios of lift to drag and strength per weight; in the design problems of balance, stability, speed, and maneuverability; and in the navigation problems of dead reckoning and celestial navigation.

7. In presenting curricular suggestions at the college level under the conditions of the present national emergency, Professor Hart first recommended self-instruction by the mathematics teachers in material selected from the following fields: exterior ballistics, artillery fire control and orientation, navigation, aerodynamics and meteorology, cryptanalysis. He gave a brief discussion of the usefulness of such study, and cited texts available for such work. He then made the following recommendations: (1) At an advanced undergraduate or the graduate level, introduction of a course in exterior ballistics and, in the absence of competition from any engineering department, courses in aerodynamics and meteorology. (2) Introduction of a joint senior college major in applied mathematics, perhaps demanding a fifth college year, through coöperation among the departments of mathematics and the physical sciences. (3) No essential emergency changes in standard courses in mathematics. (4) No special "war" courses at an elementary level except possibly for those who have had trigonometry and college algebra, a refresher course in mathematics, and including some solid geometry, spherical trigonometry, and probability, for men entering military service.

9. Kansas Mathematics Test Number Two was prepared and analyzed by the same committee of the Kansas Section which prepared Kansas Mathematics Test Number One. (See the report in this MONTHLY, vol. 47, pp. 509–510.) This Test consisted of twenty simple questions in arithmetic, and twenty in elementary algebra. It was given to 4317 students, 3992 of whom were freshmen, in twenty-nine Kansas colleges.

a. Dean E. B. Stouffer presented the part of the report which had been prepared under the supervision of Professor Mitchell, who was temporarily absent from the state. This included the number of students making each possible score for the several groups with various high school preparatory credits in algebra, and also a complete diagnostic analysis for the entire group of the replies to each of the forty questions in the Test. Results bore many similarities to those obtained for Kansas Mathematics Test Number One, as published in detail in the *Bulletin* of the Kansas Association of Teachers of Mathematics, vol. 15, no. 1, October, 1940, pp. 3–17.

b. Professor Stratton gave the comparative standing of the large high schools based on the scores made on the Tests. His report showed that of the schools where approximately fifty had taken the tests, those ranking high on the first were high on the second, and the school last on the first, retained that rank on the second. All the schools used the same text-books, they presented the same amount of preparation, the teachers in all the schools have had about the same preparation for the work and are considered good teachers. The speaker did not attempt to give a detailed explanation for the wide variation, but suggested that an explanation may be found in the schools themselves and the attitude of the administration toward the subject.

c. Professor Garrett presented the results of a study showing the correlation of the grades made in freshman algebra in twenty-four colleges and universities of Kansas the first semester of 1940–1941 with the scores made in the Kansas Test given at the beginning of the semester. Two charts were presented showing the distribution of grades in the freshman classes in algebra and the relation of these grades to the test scores. The data included the records of fifty-five sections in five-hour algebra totaling 1073 students, and forty-five sections in three-hour algebra totaling 972 students, There was a marked similarity in the results of Test Two with those of Test One.

10. Professor Peterson reported on the progress of the Association during the past year and indicated the program and problems which lie ahead. He gave a brief account of certain aspects of the Association's summer meeting in Hanover, N. H., and the winter meeting in Baton Rouge, La. He outlined significant changes in the organization which followed the recommendations of the Langer committee, called attention to the newly instituted project of the Herbert Ellsworth Slaught Memorial Papers, and discussed briefly the relationship of junior colleges with the Association.

LUCY T. DOUGHERTY, *Secretary*

ON THE MODERN DEVELOPMENT OF CELESTIAL MECHANICS*

C. L. SIEGEL, Institute for Advanced Study

Celestial mechanics deals with the problem of n bodies, or in other words, with the theory of the motion of n particles $P_1, \cdots, P_n$ in three-dimensional euclidean space attracting each other according to Newton's law of gravitation. If we denote by m_k the mass of the particle P_k and by r_{kl} the distance P_kP_l, the potential of gravitation of the system is

$$-U = -\sum_{1\leqq k<l\leqq n} m_k m_l r_{kl}^{-1}.$$

Let x_k, y_k, z_k be the rectangular cartesian coördinates of P_k; then the differential equations of the motion of P_k are

$$m_k\ddot{x}_k = \frac{\partial U}{\partial x_k}, \qquad m_k\ddot{y}_k = \frac{\partial U}{\partial y_k}, \qquad m_k\ddot{z}_k = \frac{\partial U}{\partial z_k}, \qquad (k = 1, \cdots, n).$$

This is a system of $3n$ ordinary differential equations of the second order. If we introduce the components of velocity u_k, v_k, w_k, the equations of motion may be written as a system of $6n$ differential equations of the first order, namely,

$$(1)\quad \dot{x}_k = u_k,\ \dot{y}_k = v_k,\ \dot{z}_k = w_k,\ \dot{u}_k = \frac{1}{m_k}\frac{\partial U}{\partial x_k},\ \dot{v}_k = \frac{1}{m_k}\frac{\partial U}{\partial y_k},\ \dot{w}_k = \frac{1}{m_k}\frac{\partial U}{\partial z_k}.$$

We consider the $6n$ real values $x_k, y_k, z_k, u_k, v_k, w_k$, $(k=1, \cdots, n)$, as the coördinates of a point Q in a space of $6n$ dimensions and we denote by S the manifold of all points Q for which the $n(n+1)/2$ distances r_{kl} are different from 0. The theorem of existence for the solutions of differential equations asserts that through any point Q of S passes exactly one curve of motion. It is the main problem of celestial mechanics to study the topological and analytical properties of this manifold of stream lines in the $6n$-dimensional space S. The complete solution of this problem seems to be far beyond the power of the known mathematical methods, but interesting special results have been obtained during the last 60 years, since the original discoveries of Hill in lunar theory. I will try to give an account of some of the more important of these modern results. They are connected with the names of Bruns, Poincaré, and Sundman.

Let us begin with the investigations of Bruns. They are concerned with the integrals of the system of differential equations (1). If

$$(2)\qquad \dot{\xi}_k = f_k(\xi_1, \cdots, \xi_m, t), \qquad (k = 1, \cdots, m),$$

is a system of differential equations of the first order, then an integral of this system is any function $\phi(\xi_1, \cdots, \xi_m, t)$ which is constant for all solutions of (2). From the condition $\dot{\phi}=0$ we infer the relationship

* Delivered at Rutgers University, February 11, 1941, as a symposium lecture on celestial mechanics, given during the celebration of the 175th anniversary of the founding of the university.

$$\frac{\partial\phi}{\partial t} + \sum_{k=1}^{m} f_k \frac{\partial\phi}{\partial\xi_k} = 0;$$

hence an integral of (2) is any solution ϕ of this partial differential equation. It is proved in the theory of differential equations, that the system (2) of m differential equations of the first order reduces to a system of only $m-1$ differential equations of the first order, if we know any integral which is not identically constant. More generally, if we know r independent integrals of (2), this system may be replaced by a system of only $m-r$ differential equations of the first order, and if we find m independent integrals, then (2) is completely solved.

Since the researches of Euler and Lagrange we know 10 independent integrals of the system (2), namely, the 6 integrals of momentum, the 3 integrals of angular momentum, and the integral of energy. The integrals of momentum are

$$\phi_1 = \sum_{k=1}^{n} m_k u_k, \qquad \phi_2 = \sum_{k=1}^{n} m_k v_k, \qquad \phi_3 = \sum_{k=1}^{n} m_k w_k,$$

$$\phi_4 = t\phi_1 - \sum_{k=1}^{n} m_k x_k, \qquad \phi_5 = t\phi_2 - \sum_{k=1}^{n} m_k y_k, \qquad \phi_6 = t\phi_3 - \sum_{k=1}^{n} m_k z_k;$$

they assert that the center of gravity of the system of particles $P_1, \cdots, P_n$ moves on a straight line with constant velocity. The integrals of angular momentum are

$$\phi_7 = \sum_{k=1}^{n} m_k(y_k w_k - z_k v_k), \quad \phi_8 = \sum_{k=1}^{n} m_k(z_k u_k - x_k w_k), \quad \phi_9 = \sum_{k=1}^{n} m_k(x_k v_k - y_k u_k),$$

and the integral of energy is

$$\phi_{10} = T - U,$$

where

$$T = \tfrac{1}{2} \sum_{k=1}^{n} m_k(u_k^2 + v_k^2 + w_k^2)$$

denotes the kinetic energy of the system of particles. These 10 integrals of (1) have the special property that they are algebraic integrals, *i.e.*, algebraic functions of the variables $t, x_1, \cdots, w_n$. For a long time, mathematicians and astronomers tried in vain to find other simple integrals. Finally Bruns proved that there are no other independent algebraic integrals of the problem of n bodies; in other words, that any algebraic integral of the problem of n bodies is an algebraic function of the known integrals $\phi_1, \cdots, \phi_{10}$. This theorem of Bruns shows us that a further reduction of the problem of n bodies cannot be obtained by algebraic methods. The proof of Bruns's theorem is rather difficult; it uses the same ideas which led Liouville to his theorem that the elliptic functions cannot be expressed as a finite combination of exponential, logarithmic, and algebraic functions.

The researches of Sundman deal only with the special case $n=3$, the problem of three bodies. Since the right-hand sides in (1) are analytic functions of the $6n$ variables $x_1, \cdots, w_n$, we conclude from Cauchy's existence theorem that the solutions of (1) are analytic functions of the independent variable t. We choose a fixed real value t_0 of t and consider the coördinates x_k, y_k, z_k, $(k=1, 2, 3)$, on any curve of motion for increasing real values of $t>t_0$. There are two possibilities: Either these coördinates are regular for all values of $t>t_0$ or there exists a finite number $t_1>t_0$ such that all coördinates are regular for $t_0 \leqq t < t_1$, but at least one coördinate is singular for $t=t_1$. Let us now investigate the behaviour of the motion of the three bodies for $t \rightarrow t_1$. Sundman proved that in the moment $t=t_1$ we have either a simple collision or a general collision; this means that either two of the three bodies dash together or all three of them dash together, at a certain point of the space. Moreover, he found that a general collision can only occur if the three integrals of angular momentum have the value zero. If we assume that this is not the case, we have a simple collision for $t=t_1$. Sundman proved that then the coördinates, as functions of t, have a branch-point of the second order for $t=t_1$; this means that they can be represented in the neighborhood of $t=t_1$ by convergent power series of the uniformizing variable $(t-t_1)^{1/3}$ with real coefficients. Hence, we may consider the analytic continuation of these functions beyond the branch-point $t=t_1$. According to the three possible determinations of the cube root, we find three different analytic continuations beyond the branch-point, and exactly one of these branches will be real for real values $t>t_1$. Choosing this real branch for every one of the nine coördinates, we obtain a real analytic continuation of the motion beyond the point of simple collision. Of course this analytic continuation has no physical meaning, but it is important for the mathematical investigation of the differential equations.

Consider now the behaviour of the analytic functions x_k, y_k, z_k for increasing real values of $t>t_1$. There are again two possibilities: Either they are regular for all finite $t>t_1$ or there exists a first singularity $t=t_2$. Since we have assumed that the integrals of angular momentum are not all zero on our orbit, the singularity $t=t_2$ is again a simple collision; that means a branch-point of the second order, and we may construct the real analytic continuation of the motion beyond this branch-point $t=t_2$. It may happen that we find in this manner an infinite number of times $t_1, t_2, t_3, \cdots$ of simple collisions. Now Sundman proved that this infinite increasing sequence $t_1, t_2, t_3, \cdots$ does not tend to a finite limit; in other words, that the times of the single collisions do not cluster at a finite value of the time. Consequently the motion may be continued for all real finite values of the time greater than the initial value $t=t_0$. The same is obviously true for decreasing values of $t<t_0$. Therefore we have a real analytic continuation of the motion for all finite real values of the time t with the following property: If $t=\tau$ is no point of collision, then the coördinates are power series of the variable $t-\tau$ in the neighborhood of $t=\tau$; if $t=\tau$ is a point of collision, then the coördinates are, in this neighborhood, power series of the variable $(t-\tau)^{1/3}$.

These power series will not converge for all values of t, but only in a certain

neighborhood of the special value τ. Sundman made the important discovery that the whole motion may be represented by one single power series, if we introduce instead of $t-\tau$ or $(t-\tau)^{1/3}$ a certain new uniformizing variable s defined by

$$s = \int_{t_0}^{t} (U + 1)dt. \tag{3}$$

If t runs from $-\infty$ to $+\infty$, the new variable s does the same. The coördinates x_k, y_k, z_k are now regular functions of s, for all finite real values of s, and the same holds for t as a function of s. Sundman proved that the singularities of x_k, y_k, z_k, t as functions of the complex variable s do not cluster towards the real s-axis; in other words, that there exists a certain strip containing the real s-axis which is completely free from singularities of those functions. The proof of this statement depends upon two lemmas which are of special interest. The first lemma asserts that throughout the whole motion the perimeter of the triangle $P_1P_2P_3$ has a positive lower bound not involving the time t, and the second lemma is the following one: Consider for any moment $t=\tau$ that point P_k which is opposite to the smallest side of the triangle, then the velocity $(u_k^2+v_k^2+w_k^2)^{1/2}$ of this point has a finite upper bound not depending upon τ. Applying these two lemmas, Sundman proved the existence of a strip $-\delta < I(s) < \delta$ not containing any singularity of x_k, y_k, z_k, t; here δ is a positive number depending only upon the initial conditions, and $I(s)$ is the imaginary part of s. By the substitution

$$p = \frac{e^{\pi s/2\delta} - 1}{e^{\pi s/2\delta} + 1}, \qquad s = \frac{2\delta}{\pi} \log \frac{1+p}{1-p}, \tag{4}$$

the strip is conformally mapped onto the unit circle $|p| < 1$. The segment $-1 < p < 1$ corresponds to the real s-axis, hence also to the real t-axis, and x_k, y_k, z_k, t are regular functions of the uniformizing parameter p in the whole unit circle $|p| < 1$. Therefore they may be expressed by power series of the variable p converging for $|p| < 1$. If p runs from -1 to $+1$, the time t runs from $-\infty$ to $+\infty$ and the whole motion is represented by those power series. This is Sundman's final result:

If the three integrals of angular momentum are not all zero, then the coördinates x_k, y_k, z_k, $(k=1, 2, 3)$, *and the time t can be represented by power series of the parameter p defined in* (3) *and* (4). *The power series converge for* $|p| < 1$, *and we obtain the whole curve of motion for* $-1 < p < 1$.

I have spoken rather explicitly of the methods and the results of Sundman, because his important papers have been studied by only very few people. The researches of Poincaré are more widely known; they were published in his famous *Méthodes nouvelles de la mécanique céleste*. It is impossible to give in brief a complete account of the different ingenious and fertile ideas of his work, and I will restrict myself to a sketch of his investigations concerning periodical orbits.

We consider again a system of differential equations of the first order,

$$\dot{\xi}_k = f_k(\xi_1, \cdots, \xi_m), \qquad (k = 1, \cdots, m), \tag{5}$$

and assume now that the functions f_k do not involve explicitly the independent variable t and that they have continuous partial derivatives of the first order, in a certain domain D. Moreover, we assume that there exists an integral not depending upon t, *i.e.*, a function $\phi(\xi_1, \cdots, \xi_m)$ which is constant for any solution of (5); let also ϕ have continuous partial derivatives in any point of D. These conditions are fulfilled in the special case of our system (1). The general solution of (5) for the initial conditions $t=0$, $\xi_k=\alpha_k$, $(k=1, \cdots, m)$, has the form

$$\xi_k = g_k(t, \alpha_1, \cdots, \alpha_m), \qquad g_k(0, \alpha_1, \cdots, \alpha_m) = \alpha_k, \qquad (k = 1, \cdots, m).$$

If we know in D a periodical solution with the period $\tau > 0$ and the initial values $\xi_k = \beta_k$, $(k=1, \cdots, m)$, then the relationship

$$g_k(\tau, \beta_1, \cdots, \beta_m) = \beta_k, \qquad (k = 1, \cdots, m), \tag{6}$$

holds. By the theorem of uniqueness, the condition (6) is also sufficient for periodicity with the period τ. We consider all orbits through points in the neighborhood of the point $Q_0 = (\beta_1, \cdots, \beta_m)$ and try to find other periodical solutions in D with a slightly different period σ. Let us assume that the given closed orbit through Q_0 is not tangential to the plane $\xi_1 = \beta_1$; this means that $f_1(\beta_1, \cdots, \beta_m) \neq 0$. The solution passing for $t=0$ through the point $Q = (\beta_1, \alpha_2, \cdots, \alpha_m)$ of that plane cuts it for a second time $t = \sigma > 0$ in a point $(\beta_1, \xi_2, \cdots, \xi_m)$, and σ lies in an arbitrarily small neighborhood of τ, if only the differences $\beta_k - \alpha_k$, $(k=2, \cdots, m)$, are sufficiently small. This orbit through Q will be closed if $\sigma, \alpha_2, \cdots, \alpha_m$ satisfy the m equations $h_1 = 0, \cdots, h_m = 0$, where $h_1, \cdots, h_m$ denote the following functions of $\sigma, \alpha_2, \cdots, \alpha_m$:

$$h_1 = g_1(\sigma, \beta_1, \alpha_2, \cdots, \alpha_m) - \beta_1, \qquad h_k = g_k(\sigma, \beta_1, \alpha_2, \cdots, \alpha_m) - \alpha_k,$$
$$(k = 2, \cdots, m).$$

If we assume that not all partial derivatives $\partial\phi/\partial\xi_k$, $(k=1, \cdots, m)$, of the integral ϕ vanish at the point Q_0, we infer from the relationship

$$\phi(h_1 + \beta_1, h_2 + \alpha_2, \cdots, h_m + \alpha_m) = \phi(\beta_1, \alpha_2, \cdots, \alpha_m)$$

that one of the m equations $h_k = 0$ follows from the other $m-1$, for sufficiently small $\beta_k - \alpha_k$. Consequently we have only to solve $m-1$ equations $h_k = 0$ for the m unknown quantities $\sigma, \alpha_2, \cdots, \alpha_m$, and we know the particular solution $\sigma = \tau$, $\alpha_k = \beta_k$, $(k=2, \cdots, m)$. By a well known theorem concerning implicit functions, our system of equations has for any given σ in a sufficiently small neighborhood of τ a uniquely determined solution $\alpha_2, \cdots, \alpha_m$, if the functional determinant of the $m-1$ left-hand sides h_k as functions of the variables $\alpha_2, \cdots, \alpha_m$ does not vanish for $\sigma = \tau$, $\alpha_2 = \beta_2, \cdots, \alpha_m = \beta_m$. Under this last assumption we obtain a one-parameter manifold of closed orbits in the neighborhood of the given closed orbit.

If we want to apply this method of Poincaré, we have to know already a periodical solution, and the problem arises how to find such an initial solution. This problem is of a different character, and Poincaré tried to solve it by topological methods. Let us consider the solution of (5) through any point Q of the surface $\xi_1=\beta_1$ for increasing values of t, and let us assume that it cuts again this surface at a point Q'. In this manner a topological mapping of the surface onto itself is defined, and obviously the periodical solutions correspond to the fixed points $Q=Q'$ of this mapping. The problem of finding closed orbits is therefore transformed into the problem of proving the existence of fixed points under a topological mapping of a surface onto itself. Poincaré suggested that under certain conditions a fixed point will really exist, and Birkhoff later proved this suggestion. In his researches on surface transformations, Birkhoff obtained several other results which have interesting applications to dynamical problems.

I hope to have explained that some important steps have been made since the first ingenious researches of Hill. However, there remain still a great number of unsolved problems in celestial mechanics, *e.g.*, the problems of stability and transitivity, and it seems that the solution of the main problems will require new methods of analysis.

PROPERTIES OF POINTS, LINES, AND CIRCLES ASSOCIATED WITH A POINT ON AN ELLIPSE

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Properties of certain points and straight lines associated with a point on a parabola and a point on a hyperbola have been discussed by J. R. Musselman and the author [1], [2]. It is the purpose here to discuss the properties of certain points, straight lines, and circles associated with a point on an ellipse.

1. Points. If we use complex coördinates, the map equation of an ellipse may be written in the form

$$z=\frac{k^2t^2-1}{2t}, \tag{1}$$

where t is a turn and k is real. Equation (1) means that as t runs around the unit circle in the t-plane, z runs around an ellipse in the z-plane. The self-conjugate equation of this ellipse may be obtained by writing the conjugate of equation (1) and then eliminating t from (1) and its conjugate. If this is done we obtain

$$4(z+k^2\bar{z})(k^2z+\bar{z})=(k^4-1)^2. \tag{2}$$

The center of this ellipse is at the origin, its minor axis is along the axis of reals, the lengths of its major and minor axes are k^2+1 and k^2-1, respectively, and its foci are at $\pm ki$. If we now consider the vertices of a regular polygon of n sides inscribed in the unit circle in the t-plane, these vertices P_a may be represented

by the values $t=t_1T^a$, $(a=0, 1, 2, \cdots, n-1)$, where t_1 and T are turns, t_1 is fixed, and $T^n=1$. The n points P_a on the unit circle in the t-plane determine n points Q_a on the ellipse in the z-plane. The points Q_a are determined by the map equation

$$z = \frac{k^2t_1^2T^{2a} - 1}{2t_1^2T^{2a}}. \tag{3}$$

Furthermore, the equations of the major auxiliary and minor auxiliary circles of the ellipse are

$$4z\bar{z} = (k^2 + 1)^2, \tag{4}$$

and

$$4z\bar{z} = (k^2 - 1)^2, \tag{5}$$

respectively.

LEMMA. *If z_1, z_2, z_3 are the complex coördinates of any three distinct points, the area deterined by the three points is given by*

$$A = -\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}, \tag{6}$$

where $i^2=-1$.

Let Q_a and Q_b be any two points of the set determined by equation (3). Then the area determined by Q_a, Q_b, and the center of the ellipse is given by

$$A = -\frac{1}{4i}\begin{vmatrix} \dfrac{k^2t_1^2T^{2a} - 1}{2t_1T^a} & \dfrac{k^2 - t_1^2T^{2a}}{2t_1T^a} & 1 \\ \dfrac{k^2t_1^2T^{2b} - 1}{2t_1T^b} & \dfrac{k^2 - t_1^2T^{2b}}{2t_1T^b} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{(1 - k^4)(T^{a-b} - T^{b-a})}{16i}. \tag{7}$$

Since this result is independent of t_1, it will be constant if $a-b$ is constant. Hence we have the following:

THEOREM. *The area of the triangle formed by Q_a, Q_b, and the center of the ellipse will be constant if $a-b$ is constant.*

If $a-b=1$, the area of the triangle becomes $(1-k^4)(1-T^2)/16iT$, and the area of the inscribed polygon determined by the n points Q_a is $n(1-k^4)(1-T^2)/16iT$.

Suppose now that we wish to have Q_a and Q_b to be the ends of a diameter. We must then have

$$\frac{k^2t_1^2T^{2a}-1}{2t_1T^a}=-\frac{k^2t_1^2T^{2b}-1}{2t_1T^b},$$

from which $T^a+T^b=0$. This requires that $|a-b|=n/2$. Hence Q_a and Q_b are the ends of a diameter if, and only if, n is an even number and the difference between a and b is equal to one-half of n. If $n=4$, the four points are the ends of two diameters, and it can be readily shown that the tangents at the ends of one diameter are parallel to the other diameter. Hence the two diameters are conjugate. The above facts lead to the well known result, that the triangle formed by joining the extremities of conjugate diameters of an ellipse has a constant area [**3**].

2. **Straight lines.** Let us now write the equation

$$z=\frac{k^2tt_1-1}{t+t_1}, \tag{8}$$

and consider t_1 as fixed. For a variable t, this represents a straight line which passes through Q_0 [4]. Its self-conjugate equation, obtained by writing the conjugate equation of (8) and eliminating t between (8) and its conjugate, is

$$z(k^2+t_1^2)+\bar{z}(k^2t_1^2+1)=(k^4-1)t_1. \tag{8'}$$

Let us now consider t_1 as a parameter in equation (8′) and find the envelope of the family of straight lines thus defined. The partial derivative of (8′) with respect to t_1 is

$$2t_1z+2t_1k^2\bar{z}=k^4-1. \tag{9}$$

If we eliminate t_1 from equations (8′) and (9) we obtain equation (2) which is the equation of the required envelope. Hence the straight line (8′) is tangent to the ellipse (2) at Q_0.

From equation (8′) we find that the clinant of this tangent line is $-(k^2t_1^2+1)/(k^2+t_1^2)$. Also, the clinant of the tangent to the ellipse at the point Q_a is $-(k^2t_1^2T^{2a}+1)/(k^2+t_1^2T^{2a})$. If these two tangents are to be parallel, we must have

$$\frac{k^2t_1^2+1}{k^2+t_1^2}=\frac{k^2t_1^2T^{2a}+1}{k^2+t_1^2T^{2a}},$$

which gives $T^{2a}=1$. Therefore $a=n/2$. Hence, if two distinct tangents to the ellipse at the points Q_a are to be parallel, n must be an even number.

The equation of the straight line through Q_a and Q_b is

$$2z(k^2+t_1^2T^{a+b})+2\bar{z}(k^2t_1^2T^{a+b}+1)=t_1(k^4-1)(T^a+T^b).$$

If this straight line is to be parallel to the tangent at Q_0, we must have

$$\frac{k^2t_1^2T^{a+b}+1}{k^2+t_1^2T^{a+b}}=\frac{k^2t_1^2+1}{k^2+t_1^2},$$

which leads to the equation $T^{a+b}=1$. Hence $a+b=n$.

The intersection of the tangents at Q_0 and Q_1 is given by

$$z=\frac{k^2t_1^2T-1}{t_1(1+T)},$$

and of the tangents at Q_1 and Q_2 by

$$z=\frac{k^2t_1^2T^3-1}{t_1T(1+T)}.$$

The area of the triangle formed by these two points and the center of the ellipse is

$$A=\frac{(k^4-1)(1-T)}{4i(1+T)}.$$

Since this result is independent of t_1 we have the following:

THEOREM. *The polygon formed by the tangents to the ellipse at the points Q_a has a constant area.*

Since

$$z=\frac{at+b}{ct+d}$$

is the map equation of a straight line if d/c is a turn, one may write any number of equations, each of which will reduce to

$$z=\frac{k^2t_1^2-1}{2t_1}$$

when $t=t_1$. Each equation will then represent a straight line through Q_0, and will have certain properties in connection with the ellipse. Equation (8) was one such equation, which represented the tangent to the ellipse at Q_0. We shall now exhibit some others.

Let us now consider the equations

$$z=\frac{(kt-1)(kt_1+1)}{t+t_1}, \tag{10}$$

$$z=\frac{(kt_1-1)(kt+1)}{t+t_1}. \tag{11}$$

These equations represent two straight lines through Q_0. Their self-conjugate equations are

$$(10') \qquad z(k + t_1)^2 + \bar{z}(kt_1 + 1)^2 = (k^2 - 1)(kt_1 + 1)(k + t_1),$$

$$(11') \qquad z(k - t_1)^2 + \bar{z}(kt_1 - 1)^2 = (k^2 - 1)(kt_1 - 1)(k - t_1),$$

respectively. We may now prove the following:

THEOREM. *The straight lines* (10) *and* (11) *have the same envelope, which is the minor auxiliary circle of the ellipse.*

The proof is carried through by finding the partial derivative of equation (10′) or (11′) with respect to t_1 and then eliminating t_1 from (10′) or (11′) and the derived equation. When this is done, we obtain equation (5) and the theorem is proved.

If we solve equations (10′) and (5) simultaneously and (11′) and (5) simultaneously, we will obtain the points of contact of these two straight lines with their envelope. These points are given by

$$Q_{34}: \quad z = \frac{(k^2 - 1)(kt_1 + 1)}{2(k + t_1)}, \qquad Q_{56}: \quad z = \frac{(k^2 - 1)(2t_1 - 1)}{2(k - t_1)},$$

respectively. The equation of the straight line through the points Q_{34} and Q_{56} is

$$(12) \qquad z(k^2 - t_1^2) + \bar{z}(k^2 t_1^2 - 1) = (k^2 - 1)^2 t_1.$$

If we solve equations (8′) and (12) for z we obtain

$$M: \quad z = \frac{(k^2 - 1)t_1}{t_1^2 + 1}.$$

Since the coördinate of M is real and the minor axis of the ellipse lies along the real axis, we have the following:

THEOREM. *The polar of Q_0 with respect to the minor auxiliary circle and the tangent to the ellipse at Q_0 intersect on the minor axis of the ellipse.*

Let Q_0 and Q be any two points on the ellipse determined by $t = t_1$ and $t = t_2$, respectively. Tangents to the ellipse at Q_0 and Q intersect in the point

$$A: \quad z = \frac{k^2 t_1 t_2 - 1}{t_1 + t_2}.$$

The intersection of the polars of Q_0 and Q with respect to the minor auxiliary circle of the ellipse is the point

$$B: \quad z = \frac{(k^2 - 1)(k^2 t_1 t_2 + 1)}{(k^2 + 1)(t_1 + t_2)}.$$

The equation of the straight line AB is

$$(13) \qquad (z + \bar{z})(t_1 + t_2) = (k^2 - 1)(1 + t_1 t_2).$$

Hence the straight line determined by the intersection of any two tangents to the ellipse and the intersection of the polars of the points of contact with respect to the minor auxiliary circle of the ellipse is always perpendicular to the minor axis of the ellipse.

Since the foci of the ellipse are given by $z = \pm ki$, the equations of the straight lines through Q_0 and the foci are

$$z(k \pm it_1)^2 + \bar{z}(kit_1 \pm 1)^2 = 2(k^2 - 1)i(t_1^2 + 1).$$

The straight lines associated with the ends of a diameter conjugate to the diameter through Q_0 and corresponding to the straight lines (10) and (11) are given by the equations

$$z(k \pm it_1)^2 + \bar{z}(kit_1 \pm 1)^2 = (k^2 - 1)(k \pm it_1)(kit_1 \pm 1).$$

Hence we have the following:

THEOREM. *The straight lines drawn through a point Q_0 on an ellipse and the foci of the ellipse are parallel to the straight lines drawn through the ends of a diameter conjugate to the diameter through Q_0 and tangent to the minor auxiliary circle.*

Let us now consider the equation

$$z = \frac{k^2t_1^2 - 1}{t + t_1}. \tag{14}$$

It may be easily shown that this represents a straight line through Q_0 and perpendicular to the diameter of the ellipse through Q_0.

3. Circles. With each one of the straight lines (8), (10), (11), (14) there are associated two circles. We shall find their equations.

Consider the equations

$$z = \frac{k^2tt_1 - 1}{2t}, \tag{15}$$

$$z = \frac{k^2tt_1 - 1}{2t_1}. \tag{16}$$

They represent two circles whose self-conjugate equations may be found in the usual way to be

$$4t_1z\bar{z} - 2k^2z - 2k^2t_1^2\bar{z} + (k^4 - 1)t_1 = 0, \tag{15'}$$

$$4t_1z\bar{z} + 2t_1^2z + 2\bar{z} - (k^4 - 1)t_1 = 0, \tag{16'}$$

respectively. If we subtract equation (15′) from equation (16′) we obtain equation (8′). Hence, the radical axis of the circles (10) and (11) is the tangent to the ellipse at the point Q_0. The common points of these circles are Q_0 and the foot of the perpendicular drawn from the center of the ellipse to the above tangent.

This result is easily obtained from equations (15′) and (16′). We are now in position to prove the following:

THEOREM. *The circles* (15′) *and* (16′) *are tangent to the major and minor auxiliary circles of the ellipse and the points of contact lie on two straight lines which pass through* Q_0 *and are parallel to the axes of the ellipse.*

If, in equations (15′) or (16′), we consider t_1 as a parameter, we find in the usual way that the envelope in either case consists of the two circles (4) and (5). This proves the first part of the theorem. Simple computations show that the points of contact with the minor auxiliary circle are the points $z=(k^2-1)t_1/2$ and $z=(k^2-1)/2t_1$, and that these two points determine a straight line through Q_0 perpendicular to the minor axis of the ellipse; similarly, the points of contact with the major auxiliary circle are $z=(k^2+1)t_1/2$ and $z=-(k^2+1)/2t_1$, and these points determine a straight line through Q_0 perpendicular to the major axis of the ellipse.

Let us now consider the set of points Q_a. With each of these points there will be associated a circle corresponding to the circle (15). Call this circle (15_a). Similarly, there will be a circle (16_b) corresponding to the circle (16). We now wish to find the conditions under which the circle (15_a) will be tangent to the circle (16_b). The equations of these circles are

$$4t_1T^a z\bar{z} - 2k^2 z - 2k^2t_1^2T^{2a} + (k^4 - 1)t_1 = 0, \tag{17}$$

$$4t_1T^b z\bar{z} + 2t_1^2T^{2b} z + 2\bar{z} - (k^4 - 1)t_1 = 0, \tag{18}$$

respectively. If we solve equations (17) and (18) for z and set up the conditions that both values of z shall be equal, we obtain

$$t_1^2T^{a+b} = \pm 1. \tag{19}$$

Equation (19) shows that if $a+b=n$ and we use the ends of the axes of the ellipse for starting-points for the series of points Q_a, the tangency conditions will be fulfilled.

If we consider any two points Q_0 and Q on the ellipse, determined by $t=t_1$ and $t=t_2$, respectively, the condition that the circle (15) at Q_0 is tangent to the circle (16) at Q is $t_1t_2=\pm 1$.

Let us now consider the equations

$$z = \frac{(kt - 1)(kt_1 + 1)}{2t}, \tag{20}$$

$$z = \frac{(kt - 1)(kt_1 + 1)}{2t_1}\cdot \tag{21}$$

These are the map equations of two circles through Q_0 whose self-conjugate equations are

$$(20') \qquad 4t_1z\bar{z} - 2k(k + t_1)z - 2kt_1(kt_1 + 1)\bar{z} + (k^2 - 1)(k + t_1)(kt_1 + 1) = 0,$$

$$(21') \qquad 4t_1z\bar{z} + 2t_1(k + t_1)z + 2(kt_1 + 1)\bar{z} - (k^2 - 1)(k + t_1)(kt_1 + 1) = 0.$$

If we subtract equation (20′) from equation (21′) we obtain equation (10′). Hence, the straight line (10′) is the radical axis of the circles (20′) and (21′). It may also be easily shown that the points of intersection of these two circles are Q_0 and Q_{34}.

Similarly, the map equations

$$(22) \qquad z = \frac{(kt_1 - 1)(kt + 1)}{2t},$$

$$(23) \qquad z = \frac{(kt_1 - 1)(kt + 1)}{2t_1},$$

represent two circles whose self-conjugate equations are

$$(22') \qquad 4t_1z\bar{z} - 2k(k - t_1)z - 2kt_1(kt_1 - 1)\bar{z} + (k^2 - 1)(kt_1 - 1)(k - t_1) = 0,$$

$$(23') \qquad 4t_1z\bar{z} - 2t_1(k - t_1)z - 2(kt_1 - 1)\bar{z} - (k^2 - 1)(kt_1 - 1)(k - t_1) = 0.$$

Then, as above, it may be shown that the radical axis of these two circles is the straight line (11′), and that the second point of intersection is the point Q_{56}.

Finally, we note that the map equations

$$(24) \qquad z = \frac{k^2t_1^2 - 1}{2t_1},$$

$$(25) \qquad z = \frac{k^2t_1^2 - 1}{2t},$$

determine two circles, equation (24) determining the point circle Q_0. Equation (25) determines a circle through Q_0 with its center at the center of the ellipse. Its self-conjugate equation is

$$(25') \qquad 4t_1^2z\bar{z} = (k^2 - t_1^2)(k^2t_1^2 - 1).$$

The radical axis of these two circles is the straight line (14).

We have now listed four pairs of circles. They have a number of interesting properties which are easily established. We list a few of them without proof.

1. The intersections of each pair of circles lie on the circle with Q_0O as diameter, where O is the center of the ellipse.

2. The eight centers lie by threes on four straight lines which determine a parallelogram. Four of the centers are the vertices of this parallelogram, and the other four centers determine another parallelogram.

The figure also has a wealth of concurrency properties such as the following: The polar of Q_0 with respect to the minor auxiliary circle, the perpendicular from

Q_0 to the minor axis, and the perpendicular from the center of the ellipse to the tangent at Q_0 are concurrent.

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A SIMPLE MECHANICAL APPLICATION OF AN ELLIPTIC INTEGRAL*

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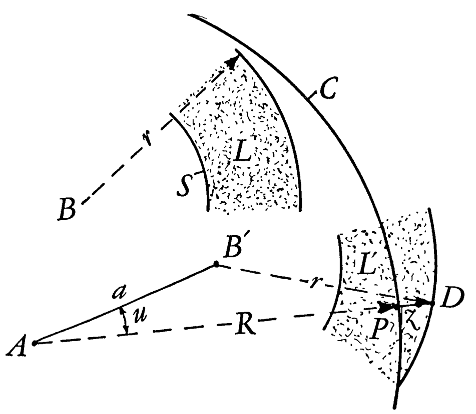
In the present paper a very simple mechanical application of the Legendre integral of the second kind is given. It concerns the evaluation of forces in a mechanical brake of a type commonly used in automobiles, which diagrammatically may be represented in the following manner: A rigid hollow circular cylinder C rotates about its geometric axis A. (In the diagram, only a part of the cylinder is represented for clarity.) A circular cylindrical surface S with axis B is inside C; B is parallel to A; and C and S are considered as rigid surfaces. An elastic strip L of constant rectangular cross-section is fixed on S. When the brake



is put into action, L is displaced in a direction perpendicular to the axis B and is pressed against the cylinder C. Let B' be the position of B after such a displacement and L' the position which L would take if there were no cylinder C. It is clear that since C actually exists, the elastic strip will be compressed and its deformation z at a point P of C, measured along the radius through A, is $z=\overline{PD}$. If R is the radius of C (*e.g.*, $R=\overline{AP}$) and r the external radius of L (*e.g.*, $r=\overline{B'D}$), it is clear from the triangle $AB'D$ that

* Presented at the meeting of the Minnesota Section of the Mathematical Association of America at Mankato, Minn., May 4, 1940.

$$z = a \cos u + (r^2 - a^2 \sin^2 u)^{1/2} - R,$$

if $a = \overline{AB'}$, $u = \measuredangle B'AP$. Applying now Hooke's law in an approximate way familiar to many engineering problems, we may say that the radial pressure acting at an infinitesimal arc $R\,du$ at P, in consequence of the compression of L against C, is $kz\,du$, k being a constant of proportionality. Due to the rotation of C, a friction force $fkz\,du$ tangential to C acts at P, where f is the friction coefficient between L and C. The braking action is determined by the resultant moment of these friction forces about A which coincides with the wheel axis of the automobile. This moment is

$$fkR \int_{u_1}^{u_2} z\,du = fkR[a \sin u_2 - a \sin u_1 - Ru_2 + Ru_1 + rE(u_2, a/r) - rE(u_1, a/r)],$$

where u_1 and u_2 are the values of u corresponding to the end points of contact between the elastic strip and the cylinder, and E is the Legendre integral of the second kind of modulus a/r.

AN EXTENSION OF THE LAURENT EXPANSION*

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1. Introduction. In a paper published in 1931 Flora Streetman and L. R. Ford developed a polynomial expansion in a complex variable.† The series obtained by them is an extension of the Maclaurin series for a function analytic at the origin. They introduced the construction of major and minor circles for defining the region of convergence and obtaining the expansion.

The results of the present paper give an expansion which provides a method for representing a function with finite singularities in a region exceeding the ring of convergence of the Laurent series. The construction of major and minor circles is again employed, three new theorems being proved on the interrelation of these circles. The definitions of major and minor circles and theorems concerning them are given here for reference.

The major circle $M(z)$ of a point z in the complex plane is the circle with center $-hz$ and radius $(1+h)|z|$, where h is a positive real number.

The minor circle $m(z)$ of a point z in the complex plane is the circle with center $hz/(1+2h)$ and radius $(1+h)|z|/(1+2h)$, where h is a positive real number.

THEOREM 1. *If z' is outside, on, or inside $m(z'')$, then z'' is respectively inside, on, or outside $M(z')$.*‡

* Suggested to the writer by J. E. Powell at Michigan State College. Presented to the American Mathematical Society on April 26, 1940 under the title, A method for analytic continuation.

† Flora Streetman and L. R. Ford, A certain polynomial expansion, this MONTHLY, vol. 38, 1931, pp. 198–201.

‡ *Ibid.*, p. 200.

THEOREM 2. *If z' is outside $m(z'')$ a distance p, then z'' is inside $M(z')$ a distance greater than or equal to p.*

Proof. For the proof of the theorem we take $z'=x+iy$ and $z''=m+in$. If z' is outside the minor circle of z'' a distance p, then it follows that

$$\left| z' - \frac{h}{1+2h} z'' \right| = \frac{1+h}{1+2h} | z'' | + p.$$

This relation reduces to the form

$$(1+h)^2(x^2+y^2) = (m+hx)^2 + (n+hy)^2 + 2p(1+h)\sqrt{m^2+n^2} + p^2(1+2h).$$

This can be written as

$$\begin{aligned}(m+hx)^2 + (n+hy)^2 = {} & (1+h)^2(x^2+y^2) - 2p(1+h)\sqrt{x^2+y^2} + p^2 \\ & - 2p(1+h)\sqrt{m^2+n^2} + 2p(1+h)\sqrt{x^2+y^2} - p^2(2+2h).\end{aligned}$$

But it is evident that the quantity composed of the last three terms of the last expression is negative since it is a positive multiple of $|z'| - |z''| - p$, which is negative by hypothesis. From this it follows that

$$| z'' + hz' | \leqq (1+h) | z' | - p,$$

which proves the theorem.

THEOREM 3. *If z'' is inside $M(z')$ a distance q, then z' is outside $m(z'')$ a distance greater than or equal to $q/(1+2h)$.*

Proof. By the hypothesis of the theorem we have, using the same notation as before,

$$(m+hx)^2 + (n+hy)^2 = (1+h)^2(x^2+y^2) - 2q(1+h)\sqrt{x^2+y^2} + q^2.$$

This is expressible in the form

$$\begin{aligned}\left| z' - \frac{h}{1+2h} z'' \right|^2 = {} & \left(\frac{1+h}{1+2h}\right)^2 | z'' |^2 + \frac{2q(1+h)}{(1+2h)^2} | z'' | + \frac{q}{(1+2h)^2} \\ & + 2q\left(\frac{1+h}{1+2h}\right) | z' | - \frac{2q(1+h)}{(1+2h)^2} | z'' | - \frac{2q(1+h)}{(1+2h)^2}.\end{aligned}$$

In this case it is evident that the quantity composed of the last three terms of this equation is positive, and from this it follows that

$$\left| z' - \frac{h}{1+2h} z'' \right| \geqq \frac{1+h}{1+2h} | z'' | + \frac{q}{1+2h},$$

which proves the theorem.

THEOREM 4. *If $|z'| > |z''|$, then $M(z')$ contains $m(z'')$ completely.*

2. The expansion for a function analytic in a region. Consider any function $f(z)$ which is single-valued and analytic in and on the contour of a region S. The region S is such that it is bounded by two regular closed curves about the origin, C' and C'', where every point on C' is less in absolute value than any point on C''. Construct the major circles for all points t'' on C'' and the minor circles for all points t' on C'. Since $|t'|<|t''|$ for all t' and t'', then by Theorem 4 all the minor circles $m(t')$ are contained completely in all the major circles $M(t'')$. Therefore there exists a region R bounded on the outside by the $M(t'')$ and on the inside by the $m(t')$, such that R encircles the origin. Also, R is contained in S.

For any z in R it follows by Cauchy's integral formula that

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C''}\frac{f(t)dt}{(t-z)} - \frac{1}{2\pi i}\int_{C'}\frac{f(t)dt}{(t-z)} \\
&= \frac{1}{2\pi i}\int_{C''}\frac{f(t)dt}{(t-z)} + \frac{1}{2\pi i}\int_{C'}\frac{f(t)dt}{(z-t)} .
\end{aligned}
\tag{1}
$$

For the first integral of (1), use the expansion

$$
\begin{aligned}
\frac{1}{(t-z)} &= \frac{1}{(1+h)t-(z+ht)} \\
&= \frac{1}{(t+ht)}\left\{1+\frac{z+ht}{t+ht}+\cdots+\left(\frac{z+ht}{t+ht}\right)^n+\cdots\right\}.
\end{aligned}
\tag{2}
$$

Since z is in R, it is inside the major circle of every t on C''. The distance of z from the center of the major circle of any t on C'' is at least some value p less than the radius of the major circle. Then

$$|z+ht| \leqq (1+h)|t| - p,$$

or

$$\left|\frac{z+ht}{t+ht}\right| = 1 - \frac{p}{(1+h)|t|} = r' < 1. \tag{3}$$

Relation (3) holds for all values of t on C''.

It follows that each term in expansion (2) is less than or equal to the corresponding term in the convergent constant term series

$$\sum_{n=0}^{\infty}\frac{(r')^n}{(1+h)|t''|},$$

where t'' is the maximum value of t on C''. By the Weierstrass test, (2) is uniformly convergent in t on C'' for a chosen z in R. Since $f(t)$ is bounded for t on C'', the uniform convergence is not affected if the series is multiplied by $f(t)$. Integration term by term can be carried out to obtain

$$\int_{C''} \frac{f(t)dt}{(t-z)} = \sum_{n=0}^{\infty} \int_{C''} \frac{(z+ht)^n f(t)dt}{(t+ht)^{n+1}}. \tag{4}$$

For the second integral of (1), consider the expansion

$$\begin{aligned} \frac{1}{(z-t)} &= \frac{1}{(1+h)z - (t+hz)} \\ &= \frac{1}{(z+hz)} \left\{ 1 + \frac{t+hz}{z+hz} + \cdots + \left(\frac{t+hz}{z+hz}\right)^n + \cdots \right\}. \end{aligned} \tag{5}$$

Since z is in R, it is outside the minor circle of every t on C' by at least some distance q. Then by Theorem 2, any t on C' is inside the major circle of z by at least q. This means that

$$|t+hz| + q \leqq |z+hz|,$$

or

$$\left| \frac{t+hz}{z+hz} \right| \leqq 1 - \frac{q}{(1+h)|z|} = r'' < 1. \tag{6}$$

Relation (6) holds for all values of t on C'.

Again by the Weierstrass test, series (5) is uniformly convergent for t on C' and any chosen z in R. Multiplying (5) by $f(t)$ which is bounded on C', and integrating the series term by term gives

$$\int_{C'} \frac{f(t)dt}{(z-t)} = \sum_{n=0}^{\infty} \int_{C'} \frac{(t+hz)^n f(t)dt}{(z+hz)^{n+1}}. \tag{7}$$

Since $f(z)$ is analytic throughout the region S, the contours of integration, C' and C'', may be deformed into any regular curve C lying completely in S and encircling the origin. Then for any z in R, equations (4) and (7) can be combined to give the expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(z+ht)^n f(t)dt}{(t+ht)^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(t+hz)^n f(t)dt}{(z+hz)^{n+1}}. \tag{8}$$

3. The definition of a region of convergence. In the case of the Laurent expansion about the origin, the region of convergence for a function possessing finite singular points is the ring between concentric circles whose radii are the moduli of singular points having different magnitudes. A similar construction gives the region of convergence for the present expansion of a function possessing a denumerable number of finite singularities. Select any one of the singular points which is not a limit point of singularities of greater absolute value. Let us call this point a'. For all the singular points $\cdots, c', b', a'$, which are less than or equal to $|a'|$ in magnitude, construct the minor circles $\cdots, m(c'), m(b'), m(a')$. For the remaining finite singular points $a, b, c, \cdots$, construct the major cir-

cles $M(a)$, $M(b)$, $M(c), \cdots$. Since we have established the relation that $\cdots, |c'|, |b'|, |a'| < |a|, |b|, |c|, \cdots$, then by Theorem 4 all the minor circles are completely inside all the major circles. Therefore, there exists a domain D whose inner boundary consists of arcs of the minor circles of certain singularities and whose outer boundary consists of arcs of the major circles of other singularities. By the manner of definition of major and minor circles, D encircles the origin. It can be shown that expansion (8) holds for any point z in the interior of D with C in D about the origin.

4. **Discussion of the expansion.** Let us denote the two series in the expansion of $f(z)$ by A and B, respectively, so that we have

$$f(z) = A + B.$$

THEOREM 5. *The $(k+1)$st term of series A is a polynomial in z of degree k.*

This theorem is proved by Streetman and Ford. It is also discussed in a more general case by Goursat.*

THEOREM 6. *The $(k+1)$st term of series B is a polynomial of degree $(k+1)$ in $1/z$.*

This theorem follows immediately upon considering the $(k+1)$st term when $n=k$. It can be written in the form

$$\frac{1}{2\pi i(1+h)^{k+1}} \frac{1}{z^{k+1}} \int_C (t+hz)^k f(t)dt.$$

By expanding the binomial in the integrand and integrating term by term, we can write it in the form

$$\frac{1}{2\pi i(1+h)^{k+1}} \sum_{j=0}^{k} \frac{b_j}{z^{j+1}},$$

where

$$b_j = \binom{k}{j} (h^{k-j}) \int_C t^j f(t)dt.$$

In this form the theorem is evident.

It is of particular interest to note the effect upon the series (8) and upon the region of convergence of the choice of different values for h. For the case when $h=0$, the series takes the form of the Laurent expansion of $f(z)$ about the origin. It is readily verified that the region of convergence is now defined as the ring of convergence of the Laurent series. It is bounded on the inside by the circle with center at the origin and radius $|a'|$. It is bounded on the outside by the circle with center at the origin and radius the minimum of $|a|, |b|, |c|, \cdots$. We

* Goursat, E., translated by Hedrick and Dunkel, Mathematical Analysis, vol. II, part I, New York, Ginn and Company, 1916, pp. 85–86.

next observe the effect of assigning large values to h. The region of convergence D is increased in size with larger h, for the minor circles of the inner singularities become smaller and the major circles of the outer singularities become larger. Consider, for example, the minor circle $m(a')$. Since

$$\lim_{h=\infty} \frac{ha'}{1+2h} = \tfrac{1}{2}a',$$

and

$$\lim_{h=\infty} \frac{1+h}{1+2h} | a' | = \tfrac{1}{2} | a' |,$$

then $m(a')$ approaches as a limiting position for increasing h the circle with center $\frac{1}{2}a'$ and radius $\frac{1}{2}|a'|$. Likewise, for the major circle $M(a)$,

$$\lim_{h=\infty} (-ha) = \infty,$$

and

$$\lim_{h=\infty} (1+h) | a | = \infty.$$

Thus in its limiting form, D has an inner boundary composed of the circles through the origin and the inner singular points $\cdots, c', b', a'$. Its outer boundary is composed of the straight lines through the outer singular points $a, b, c, \cdots$, perpendicular to the radius vectors of the respective points.

To observe the effect of different choices for h on the expansion itself, we note that in series (2) and (5) from which (8) is derived, the ratios determining the convergence are functions of h. It is evident that the two series converge most rapidly for $h=0$ and less rapidly as h is increased in value. Thus the convergence of the Laurent series is more rapid than the convergence of (8) for any other choice for h. The advantage in using (8) lies in the fact that the region of convergence is considerably enlarged for larger h as the following application will show.

5. An application. We shall apply the expansion to the function

$$f(z) = \frac{1}{(z-1)(z-2)},$$

having simple poles at $z=1$ and $z=2$. The region of convergence is to be the domain bounded by the minor circle of $z=1$ and the major circle of $z=2$. From equation (8), the general form of the expansion for this function is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(z+ht)^n dt}{(1+h)^{n+1} t^{n+1} (t-1)(t-2)} + \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(t+hz)^n dt}{(z+hz)^{n+1} (t-1)(t-2)}. \tag{9}$$

Evaluation of the general terms in each of these series gives a representation

$$f(z) = -\sum_{n=0}^{\infty} \frac{(z+2h)^n}{(2+2h)^{n+1}} - \sum_{n=0}^{\infty} \frac{(hz+1)^n}{(z+hz)^{n+1}}.$$

It can be easily noted in the expansion thus displayed that the first sum is a series of polynomials in z and that the second sum is a series of polynomials in $1/z$, thus demonstrating Theorem 5 and Theorem 6. From the discussion in section 3 it follows that for a sufficiently large choice of h, the expansion represents the function for any finite value of z such that

$$|z - \tfrac{1}{2}| \geqq r' > \tfrac{1}{2},$$

where the real part of z is less than or equal to r'' which is less than 2.

HEURISTIC REASONING AND THE THEORY OF PROBABILITY*

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1. Introduction. Heuristic reasoning is encountered in all fields, theoretical or practical. Rigorous, precise, properly so-called logical reasoning is found in its pure form only in mathematics. To compare these two types of reasoning, it is advantageous to consider first their uses in mathematics.

The properly so-called logical type of reasoning appears generally by itself on the pages of mathematical treatises; the heuristic reasoning which in general guided the invention of the logical reasoning is omitted. The rigorous demonstration establishes the truth, the rigorous refutation establishes the falsity of the theorem under consideration. Heuristic reasoning cannot demonstrate the truth or the falsity of a theorem; it can only augment or diminish our confidence in a theorem which is still only a conjecture neither proved nor disproved. The correct proof is definitive, it establishes irrefutably the truth of the theorem—once for all. An heuristic proof is provisional; the one I find today increases my confidence which may be shaken tomorrow by another heuristic proof and definitely shattered the following day by the rigorous refutation of the theorem under consideration; nevertheless, the heuristic proof which I find today may be completely "correct," completely reasonable, in the sense that it yields the best that can be obtained in the light of the actual state of my knowledge.

There is something impersonal about precise reasoning; everyone does it in essentially the same way, in the "proper" way, and I am inclined to agree with

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* This paper was first written in French, in August 1939, but the publication has been prevented by the war. For the English translation of the paper in its present shape, I am indebted to the kind help of M. R. Demers. The paper was presented at the Stanford University Symposium August 11, 1941 and its contents will be incorporated into a book the author is writing on the solution of mathematical problems.

Descartes* that "the deduction, or the immediate inference of one thing from another may well be omitted if one fails to observe it, but it cannot be done wrongly—not even by the crudest intellect." In any case, the impersonal characteristics of precise reasoning can be the subject of a study of mathematical order, of logic. It seems to me that heuristic reasoning likewise has something impersonal about it, something which everyone does in the same way, and in what follows, I will present a very modest mathematical study of some of the impersonal characteristics of heuristic reasoning.

I will use some formulas from the theory of probability to express clearly the essence of some simple heuristic conclusions frequently encountered. This goal can be attained if we agree on a convention when using these formulas. That one of the aims of the theory of probability is the formulation of heuristic conclusions or the "perfection of the art of reliably appraising conjectures," has been stated and restated many times since Jacques Bernoulli. But the simple convention which can bring us nearer to this goal and which I will formulate in section 5 has never to my knowledge been clearly indicated before.

Sections 2 and 3 of this paper present some preliminary remarks on two subjects which will later be brought together: the theory of probability and the solution of problems. Section 4 introduces historical evidence. The three following sections, 5, 6, and 7, contain the principal part: the first presents the fundamental convention, and the next two apply it to certain heuristic reasonings. Section 7 touches upon inductive reasoning. The last paragraph, 8, advances some general theses.

I was led to the considerations which follow by some reflections on the psychology and logic of research.† I owe much to a conversation which I had the pleasure of having with Bruno de Finetti.‡

2. Some views on the rôle of the theory of probability. Probability and plausibility. Two different points of view may be assumed to explain the fundamental notions of the theory of probability: the "objective" point of view, and the "subjective" point of view. Let us recall briefly these two points of view.

From the objective point of view, we are especially concerned with *frequencies*. The frequency of an event is a fraction whose denominator is the total number of cases observed in a certain investigation, and the numerator is the number of those cases in which the event in question occurred. For example, if we count 1300 boys out of 2500 registered births, the resulting frequency of the birth of a boy is $1300/2500 = 0.52$. From the objective view-point, the aim of the

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* Oeuvres de Descartes, edited by Adam and Tannery, vol. 10, 1908, p. 365. See also the new edition (text and translation) of Descartes's Regulae ad directionem ingenii, by G. Le Roy, Paris, p. 13.

† See two papers by the author, having the same title: Wie sucht man die Lösung mathematischer Aufgaben?, Zeitschrift für mathemat. und naturwissensch. Unterricht, vol. 63, 1932, pp. 159–169 and Acta Psychologica, vol. 4, 1938, pp. 113–170.

‡ See B. de Finetti, Compte rendu critique du Colloque de Genève sur la théorie des probabilités, Actualités scientifiques et industrielles, Nr. 766, pp. 27–28.

theory of probability is the *prediction of observable frequencies* based on suitable hypotheses, or on already observed frequencies.

From the subjective view-point, the aim of the theory of probability is *to measure the degree of belief.* It is customary to add that we deal with reasonable beliefs, and that, if they are reasonable, our beliefs change with the state of our knowledge.

From the objective view-point, probabilities are merely idealized frequencies. The adherents of this point of view tell us that in order not to lose contact with reality we must consider all the probabilities which occur in any theoretical consideration as replaceable by concrete numerical frequencies obtained from observations. On the other hand, from the subjective point of view, each probability is to be considered as the measure of a degree of belief.

I will not elaborate further on these two points of view, but I will make a proposition on terminology. To distinguish more clearly these two points of view, let us agree to reserve the word *probability* for an idealized frequency, and the word *plausibility* for the measure of a degree of belief. Thus we are led to speak of probability only from the objective point of view, and of plausibility only from the subjective point of view.

I certainly do not want to discuss here the respective merits of these two view-points; however, it is impossible to ignore the fact that all the applications of the theory of probability, so numerous today, which arrive in a reasonable way at numerical values for the probabilities concerned are all made from the objective view-point. Is the subjective view-point without concrete and palpable applications? I believe not, and it is precisely in the solution of problems, and in particular, in the solution of determinate mathematical problems, that I propose to discover such applications.

3. Remarks on the solution of problems. Degrees of proximity and degrees of belief. One often hears it said by those who are habitually solving problems—mathematical problems, chess problems, or cross-word puzzles—that the search for the solution was directed by a kind of feeling. Each important step of the research, each turn of the way leading to the solution is accompanied by a change of feeling. This feeling for the problem may be infinitely modulated, but it is convenient to distinguish two components: the degree of proximity and the degree of belief.

If a person is occupied with a problem of the type to which he is accustomed—in chess, cross-words, or mathematics—he soon adopts some attitude: "Now this problem means something to me—I like it—I have something here," he will think, or perhaps just the opposite: "This problem conveys nothing to me—I don't like it—I don't see anything in it." These locutions like many others express the intensity of the echo aroused in us by the problem, or the degree of our hope of solving it, or more briefly yet, the *degree of proximity* to the solution.

But there is more. Let us consider the mathematical problem of proving or disproving a given theorem (such as the theorem of Pythagoras, or Fermat's last

theorem, or the famous Riemann hypothesis). Which course will we pursue? Will we begin by trying to prove the theorem, or will we first try to disprove it? That depends on the degree of our belief in the proposed theorem; we try to establish the truth or the falsity of the theorem according to which seems to us the more or less likely. The degree of belief will change after each essential observation we succeed in noting about the given theorem. If some one perseveres for hours, days, or even years trying to prove a theorem, then necessarily he must have a very high *degree of belief* in the theorem.

There is no doubt that it is very important for the investigator to distinguish degrees of proximity and to distinguish degrees of belief. It seems to me that an essential part of scientific talent consists in a particularly intense reaction, a particularly lively sensitiveness to degrees of proximity and belief.

There is something intimately personal in our evaluation of the degrees of proximity and of belief. An observation which gives me the impression of being a prodigious step towards the solution of a problem leaves my neighbor completely cold—naturally, our experiences and temperaments are different. Nevertheless, there are certain impersonal traits in the evaluation of degrees of belief; let us hear the testimony of a great mathematician on this subject.

4. An example of inductive research in a mathematical subject. The totality of the processes by which the natural sciences establish their laws is often called induction. We cannot discuss here the nature of induction, but it seems to me indubitable that an essential part of the inductive method consists in the examination of a general law by its particular consequences. This inductive method is readily employed not only in the natural sciences but also in mathematical research, and sometimes, with certain scientists and in certain branches of science, this method assumes an empirical and heuristic character which deserves our attention.

Empirical researches and heuristic arguments are rarely mentioned in the printed works of mathematicians. But there are some exceptions. On the occasion of a still celebrated piece of research, Euler devoted a whole memoir to the exposition of the heuristic motives for his belief in the truth of a theorem which he was unable to prove. This memoir which is entitled *Découverte d'une loi tout extraordinaire des nombres par rapport à la somme de leurs diviseurs** should be read in its entirety by those who would understand either the notion of probability, the inductive processes, or the psychology of research.

In what follows I will give a schematic extract of Euler's memoir. The theorem investigated by Euler is remarkable in several respects and even today it is of great mathematical interest. However, we are not concerned here with the mathematical content of this theorem, but with the reasons which induced Euler to believe in the theorem when it was still unproved. I will ignore here the mathematical content of the theorem; I will designate it by T; I will speak of theorem T and I will speak of other theorems having certain logical relations with theo-

* Leonhardi Euleri Opera Omnia, ser. 1, vol. 2, 1915. See pp. 241–253.

rem T in the same way, *i.e.*, by ignoring their concrete contents and designating them by appropriate letters. I will give Euler's text as much as is possible, word for word. But since I am setting aside the concrete content of his research, and since I am only interested in the nature of some of his reasons, I must change some of his words, replace certain mathematical propositions by abstract descriptions, and alter slightly the order of presentation of the selected portions of Euler's text. The necessary references will be indicated by footnotes and the italics will be reserved for those phrases which are not due to Euler.

SCHEMATIC EXTRACT OF EULER'S MEMOIR

Theorem T is of such a nature that we can be assured of its truth without giving it a perfect demonstration. Nevertheless, I will present evidence for it of such a character that it might be regarded as almost equivalent to a rigorous demonstration.†

Theorem T includes an infinite number of particular cases: $C_1, C_2, C_3, \cdots$. *Conversely, the infinite set of these particular cases* $C_1, C_2, C_3, \cdots$ *is equivalent to theorem T. We can find out by a simple calculation whether* C_1 *is true or not. Another simple calculation determines whether* C_2 *is true or not, and similarly for* C_3, *and so on. I have made these calculations and I find that* $C_1, C_2, C_3, \cdots, C_{40}$ *are all true.* It suffices to undertake *these calculations* and to continue *them* as far as is deemed proper to become convinced of the truth of this sequence *continued indefinitely.* But I have no other evidence for this, except a long induction which I have carried out so far that I cannot in any way doubt the law of which $C_1, C_2, \cdots$ *are the particular cases.* I have long searched in vain for a rigorous demonstration of *theorem T*, and I have proposed the same question to some of my friends with whose ability in these matters I am familiar but all have agreed with me on the truth of *theorem T* without being able to unearth any clue of a demonstration. Thus it will be a known truth, but not yet demonstrated; for each of us can convince himself of this truth by the *actual calculation of the cases* $C_1, C_2, C_3, \cdots$ as far as he may wish; and it seems impossible that the law which has been discovered to hold for 20 terms, for example, would not be observed in the terms that follow.‡

Having thus discovered the truth of *theorem T* even though it has not been possible to demonstrate it, all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the

† Euler, *loc. cit.*, p. 242, lines 1–4. In fact these lines are concerned with theorem T^* (see the first footnote p. 455), but T^* is equivalent to T. By T I mean the theorem which in modern notation reads as follows:

$$\prod_{m=1}^{\infty}(1 - x^m) = \sum_{m=-\infty}^{+\infty}(-1)^m x^{(3m^2+m)/2}.$$

By C_n, I mean the assertion that the coefficient of x^n is the same in both members of the equation above.

‡ Euler, *loc. cit.*, p. 249, line 5 to p. 250, line 4.

demonstration of *theorem T*; and it was with this in mind that I maneuvered *theorem T* in many ways and so discovered among others *theorem* T^* whose truth must be as certain as that of *theorem T*.†

Theorems T and T^* *are equivalent*; *they are both true or both false*; *they stand or fall together. Like T, theorem* T^* *includes an infinity of particular cases* C_1^*, C_2^*, C_3^*, $\cdots$, *and this sequence of particular cases is equivalent to theorem* T^*. *Here again, a simple calculation shows whether* C_1^* *is true or not. Similarly, it is possible to determine whether* C_2^* *is true or not, and so on.* It is not difficult to apply *theorem* T^* to any given particular case, and so become convinced of its truth by as many examples as one may wish to develop. And since I must admit that I am not in a position to give it a rigorous demonstration, I will justify it by a sufficiently large number of examples, by C_1^*, C_2^*, $\cdots$, C_{20}^*. I think these examples are sufficient to discourage anyone from imagining that it is by pure chance that my rule is in agreement with the truth.‡

If one still doubts that the law is precisely that one which I have indicated, I will give some examples with larger numbers. *By examination, I find that* C_{101}^* *and* C_{301}^* *are true, and so I find that theorem* T^* *is valid even for these cases which are far removed from those which I examined earlier.*§ These examples which I have just developed undoubtedly will dispel any qualms which we might have had about the truth of *theorems T and* T^*.||

Few mathematicians have made "inductive" researches comparable in extent and importance to those of Euler. Few mathematicians, I am inclined to believe, have reflected as seriously as Euler has on the signification and justification of their inductive processes.¶ To my knowledge, no mathematician has ever described his inductive methods with such charm and candor as Euler has done in the memoir from which I have just given a schematic extract.

But every mathematician with some experience uses readily and effectively the same method that Euler used which is basically the following: To examine a theorem T, we deduce from it some easily verifiable consequences $C_1, C_2, C_3, \cdots$. If one of these consequences is found to be false, theorem T is refuted and the question is decided. But if all the consequences $C_1, C_2, C_3, \cdots$ happen to be

† Euler, *loc. cit.*, p. 250, lines 5–14. By T^* I mean the general recurrence formula for the sum of the divisors, discovered by Euler, and which is the main object of the memoir. By C_n^* I understand the particular case of this formula which expresses the sum of the divisors of n in terms of the analogous sums for $n-1$, $n-2$, $\cdots$.

‡ Euler, *loc. cit.*, p. 246, lines 1–5, and p. 247, lines 1–2.

§ Euler, *loc. cit.*, p. 247, line 3 and the following lines. Here our schematization does not completely convey an important nuance in the inductive reasoning of Euler.

|| Euler, *loc. cit.*, p. 248, lines 12–13. I add some dates: Theorem T was discovered by Euler in 1740, theorem T^* in 1747, and a proof of theorem T (implying one for T^*) in 1750. See Euler, *loc. cit.*, the following footnotes by the editor: 3) on p. XVIII, 1) on p. 191, 2) on p. XXIII, and 1) on p. 390. Besides the passages mentioned, the following pages in volume 2 of the Opera deal with the same topic: pp. 191–193, 280–284, 373–398.

¶ See, *e.g.*, Euler, *loc. cit.*, pp. 459–492, and especially the "summarium," pp. 459–460.

valid, we are led after a more or less lengthy sequence of verifications to an "inductive" conviction of the validity of theorem T. We attain a degree of belief so strong that it seems superfluous to make any ulterior verifications. The slight increase in our confidence in the theorem which could result from the confirmation of the next consequence is not worth the time and effort which this verification would necessitate. And thus the mathematician arrives at that point where the physicist must always stop. He ceases to doubt the theorem actively since the possibilities of verification appear, at least for the moment, to be exhausted.

Moreover, it is remarkable how few tries will sometimes suffice to convince us of theorems of a complicated nature. The reader will perhaps be astonished to read the following passage due to Descartes: "In order to show by enumeration that the area of a circle is greater than that of any other figure of the same perimeter, we do not need to make a general investigation of all the possible figures, but it suffices to prove it for a few particular figures whence we can conclude the same thing, by induction, for all the other figures."†

The modern scientist is more cautious, but still follows, without necessarily having read it, the advice of Descartes. The eminent physicist Lord Rayleigh calculates numerically the fundamental frequencies of ten membranes of different shapes but of equal area and subject to the same physical conditions. He finds that the circular membrane has the lowest fundamental frequency and leaves the conclusion to the reader.‡ The fact that such a conclusion may be left to the reader proves that there must be something impersonal about the conclusion.

5. Application of the theory of probability. Fundamental convention. Is the degree of belief which the scientist has in a theorem he is investigating amenable to a calculus? A calculus has been applied often and with success to probabilities, a probability being interpreted as a frequency drawn from statistical observation. But is it possible to apply this same calculus to plausibilities, by interpreting plausibility as a degree of belief, the belief which the investigator has in the theorem which he is examining?

There is a basic difficulty. It has never been possible to give in a reasonable way a determinate numerical fraction which would measure the chance for the validity of a determinate theorem under scrutiny—be it a mathematical, physical, or theological theorem. It appears impossible to give a numerical value to the degree of confidence which Euler must have had in his "theorem T" after having verified the first forty particular cases $C_1, C_2, \cdots, C_{40}$. It seems fantastic or puerile to try to determine, to a given number of decimal places, the plausibility of the Riemann hypothesis in 1939 or the plausibility of the newtonian law of gravitation in 1900.

The impossible must not be attempted. This impossibility of attaching a determinate numerical value to a plausibility with which an investigator may be concerned seems to me to be so fundamental that in my opinion there is but one

† See Descartes, Oeuvres, *loc. cit.*, p. 390, and G. Le Roy, p. 65.

‡ Lord Rayleigh, The Theory of Sound, vol. 1, 1877, p. 289.

course: to formulate this impossibility as a principle, and taboo the assignment of numerical values to plausibilities. I recall that physicists formulated the principle of the impossibility of perpetual motion with remarkably fruitful results.

These considerations indicate that we must make a mathematical distinction between probability and plausibility. A *probability* is measured by a *determinate number* between 0 and 1. To a *plausibility*, we will make correspond an *indeterminate number* or a *variable* whose domain is the open interval (0, 1). Apart from this difference, the calculus of plausibilities obeys the same rules as the calculus of probabilities.

Thus we intend to subject plausibilities to algebraic manipulation, but we are forbidden to assign numerical values to plausibilities. Will a theory so restrained be fruitful? We will see; let us try. *A priori* I see no disadvantage. A plausibility characterizes a degree of belief. Must degrees of belief be measured by determinate numbers? It is not the "absolute" degree of his belief which is important for the investigator (or for the psychological understanding of his attitudes) but the increase or decrease of his belief which results from a new discovery, the change in belief during the investigation. Thus, *a priori*, I see no objection to the method which I will explain.

We wish to express by formulas certain characteristics of an heuristic argument. This reasoning, let us suppose, occurs in the investigation of certain theorems, say T, U, V, $\cdots$. All these theorems are clearly stated but, at least at the beginning of the inquiry, are not demonstrated. At each stage of our research we attach a certain plausibility to these theorems. The plausibility attached to theorem T at the inception of the investigation will be designated by $p(T)$. As the research progresses, we obtain certain results concerning this theorem, the logical situation changes and the plausibility of T may change with it. The plausibility of T, originally designated by $p(T)$, will be designated by $p_1(T)$ after the first change but before the second change in the logical situation; by $p_2(T)$ after the second change but before the third change in the situation; and so on. If successive changes have no effect on the plausibility of a certain theorem U, then $p(U)=p_1(U)=p_2(U)= \cdots$; in this case, the plausibility of U will always be designated simply by $p(U)$, its original designation. It may happen that at a certain stage theorem V is refuted; from then on the plausibility of V will be replaced by 0, and analogously, the plausibility of a theorem will replaced by 1 as soon as that theorem is proved.

All the plausibilities, $p(T)$, $p_1(T)$, $p_2(T)$, $\cdots$, $p(U)$, $p_1(U)$, $\cdots$, $p(V)$, $\cdots$, are to be considered as variables which have a common domain: the open interval (0, 1). Thus $p(T)$ is not a number, but only a variable. This variable can vary in the open interval (0, 1); it is capable of a determinate variation, an increase or a decrease; it can tend to 0 or to 1, either end-point of its domain; but no numerical value can replace it, with the possible exceptions of 0 and 1.

The theory of probability furnishes relations involving $p(T)$, $p_1(T)$, $\cdots$, $p(U)$, $\cdots$; thus all the variables cannot be independent. If these relations yield certain inequalities between these variables, or certain connections between their

limits or their variations, these results can be interpreted as impersonal heuristic conclusions.

It is not worth while to discuss at length these generalities, before examining the applications.* Only the applications can show us whether or not our anticipations of the theory make sense, and whether or not they conform, at least to a first approximation, to observable psychological reality.

6. First example of the application of the theory of probability. Conclusion to be drawn from the refutation of a hypothesis. From here on, I assume that the notation and the simplest rules of the theory of probability are known to the reader.† We will solve the following problem.

Someone is trying to decide whether or not theorem T is true. He notes that T is a consequence of a theorem H. Later he succeeds in proving that theorem H is false. How does this refutation affect the plausibility of T?

It is desirable that the reader place himself in the position of the person who is trying "to decide T" (I shall use this abbreviation instead of the longer locution "to decide whether theorem T is true or false"). This problem shows a typical situation which arises often in mathematical research.‡

Our original aim is to decide T, the given theorem. We do not know whether T is true or false; precisely, we are trying to ascertain whether it is true or not.

We hit upon the auxiliary theorem H. We do not know whether H is true or false, but we do know with certitude that T follows from H. We now have an indubitable logical connection, soundly established, between the still undecided, doubtful theorems T and H.

We change tack. Instead of continuing with T, the given theorem, we concentrate our efforts on H, the auxiliary theorem. In practice, when does this happen? When we become tired of T, when H seems to be "nearer" than T, when we see a certain plausibility for H, and especially when H seems to be a "hypothesis which gives a deeper insight into the situation," when we think we recognize in H "the real reason" for T. Of course, the original theorem T could be correct even if the hypothesis H, introduced in the course of the investigation, were false; but we become attached to the investigation of H with greater tenacity

* Our discussion was in other respects incomplete; the treatment of "conditional" plausibilities of the form $p(T/U)$, which must also be considered as variables, was not mentioned.

† Notation: $p(ABC)$ is the plausibility that theorems A, B, and C are true simultaneously (analogous notation for any number of theorems); $p(\bar{A})$ is the plausibility that A is false; $p(A/B)$ is the plausibility of A "posito B," *i.e.*, under the hypothesis that B is true. Theorems on "mutually exclusive events" and on "compound events": $p(AB)+p(A\bar{B})=p(A)$ and $p(AB)=p(A)p(B/A)$, respectively.

‡ In fact, we are here concerned with one of the three most important ways of introducing an auxiliary theorem H ("Hilfssatz"), to facilitate the decision of the proposed theorem T:

(1) T is equivalent to H (symbolically, $T\rightleftarrows H$);

(2) T is a consequence of H ($H\rightarrow T$);

(3) T implies $H(T\rightarrow H)$.

Here we are considering case (2). Case (3) will be considered in paragraph 7, but with a slightly different notation.

if it seems to us unlikely that theorem T should be true without hypothesis H being true also, that is, if in H we see the real, profound reason for theorem T.

In the proposed problem, there are two different plausibilities of T, the plausibility of T at two different phases of the inquiry. We will designate by $p(T)$ the plausibility of T *before* H was disproved, and by $p_1(T)$ the plausibility of T *after* H was disproved.

Before deciding H, we must distinguish two possible cases. In the first case, the hypothesis H is true; in the second, H is false, and hypothesis $\overline{H}$, the negation of H, is true. By the theorem on the probabilities of mutually exclusive events, we have

$$p(T) = p(TH) + p(T\overline{H}),$$

and by the theorem on compound events,

$$p(T\overline{H}) = p(\overline{H})p(T/\overline{H}) = [1 - p(H)]p(T/\overline{H}),$$
$$p(TH) = p(H)p(T/H) = p(H),$$

since T follows from H, so that T is certain if H is true and thus $p(T/H) = 1$. By combining the preceding formulas, we obtain

$$p(T) = p(T/\overline{H}) + p(H)[1 - p(T/\overline{H})],$$

or

$$p(T) = p(T/\overline{H}) + p(H)p(\overline{T}/\overline{H}).$$

After deciding H, we replace $p(T)$ by $p_1(T)$ and $p(H)$ by 0, since we know now that H is false. We find that

$$p_1(T) = p(T/\overline{H}).$$

By combining the expressions for $p(T)$ and $p_1(T)$, we obtain

$$\boxed{p(T) - p_1(T) = p(H)p(\overline{T}/\overline{H}).}$$

This formula answers precisely and completely the given problem. Let us be sure we fully understand it.

The right-hand member being the product of two plausibilities, is positive. Therefore the left-hand member of this equation is also positive; it expresses the diminution of the plausibility of T following the refutation of H.

The first factor on the right is the plausibility we attached to H before its refutation. The second factor on the right is the plausibility that T be false with hypothesis $\overline{H}$, the negation of H.* The right member, product of two plausibili-

* It is known that H implies T (or $H \rightarrow T$). If T were equivalent to H ($T \rightleftarrows H$), then the plausibility $p(\overline{T}/\overline{H})$ should be replaced by 1. It might be said that $p(\overline{T}/\overline{H})$ expresses how much stronger the connection between T and H is in our heuristic evaluation than it is in the logical implication $H \rightarrow T$; otherwise stated, it expresses the "nearness" of our heuristic bond between T and H to the more stringent logical bond $T \rightleftarrows H$. See cases (1) and (2) mentioned in the preceding footnote.

ties, increases as its factors increase.

We can thus state the result obtained as follows:

Our confidence in a theorem we are investigating can only diminish when a hypothesis, from which the given theorem can be deduced, is refuted.

The decrease in our confidence following the refutation of this hypothesis depends on two plausibilities:

the plausibility of the hypothesis before it was refuted, and

the plausibility of the impossibility of proving the given theorem by supposing the hypothesis opposite to the one which has just been refuted.

The stronger these two plausibilities are, the more serious is the decrease of our confidence.

But all this is evident; it is nothing else but common sense, you will say, as soon as you have grasped the situation to which this rule applies. In fact, let us go to the extreme cases.

When the hypothesis H, which in the end proved to be false, seemed very plausible ($p(H)\rightarrow 1$) and to be the true reason behind the theorem in question, in such a way that this theorem seemed very likely indemonstrable with the contrary hypothesis ($p(\overline{T}/\overline{H})\rightarrow 1$), then the refutation of hypothesis H is a hard blow to our confidence in the theorem under consideration.

On the other hand, if the hypothesis H which was exploded had been very implausible from the beginning ($p(H)\rightarrow 0$) and if the contrary hypothesis had not seemed particularly unfavorable for our theorem (no particular supposition for $p(\overline{T}/\overline{H})$), then our loss of confidence would have been slight.

Note that in the preceding discussion no numerical values were attributed to the plausibilities which arose. We considered $p(H)$ and $p(\overline{T}/\overline{H})$ as independent variables. To find the appropriate heuristic conclusions we considered these variables to be increasing or decreasing, tending to 1 or to 0. This conforms to the fundamental convention as stated.

7. Continuation of the applications of the theory of probabilities. On inductive reasoning. The following is a simpler problem but analogous to the one which we have just considered.

We are trying to decide whether theorem T is true or false. We note that theorem C is a consequence of T. Later we succeed in proving that theorem C is correct. What is the change in the plausibility of T?

Here is the situation. Our original goal is to decide T, the given theorem. We find an auxiliary theorem C. We do not know if either T or C is true or false, but we know with certainty that C follows from T. We change tack and, tired of T, we try to decide C. If we succeed in refuting C, we shall also refute T. But we succeed in proving C. This gives us no definite, purely logical conclusion about T—but will there be a change in the plausibility of T? Call $p(T)$ the plausibility of T *before* the demonstration of C, and $p_1(T)$ the plausibility of T *after* the demonstration.

Since C is a necessary logical consequence of T, we have $p(T)=p(TC)$, and so, from the law on the probabilities of compound events,

$$p(T) = p(C)p(T/C).$$

After having proved C, we replace $p(T)$ by $p_1(T)$ and $p(C)$ by 1. Thus $p_1(T)=p(T/C)$, and

$$\boxed{\frac{p_1(T)}{p(T)} = \frac{1}{p(C)}\,.}$$

This formula gives us immediately the following rule:

Our confidence in a theorem can only increase when a consequence of the theorem is confirmed.

The growth of our confidence will vary inversely as the plausibility of the consequence before its confirmation.

But everyone thinks in this way. It is the implausible, the surprising consequence whose confirmation adds the most to our faith in a general law from which the consequence was inferred.

Let us go further in this direction and consider a sequence of consequences instead of one. We are led to the following problem, from whose solution we can learn something about inductive methods.

We are trying to decide theorem T. We derive a sequence of consequences from T, say C_1, C_2, C_3, $\cdots$. We succeed in verifying C_1, then C_2, then C_3, and so on. What will be the effect of these successive verifications on the plausibility of theorem T?

Let us introduce a suitable notation. Designate by $p(T)$ the plausibility of theorem T before the start of the verifications; by $p_1(T)$ the plausibility of T after the verification of C_1, but before the verification of C_2; and in general, by $p_n(T)$ the plausibility of theorem T after the first n consequences, $C_1, C_2, \cdots, C_n$, of T have been proved, but before C_{n+1}, the next consequence, is verified, $(n=1, 2, 3, \cdots)$. The problem is to examine the transition from $p_{n-1}(T)$ to $p_n(T)$.

Since $C_1, C_2, \cdots$ are necessary consequences of theorem T, we have

$$p(T) = p(TC_1C_2 \cdots C_n),$$

and so, from the theorem on compound probabilities,

$$p(T) = p(C_1C_2 \cdots C_n)p(T/C_1C_2 \cdots C_n).$$

After it is known that $C_1, C_2, \cdots, C_n$ are all true, the left member of the preceding equation should be replaced by $p_n(T)$ and the first factor on the right by 1. Thus we have

$$p_n(T) = p(T/C_1C_2 \cdots C_n),$$

and, consequently,

$$p(T) = p(C_1C_2 \cdots C_n)p_n(T).$$

We transform this formula in two different ways: first, by replacing n by $n-1$, and secondly, by using once more the theorem on compound probabilities:

$$p(T) = p(C_1C_2 \cdots C_{n-1})p_{n-1}(T),$$
$$p(T) = p(C_1C_2 \cdots C_{n-1})p(C_n/C_1C_2 \cdots C_{n-1})p_n(T).$$

By comparing these last two formulas, we obtain*

$$\boxed{\frac{p_n(T)}{p_{n-1}(T)} = \frac{1}{p(C_n/C_1C_2 \cdots C_{n-1})}\,.}$$

Here is the formula which will give us precise and penetrating information on inductive reasoning. This formula shows that the quotient of the two plausibilities which interest us (plausibility of T before and after the verification of consequence C_n of T) depends on the relation of C_n to the consequences $C_1, C_2, \cdots, C_{n-1}$ verified before C_n. To consider the different cases which may arise, we will decrease the plausibility $p(C_n/C_1 \cdots C_{n-1})$ from 1 to 0.

To arrive at a clear discussion of our formula, it is advantageous to distinguish two cases. If C_n is a logical consequence of the preceding $C_1, C_2, \cdots, C_{n-1}$ taken together, we will say that C_n is *not* a *new* consequence of T; otherwise we will say that C_n is a *new* consequence.

If C_n is not a new consequence of T, the plausibility $p(C_n/C_1 \cdots C_{n-1})$ must be replaced by 1. In this case, and only in this case, $p_n(T)$ is equal to $p_{n-1}(T)$. This is clear: C_n following logically from $C_1, C_2, \cdots, C_{n-1}$ is already verified once $C_1, C_2, C_3, \cdots, C_{n-1}$ are verified; any further verification of C_n does not change the logical situation, and so does not affect the plausibility of T.

If C_n is a new consequence, the denominator of the right-hand member of our formula is less than 1 and so $p_n(T)$ is greater than $p_{n-1}(T)$. The plausibility of a theorem can only increase as a new consequence is confirmed.

In other words, each new consequence furnishes, by its confirmation, inductive evidence (augmentation of confidence) for the theorem that is being examined by its consequences. But the strength of this evidence may vary. Let us see what our formula has to say about this.

If the confirmation of the new consequence C_n has become very plausible through the verification of the preceding consequences $C_1, C_2, \cdots, C_{n-1}$ ($p(C_n/C_1C_2 \cdots C_{n-1}) \rightarrow 1$), then $p_n(T)$ is only slightly greater than $p_{n-1}(T)$, our

* This formula is found in J. M. Keynes, A treatise on probability, London, 1921, p. 235. The following is different from Keynes' discussion.

confidence in theorem T has increased only slightly by the confirmation of C_n; the inductive evidence is weak.

If, on the other hand, the confirmation of the new consequence has not become plausible through the verifications of the preceding consequences ($p(C_n/C_1 \cdots C_{n-1}) \to 0$), then our confidence in the theorem is greatly bolstered by this confirmation; the inductive evidence is strong.

Definitively, our formula has given us the following general rule:

Our confidence in a theorem can only increase as a new consequence of the theorem is established.

The increase in our confidence brought about by a new confirmation, or, if we wish, the inductive evidence furnished by this new confirmation, will vary inversely as the plausibility of the new consequence appraised in the light of the previously verified consequences.

We can give this rule another formulation which is preferable in certain respects. Note that

$$p(C_n/C_1C_2 \cdots C_{n-1}) + p(\overline{C}_n/C_1 \cdots C_{n-1}) = 1,$$

and write our formula as follows:

$$p_n(T)/p_{n-1}(T) = 1/[1 - p(\overline{C}_n/C_1 \cdots C_{n-1})].$$

Here $p(\overline{C}_n/C_1 \cdots C_{n-1})$ is the plausibility that C_n is false, based on the consequences verified up to this point. But if C_n is false, this will be borne out by its examination, and so theorem T will be refuted also, since C_n is one of its consequences. Thus, as $p(\overline{C}_n/C_1 \cdots C_{n-1})$, the plausibility of the refutation of T by the examination of C_n, increases, the left member of our equation increases. This we can state as follows:

That consequence which, on the basis of the preceding verifications, stands the best chance of refuting the given theorem will disclose the strongest inductive evidence if it is confirmed in spite of the forebodings.

But this is obvious. If a theorem succeeds in escaping unscathed from a situation presenting many pitfalls, it will be esteemed in proportion to the risk involved.

Let us now place ourselves in the position of the physicist or the mathematician who is examining a given theorem T by its consequences. To decide whether T is true or false, he notes some of its consequences and examines these.

If one consequence is discovered false, the theorem is refuted. In this respect all the consequences of the theorem are on a par.

If one of the examined consequences is found true, the plausibility of the given theorem is boosted. But in this respect the consequences of the theorem are not all equivalent; the verification of one consequence may produce a greater change of confidence, furnish stronger inductive evidence than that of another.

On what does the strength of inductive evidence depend? The two rules we

have just stated give the same answer to this question, but in two different forms.

On the one hand, the examination of a new consequence supplies strong inductive evidence when this consequence has not been made plausible by the consequences examined previously. In practice, this will be the case when this consequence has no immediate relation with the old ones, when it is removed from the preceding, when this new consequence is not only new, but of a new kind.

On the other hand, the examination of a new consequence introduces strong inductive evidence when it has a good chance of compromising the theorem. In practice, this will be the case when the examination touches upon a new aspect of the theorem, an aspect of the theorem which had not been previously considered.

But this is common sense. The prestige of the newtonian law of gravitation, based primarily on the motion of the planets, was immensely enhanced as successive verifications involved new phenomena: the movements of the comets, the tides, *etc.* Our confidence in a physical law depends in general more on the variety than on the multiplicity of its verifications. If we reread Euler's text, we shall see that there again, it was the consequences of a new kind which brought the most important evidence with which Euler convinced himself and hoped to convince the reader also.

8. Conclusions. I do not know if the ideas which have been presented have won the assent of the reader, but if they are not completely without foundation, they justify to a certain point the following theses:

1. *Impersonal heuristic conclusions exist.*

In fact, in what precedes we have seen several examples of such conclusions, of which the simplest are the following two:

The plausibility of a theorem can only increase when a consequence of the theorem is confirmed.

The plausibility of a theorem can only decrease when a hypothesis of which the given theorem is a consequence is refuted.

These conclusions differ from a syllogism, as an heuristic argument in general necessarily differs from a properly so-called logical argument. In contrast to a syllogism, they yield no definitive results, they deal only with degrees of confidence which are by their very nature of a transitory and provisional character. However, apart from this, these conclusions seem to me to be as impersonal and inevitable as a syllogism.

2. *The formulas of the theory of probability, taken qualitatively, are suitable for the presentation of impersonal heuristic conclusions.*

I insist on the clause to the effect that these formulas must be taken qualitatively, and in support of this thesis I refer to the examples already cited.

3. *The formulas of the theory of probability, taken qualitatively, are suitable to describe one component in the evolution of our feeling in the course of a research: the variation in the degree of belief.*

This thesis differs from the preceding thesis in that it is psychological while its predecessor is epistemological. I advance this psychological thesis with considerable reservation; I believe it deserves examination, but I am not sure to what extent it is correct.

4. *The calculus of probabilities is capable of a concrete interpretation from the subjective point of view. In fact, the preceding application gives such an interpretation. It must be added, however, that the interpretation given here is incomplete.*

The application given in the preceding paragraphs is incomplete since the magnitudes considered are variables for which no determinate numbers may be substituted; it must also be noted that the application is limited to the simplest formulas. The objective interpretation of the theory of probability, as the theory of frequencies, is in my opinion much more complete since here it is possible to arrive at numerical values. In advancing this thesis I am taking an intermediate position between the "objectivists" and the "subjectivists," but here I renounce any long defense of this position.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NOTE ON LOGARITHMS

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Consider the curve $XY=1$. The area under this curve from $X=a$ to $X=b$ is exactly equal to $\log b/a$, $b>a>0$. This area may be approximated by erecting ordinates aM and bN to the curve at $X=a$ and at $X=b$ and finding the area A of the trapezoid $abNM$. It is evident that

$$A = \frac{b^2 - a^2}{2ab} = \frac{1}{2}\left[\frac{b}{a} - \frac{a}{b}\right] = \frac{b-a}{2ab}(b+a). \tag{1}$$

By erecting an ordinate XP to the curve at $X=\sqrt{ab}$, we find that the area S of the pentagon $abNPM$ formed is represented by

$$S = (b-a)/\sqrt{ab}. \tag{2}$$

This is a still better approximation for $\log b/a$. By erecting an ordinate at $X=(a+b)/2$ and proceeding as above, we obtain for the area of the resulting pentagon

$$(3) \qquad B = \frac{b-a}{4}\left(\frac{1}{a} + \frac{4}{a+b} + \frac{1}{b}\right).$$

In all three cases the values obtained are greater than the area under the curve. Hence, any one of them may serve as an upper bound for $\log b/a$.

To get a bound for the error involved, we draw a line tangent to the curve at P; this is parallel to the chord MN and forms another smaller trapezoid whose area is

$$B' = \frac{b-a}{2}\left(\frac{4}{\sqrt{ab}} - \frac{1}{a} - \frac{1}{b}\right).$$

The area under the curve lies between the areas of these two trapezoids. It is not difficult to show that the error E involved in taking $\log b/a = A$ satisfies the inequality

$$E < \frac{b-a}{ab}(\sqrt{b} - \sqrt{a})^2.$$

If we take $\log b/a = S$, the error satisfies the inequality

$$E < \frac{b-a}{2ab}(\sqrt{b} - \sqrt{a})^2.$$

If we consider the curve MN as a parabolic arc and observe that the area of the figure bounded by the arc MPN and the chord MN is equal to $2/3$ of the area of the parallelogram $MKLN$, where KL is the tangent at P, and K and L are points on aM and bN, respectively, we are led to the excellent approximation

$$(4) \qquad \log b/a = F = \frac{b-a}{6ab}(8\sqrt{ab} - a - b), \qquad b/a < 4.$$

I do not have an explicit measure of the error involved, but it is far less than the error in the preceding cases, particularly when b/a is near unity. If we set $a=1$ and $b=1+X$ we get, on expanding,

$$(5) \quad F = \log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \frac{19X^5}{96} \cdots, \quad |X| < 1.$$

It is instructive to compare this with the standard formula

$$(6) \qquad \log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \frac{X^5}{5} \cdots, \quad |X| < 1.$$

By applying Simpson's rule we get another approximation, namely,

$$(7) \qquad \log b/a = H = \frac{b-a}{6}\left[\frac{1}{a} + \frac{8}{a+b} + \frac{1}{b}\right],$$

which is equivalent to

$$\log (1 + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \frac{5X^5}{24} \cdots, \quad |X| < 1, \tag{8}$$

when $a=1$, $b=1+X$ as before.

If we set $b-a=1$ in (1) and let b take on successive positive integral values, we are led to the interesting inequality

$$\log N < 1 + 1/2 + 1/3 + \cdots + 1/N - \frac{N+1}{2N}\cdot \tag{9}$$

Since $\lim_{N\to\infty} (1+1/2+1/3+ \cdots +1/N-\log N)=\gamma$, Euler's constant, it is natural to write

$$\log N = 1 + 1/2 + 1/3 + \cdots + 1/N - 1/2N - \gamma + k/N^2 \tag{10}$$

and to try to find a satisfactory value of the constant k. We are readily led to the value $k=1/12$. Hence

$$J = \log N = \sum_{n=1}^{N} 1/n - 1/2N - \gamma + 1/12N^2 \tag{11}$$

is a good approximation if N is not too small. The right side is always larger than the left side.

The four quantities A, B, S, and J serve as upper bounds as well as rough approximations. They are useful in developing certain inequalities.

Perhaps the reader would like to consider the following problems:

(a) Show that $\log \tan (\pi/4+X) \leqq \tan 2X$, and that the equality holds approximately when X is small.

(b) Show that $\log \sec X \leqq \frac{1}{2}[\sec X - \cos X]$, and that the equality holds approximately when X is small.

NOTE BY THE EDITOR

The recent article by A. A. Albert (*A rule for computing the inverse of a matrix*, this MONTHLY, March 1941, p. 198) caused considerable comment. The commenters formed two general classes; those who were already acquainted with the method and who were surprised that it was not generally known, and those who were not acquainted with it and who, after they learned of it, were also surprised that it was not generally known. In spite of the surprise exhibited by both classes, the latter group was definitely in the majority.

It was pointed out, though, that the rule has appeared in print before. On page 119 of *Elementary Matrices*, by Frazer, Duncan, and Collar (Cambridge University Press, 1938), the rule is stated in essentially the same form as that given by Albert. It also appears in a set of planographed *Notes on Higher Algebra*, by Saunders Mac Lane, and is included in Mac Lane and Birkhoff's *Survey of Modern Algebra*, (Macmillan, 1941). R.J.W.

NOTE ON GROUPS OF SUBTRACTION AND DIVISION

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Necessary and sufficient conditions that a finite group be generated by the two operators R_1 and R_2, defined by $R_1(z)=r/z$ and $R_2(z)=s-z$, where r, s are complex numbers, were given in a recent paper by E. J. Finan,* extending results of an earlier paper by G. A. Miller.† It was shown that the complex numbers r and s must be such that $x=s^2/r$ is a real root of one of two algebraic equations:

$$x^k-\binom{2k-1}{1}x^{k-1}+\binom{2k-2}{2}x^{k-2}-\cdots+(-1)^k\binom{k}{k}=0, \tag{1}$$

$$x^{k-1}-\binom{2k-2}{1}x^{k-2}+\binom{2k-3}{2}x^{k-3}-\cdots+(-1)^{k-1}\binom{k}{k-1}=0. \tag{2}$$

The object of this note is to show how the proofs of Finan and Miller may be shortened and their results extended by obtaining a simple trigonometric expression for the quantity s^2/r.

By introducing homogeneous variables z_1 and z_2, such that $z=z_1/z_2$ and $z'=z_1'/z_2'$, each of the two generating transformations R_1 and R_2 of the group can be represented by a pair of two-dimensional linear transformations of determinant -1, and their product R_2R_1 will be represented by a pair of such transformations of determinant $+1$, as follows:

$$\begin{aligned} &R_1:\ \begin{aligned}\pm z_1' &= r^{1/2}z_2,\\ \pm z_2' &= r^{-1/2}z_1;\end{aligned} \qquad R_2:\ \begin{aligned}\pm z_1' &= -z_1+sz_2,\\ \pm z_2' &= z_2;\end{aligned}\\ &R_2R_1:\ \begin{aligned}\pm z_1' &= sr^{-1/2}z_1-r^{1/2}z_2,\\ \pm z_2' &= r^{-1/2}z_1.\end{aligned}\end{aligned} \tag{3}$$

The transformation R_2R_1 may next be represented in the diagonal form

$$R_2R_1:\ \pm w_1'=\lambda_1w_1,\ \pm w_2'=\lambda_2w_2, \tag{4}$$

by setting

$$r^{-1/2}z_1=\lambda_1w_1+\lambda_2w_2, \qquad z_2=w_1+w_2, \tag{5}$$

where λ_1 and λ_2 are the roots of the characteristic equation of R_2R_1, namely,

$$\begin{vmatrix} sr^{-1/2}-\lambda & -r^{1/2}\\ r^{-1/2} & -\lambda\end{vmatrix}\equiv\lambda^2-sr^{-1/2}\lambda+1=0. \tag{6}$$

A necessary and sufficient condition that R_2R_1 be of finite period n is that either λ_1 and λ_2 or $-\lambda_1$ and $-\lambda_2$ be primitive nth roots of -1. Since $\lambda_1\lambda_2=1$, we must have

* E. J. Finan, On groups of subtraction and division, this MONTHLY, vol. 48, 1941, p. 3.

† G. A. Miller, Groups of subtraction and division, Quarterly Jour. of Math., 1905, pp. 80–87.

$$\pm\, sr^{-1/2} = \pm\,(\lambda_1+\lambda_2) = e^{i\theta}+e^{-i\theta} = 2\cos\theta;\qquad \theta = \nu\pi/n, \tag{7}$$

where ν and n are relatively prime integers. Thus we may express the quantity $x=s^2/r$ in the form

$$x = s^2/r = 4\cos^2 \nu\pi/n = 2\cos(2\nu\pi/n) + 2. \tag{8}$$

It follows immediately that s^2/r is an algebraic integer whose only rational values for integral n are 0, 1, 2, 3, 4. These correspond to the integers $n=2, 3, 4, 6, 1$; $\nu=1$ or $n-1$. When the group defined by R_1 and R_2 is of finite order $2n$, $n>2$, then the real algebraic integer $x=s^2/r$ will be a root of an algebraic equation with rational integral coefficients, irreducible in the rational field, whose degree is $\frac{1}{2}\phi(n)$, where $\phi(n)$ is the Euler ϕ-function.* The roots of this irreducible equation are the algebraic integers $2\cos(2\nu\pi/n)+2$, where ν takes on the $\frac{1}{2}\phi(n)$ integral values less than $n/2$ which have no factor in common with n. Equations (1) for odd $n=2k+1$, and (2) for even $n=2k$, are easily shown to have all the roots (8), with $0<\nu<n/2$, but they are not in general irreducible.

When r and s are assigned arbitrary complex values, the form of the linear fractional transformation for $(R_2R_1)^m$ can be simply expressed by first setting $s^2/r=4\cos^2\theta$, where θ is not necessarily a real parameter. The reader may then verify that the functions

$$P_m = r^{m/2}\sin(m+1)\theta/\sin\theta \tag{9}$$

are polynomials in r and s which satisfy the recursion relations

$$P_{m+1} = P_m s - P_{m-1} r. \tag{10}$$

It can then be proved by induction that $(R_2R_1)^m$ can be written in the form

$$[(R_2R_1)^m]z = \frac{P_m z - P_{m-1} r}{P_{m-1} z - P_{m-2} r}\cdot \tag{11}$$

Proof. This formula is true for $m=1$, since $P_{-1}=0$, $P_0=1$, $P_1=s$. Assuming the formula for a given value of m, and replacing z by $[R_2R_1]z=s-(r/z)$, we obtain

$$\begin{aligned}[(R_2R_1)^{m+1}]z &= \frac{P_m[s-r/z]-P_{m-1}r}{P_{m-1}[s-r/z]-P_{m-2}r} = \frac{(P_ms-P_{m-1}r)z-P_mr}{(P_{m-1}s-P_{m-2}r)z-P_{m-1}r}\\ &= \frac{P_{m+1}z-P_mr}{P_mz-P_{m-1}r}\cdot\end{aligned}$$

Hence formula (11) holds for the exponent $m+1$, and the induction proof is complete.

* This result was not proved in the papers cited.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Basic Geometry. By G. D. Birkhoff and Ralph Beatley. Atlanta, Dallas, and New York, Scott, Foresman and Company, 1941. 294 pages. $1.32.

The Theory of Econometrics. By H. T. Davis. Bloomington, Indiana, The Principia Press, Inc., 1941. 14+482 pages. $4.00.

Elementary Functions and Applications. Revised edition. By A. S. Gale and C. W. Watkeys. New York, Henry Holt and Company, 1941. 21+409 pages. $2.25.

The Analytical Foundations of Celestial Mechanics. By Aurel Wintner. (Princeton Mathematical Series, Vol. 5.) Princeton, New Jersey, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 12+448 pages. $6.00.

Education on an International Scale. By G. W. Gray. A History of the International Education Board, 1923–1938. With an Introduction by R. B. Fosdick. New York, Harcourt, Brace and Company, 1941. 13+114 pages. $2.00.

Brief Trigonometry. By E. A. Cameron. New York, Reynal and Hitchcock, 1941. 8+121+17 pages. $1.25.

Partial Differential Equations. By F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1941. 9+259 pages. $3.00.

Elements of the Differential and Integral Calculus. Revised edition. By W. A. Granville, P. F. Smith, and W. R. Longley. Boston, Ginn and Company, 1941. 11+556 pages. $3.60.

Sur les Fonctions Orthogonales de Plusieures Complexes avec les Applications à la Théorie des Fonctions Analytiques. By S. Bergmann. New York, Interscience Publishers, Inc., 1941. 62 pages. $1.50.

Waves. A Mathematical Account of the Common Type of Wave Motion. By C. A. Coulson. Edinburgh and London, Oliver and Boyd; New York, Interscience Publishers, Inc., 1941. 12+156 pages. $1.50.

Tables of Sine, Cosine, and Exponential Integrals. Vol. II. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York. Conducted under the sponsorship of the National Bureau of Standards. New York, Work Projects Administration, 1940. 36+225 pages.

Fundamentals of Mathematics. By Moses Richardson. New York, The Macmillan Company, 1941. 17+525 pages. $3.25.

Between Physics and Philosophy. By P. Frank. Cambridge, Mass., Harvard University Press, 1941. 238 pages. $2.75.

First Year of College Mathematics. By H. J. Miles. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1941. 17+607 pages. $3.00.

Guide to Tables in the Theory of Numbers. By D. H. Lehmer. (Bulletin of the National Research Council, No. 105, February, 1941. Division of Physical Sciences, Committee on Mathematical Tables and Aids to Computation, Report 1: Report of the Sub-committee on Section F: Theory of Numbers.) Washington, D. C., National Research Council, 1941. 14+177 pages. $2.50.

On the Study and Metrology of Silver-Punched Coins. By D. D. Kosambi. (Reprinted from New Indian Antiquary, Vol. IV, Nos. 1–2.) Bombay, Karnatak Publishing House. 61 pages. Rs3-0-0.

College Geometry. By P. H. Daus. New York, Prentice-Hall, Inc., 1941. 15+200 pages. $2.50.

Mathematical Tables. By H. B. Dwight. Elementary and Some Higher Mathematical Functions including Trigonometric Functions of Decimals of Degrees and Logarithms. New York and London, McGraw-Hill Book Company, Inc., 1941. 7+231 pages. $2.50.

Barlow's Tables. Squares, Cubes, Square Roots, Cube Roots, and Reciprocals of all Integer Numbers up to 12,500. Edited by L. J. Comrie. Fourth edition. London, E. and F. M. Span, Ltd.; Brooklyn, Chemical Publishing Co., Inc., 1941. 12+258 pages. 8s 6d.

Plane-Strain Distribution of Stress in Elastic Media. By D. L. Holl. (Iowa State College, Bulletin 148, February, 1941.) Ames, Iowa, Iowa State College, 1941. 56 pages. Free.

REVIEWS

Brief Trigonometry. By A. R. Crathorne and G. E. Moore. New York, Henry Holt and Company, 1941. 5+121 pages. $1.20.

In a book carrying this title and the sub-title "A Text in Twenty Assignments" and containing 30 pages of supplementary problems, tables, and other "back of the book" material, one might expect a drastic cut in subject-matter as compared to the usual text-book in plane trigonometry. Such a shortening is not to be found. Excepting inverse functions which are barely mentioned. trigonometric equations which include only those involving functions of one angle, and logarithms which include only the base 10, the only topic omitted and commonly found in other books is that of complex numbers. Its brevity lies chiefly in its style, embodying an efficiency and elegance in language not often found in an elementary book. Discussions are complete and well illustrated but there is little of the verbosity commonly appearing under the guise of "explanation."

The reviewer can cite a few statements which he finds offensive but they are not numerous or important. He prefers to cite one statement which he has found to be especially delightful. In defining the trigonometric functions for any angle the authors have led up to and displayed the usual equations $\sin \theta = y/r$, *etc.* Then follows the remark, "These definitions hold for all angles defined by a radius vector which does not coincide with one of the axes."

Trigonometric functions are first defined for acute angles and later for any angle. Tables include four-place logarithms of numbers and of trigonometric functions, and four figure values of trigonometric functions. The usual arrangement of chapters is supplanted by "Assignments" which are somewhat less inclusive than ordinary chapters. Answers are given to odd-numbered problems only, and are available for the others. The main text is followed by 23 pages of supplementary problems arranged according to the assignments with which they go.

The book should make an excellent text for use in courses which are, of necessity, short.

O. E. BROWN

Fundamentals of Mathematics. By Moses Richardson. New York, The Macmillan Company, 1941. 18+525 pages. $3.25.

This book is intended for first-year college students who have had at least the usual course in plane geometry and one year of algebra before entering. The purpose is to acquaint the pupil with the principles of mathematics, considered as an abstract science, but geared to his level, with a minimum of technique. More material is provided than can be used in the time assigned (three hours per week for two semesters), but the subjects are so presented that considerable choice is permitted in the second half of the book.

In order to test the teachability of the book before publication, and in particular, to determine whether the reasoning was beyond the grasp of those for whom it was intended, the material was employed by fifteen teachers in classes aggregating some 1800 students. It is written for students with an inquiring mind, an attitude that is stimulated by careful presentation and gradual development of the idea under consideration. To assist in this process there are a number of biographical notes and several portraits of mathematicians.

The author has adopted a brisk, even breezy style, sometimes with debatable results. Everywhere the quality of the earlier instruction received by the pupil is cited with a disparaging remark, which may easily act as a boomerang. Apart from this, which is a real blemish, the book is well written and well adapted to the purpose for which it is designed. There are a few cases dangerously close to the pitfalls warned against, such as circular reasoning, learning by rote, "clearly."

At the end of each chapter is a list of references, all in English, for further study. This does not claim to be exhaustive. A few excellent helps are missing.

The scheme of the book is, after a well written admonition to the student as to his attitude towards the course in general, to base algebra on foundations analogous to those employed in geometry; this includes a chapter on logarithms and one on impossible and unsolved problems. These subjects cover 210 pages, and are supposed to constitute a term's work. From now on, the student is expected to have this critical attitude toward any mathematical question.

The chapter of fifty pages on analytic geometry includes the straight line, circle, a glimpse into three and more dimensions, and a discussion of construc-

tions by ruler and compasses. Then follow functions, limits, and a sketchy introduction to the calculus. Thus far no use has been made of trigonometry. The next chapter of 30 pages presents the essentials, including the laws of sines and of cosines, and solutions of general triangles, but without addition theorems or the use of identities.

Then follow chapters on probability and statistics, induction, finite and infinite cardinal numbers (too brief), euclidean and non-euclidean geometry (very well written), groups, and a concluding chapter on the nature of mathematics.

An appendix of 30 pages provides trigonometric identities and formulas, discusses the process of completing the square, combinations and permutations, with applications to the binomial theorem, and four-place tables of logarithms and trigonometric functions. Answers to the exercises and a full index are provided. The book is strikingly free from typographical errors.

In the hands of competent and sympathetic teachers, the book will accomplish its laudable purpose. It is liable to do harm when used by others.

VIRGIL SNYDER

Trigonometry. Revised edition. By N. J. Lennes and A. S. Merrill. With tables. New York, Harper and Brothers, 1939. 12+245+92 pages. $2.20.

In this revised edition, the authors retain the basic point of view which determined the organization of their *Plane Trigonometry* in 1928, to prepare a text in which "trigonometric computation is treated completely from the start, with the development of only such formulas as are needed for the purpose; this being followed by the usual trigonometric analysis." Most of the changes indicated below will be seen to further their purpose.

Part I is devoted exclusively to the solution of triangles and includes the barest minimum of discussion of trigonometric concepts. The relations between the functions of an angle, and the addition formulas, which were formerly discussed here, have been relegated to Part II, and there is a reorganization of the work on oblique triangles to include a re-ordering of topics, some changes in proofs, and some new material.

In the second part of the book, devoted to trigonometric analysis, the usual material is given with one exception, *viz.*, there is no complete presentation of the reduction formulas. On the other hand, there are several chapters devoted to material which is not usually found in trigonometry texts, including, in addition to exponential and hyperbolic functions and an historical sketch, which were in the original edition, two new chapters entitled "Miscellaneous Work on Trigonometric Identities" and "Spherical Trigonometry," respectively. In the new edition the authors have given answers to odd-numbered problems and have introduced a much-needed index.

All the excellent features of the original book have been retained. The changes have added desirable features of compactness and should increase the effectiveness of the text as a teaching instrument.

MINA REES

Brief Trigonometry. By E. A. Cameron. New York, Reynal and Hitchcock, 1941. 8+117+17 pages. $1.25.

The purpose of this book is to present the essentials of plane trigonometry in the minimum time consistent with adequate understanding. The practical side is presented as early as possible in order to create and sustain interest in the subject. Four-place tables of logarithms, trigonometric functions, and squares are provided. It is estimated that the subject can be covered in 30 lessons. The figures, 54 in number, are well drawn, and the press-work is excellent.

VIRGIL SNYDER

Punched Card Methods in Scientific Computation. By W. J. Eckert. New York, The Thomas J. Watson Astronomical Computing Bureau, Columbia University, 1940. 9+136 pages. $2.00.

Electric accounting machines are now widely used in statistical research and in business. It is the purpose of this book to show how these machines can be used in the more general field of scientific computation. The author has had marked success in adapting the machines to astronomical calculation and research. However, the specific applications to astronomy are limited to Part III of the book, and the earlier parts are written for the scientist or mathematician who may be interested in finding efficient calculational techniques. The type of material discussed in Parts I and II is indicated by the chapter headings which are Introduction; The Punched Card and the Machines; General Considerations of the Punched Card Technique; The Construction and Use of Special Tables of Tabular Functions; Interpolation, Mechanical Quadrature, and Allied Subjects; Numerical Harmonic Analysis and Synthesis; The Multiplication of Series; and The Numerical Solution of Differential Equations. The astronomer will be especially interested in Part III which has chapters entitled The Calculations Involved in the Construction of a Catalogue of Photographic Star Positions; Stellar Photometry; Numerical Lunar Theory; and The Computation of Planetary Perturbations.

The book is not a manual of machine operation but is designed to give some perspective as to the general nature of the machines, the basic principles of operation, the types of computational techniques which are now available and, in the language of the author, "to enable a scientist so to formulate his problem that any skilled operator of the machines could carry it out."

The presentation is clear and concise, and sufficiently illustrated with diagrams.

This book opens up a field which will unquestionably be developed extensively in the years to come. Though detailed applications to all fields are not presented, for example the applications to the field of statistics are not discussed extensively, the book and the punched card method which it advocates might well be studied by anyone who is seeking relief from extensive numerical computation.

P. S. DWYER

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

CONSTRUCTIONS WITH LIMITED MEANS

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Students learn in secondary school to carry out geometrical constructions with ruler and compasses. How often the idea of using either of these instruments without the other occurs to them is, of course, not known. This problem of making constructions with limited means has an interesting history. A list* of those who considered the use of compasses alone must include the following: G. Mohr (1672), a Danish mathematician [**3**]; L. Mascheroni (1797), whose book on the subject was translated into French in 1798 and into German in 1825; and A. Adler (1890). Among the mathematicians who interested themselves in the use of the ruler alone are found J. V. Poncelet (1817, 1822) and J. Steiner (1833). The work of these two groups was brought together for comment and discussion by Hilda Hudson [**4**] in a book called *Ruler and Compasses* written in English in 1916. The problem is thus seen to have merited attention in at least five languages. No attempt is made here to compile an exhaustive list of writers concerned with it.†

The words "ruler" and "compasses" have been used freely in the preceding paragraph without regard to exactly what they are and what can be done with them. Is the ruler graduated, or simply an unmarked straight-edge? Can the length of a segment be carried from one place to another at will by compasses? Or can we simply place one foot on the blackboard (or paper) and with the other describe a circle?—to quote from that inspired volume *The Second Boners Omnibus*. According to the type of instruments used, the constructions that can be carried out will differ. Hilda Hudson says that Euclid's compasses carry the length of a segment only "from one radius to another of the same circle," and his ruler cannot "carry distance" at all. Mascheroni uses compasses as dividers (*i.e.*, to transport segment-length) at will.

The various mathematicians mentioned above employ quite different theory to lead to their constructions. Mascheroni deals with reflection in a straight line to a considerable extent. Adler makes inversion with regard to a circle an important element in his construction planning. It is interesting to note that some of Mascheroni's constructions are decidedly shorter than Adler's. For example [**4**], to determine the points of intersection of a given circle and a straight line joining two given points, Adler draws sixteen circles and Mascheroni three. For euclidean compasses, five circles would be required to solve this problem. The student may wonder why any circles at all are needed. Why not simply

* See the bibliography at the end of the paper.

† For further names, see Archibald [**2**].

draw the straight line and the circle and *see* where they intersect? The answer is that when the use of a ruler is forbidden, no straight lines at all are drawn. Such a line is simply given by two points. For instance, when one point on a straight line is known and the line is required, the problem is solved by locating one more point thereon as the intersection of circular arcs.

It has been proved that all constructions possible by straight-edge and compasses are also possible by compasses alone. Moreover, they can be carried out by the straight-edge alone, provided there is given in the plane one fixed auxiliary circle whose center is known. This last is Steiner's thesis, contained in his book mentioned in the bibliography [**9**]. He comes to the actual proof on the fiftieth page (Chapter III) of this eighty-page volume, after preparing the way for it and leading up to it through the first two chapters. Poncelet and not Steiner was the originator of the idea [**2**], just as Mohr before Mascheroni was responsible for the geometry of the compasses [**3**]. One should therefore really say "Poncelet constructions" and "Mohr constructions" instead of the usual "Steiner constructions" and "Mascheroni constructions."

The books chiefly consulted in the preparation of this article are those of Mascheroni (translation into French by Carette) [**5**], Steiner [**9**], and Hilda Hudson [**4**]. Before some of the actual constructions are carried out with various instruments and auxiliary figures, a few comments from these authors may be of interest.

From Hilda Hudson comes the statement that those problems may be solved by ruler and compasses which depend on algebraic equations whose degrees are powers of 2 and whose roots may be found by rational operations and the extractions of square roots only, whereas those problems may be solved by straight-edge alone which depend on linear equations with roots obtainable by rational operations only. The student who is more attracted to analysis than to geometry will find this interesting, and will see the connection on recalling that the equation of a straight line is of the first degree, while that of a circle is of the second degree. The sixth chapter on the comparison of methods is worthy of note. The author says here: "Among many ways in which errors may arise, one of the most frequent is in determining the point of intersection of two lines, straight or curved, which meet at a very small angle." Then there follows a discussion on "economizing the chance of error" in making constructions.

Steiner asserts that upon just two problems depend all constructions usually carried out *by straight-edge and compasses*. These are the following:

(*i*) to find the intersections of a straight line and a circle, and

(*ii*) to find the intersections of two circles.

He adds that when one is trying to show that all such constructions can be carried out by straight-edge alone, provided a fixed auxiliary circle and its center are given in the plane, then the solution of all problems depends on (*i*), while (*ii*) must and can be reduced to (*i*).

Mascheroni states that he wishes to avoid the use of the straight-edge because it is wellnigh impossible to be sure of all the points it locates on account

of the difficulty of having it exactly straight everywhere. The compasses form a much more accurate instrument, needing only to have the opening firmly fixed and the points made very sharp. Geometry of the compasses is of great value to the makers of astronomical instruments, and possibly to military engineers. He has put into his "eleventh book," or eleventh chapter, a collection of mathematical recreations "for the sake of artists who will wish simply to entertain themselves with the compasses."

Suppose now, for the sake of entertainment for those who may or may not be artists, a few problems be considered. Reasons are not given here, but those underlying the majority of the constructions in the books cited can be understood by most students who have had three years of college mathematics. "Line" in what follows may always be taken to mean "straight line." The student is urged to carry out the constructions on paper while reading what follows.

Problem I. Draw through any point a line parallel to a given line, the point being not on the given line, by the following methods:

(*a*) By means of a straight-edge alone, when three points on the given line are known with one midway between the other two.

Let the three given points be B, D, and F, with $BD=DF$. We are to draw through a point H a parallel to BF. Draw BH and FH. Take any point A on BH, and draw AD and AF. Join the intersection* C of FH and AD with B, and let this line cut AF in I. Then HI is the desired line.

(*b*) By means of a straight-edge alone, when a parallelogram is given.

Let $ABCD$ be the given parallelogram, with diagonals AC and BD meeting at E. Through E draw to one of the two given pairs of parallel lines, as to AD and BC, a third parallel EF. [A construction must be indicated for bisecting a segment on one of two known parallel lines, and then (*a*) will provide this third parallel. Let a segment MN be taken on one of the two parallels. Draw from any point P to the end-points M and N of the given segment the lines PM and PN, which cut the other parallel in H and I, respectively. Draw the lines HN and IM, which meet in some point Q. Then the line PQ will pass through the middle point R of the segment MN.] This line EF lies midway between AD and BC, so that the three parallels cut every other line (not parallel to them) in three points such that one of them lies midway between the other two. Then use (*a*).

(*c*) By means of a straight-edge alone, when a fixed circle and its center are given in the plane.

* *Editorial Note.* The construction is faulty here, since if A (which is "any point" on BH) is the midpoint M, there is no point C as described. Moreover, it does not suffice to say merely, "let A be any point other than M on BH," for we must show how such a point is chosen when we do not know how to find M. To the remark, "If you happen to have selected the one point on BH that will not work, take another instead,—they are all there for you to choose from," one may reply that this assumes that geometry is an experimental science. Moreover, as a matter of trial with a finite sheet of paper, there are infinitely many points A for which FH and AD do not intersect.

E.J.M.

If the given line goes through the center of the auxiliary circle, (*a*) can be used at once.

If the given line is a secant but not a diameter of the auxiliary circle, draw lines from the intersection points C and D through the center M to meet the circle again in C_1 and D_1, respectively. Then C_1D_1 is parallel to CD, and the rest of the construction follows as in (*b*).

If the given line does not meet the circle, take any point G on it and draw the diameter ABG, meeting the circle in A and B. Through any point C of the circumference of the circle draw CD parallel to AB, meeting the circle again in D and the given line in E. Draw the diameters CMC_1 and DMD_1, and the line C_1D_1 to meet the given line at F. Then G is midway between E and F, and (*a*) can be used.

(*d*) By means of compasses alone.

Let the line be given by the points A, B. Through C we are to draw a parallel to AB. With C as center and AB as radius describe an arc. With B as center and AC as radius describe an arc. The two arcs meet in the desired point D, such that CD is parallel to AB.

Problem II. Drop a perpendicular on any given line from any given point, by the following methods:

(*a*) By means of a straight-edge alone, when a square is given.

Let $ABCD$ be the given square and E the point of intersection of its diagonals, AC and BD. Draw through E any line meeting CD in F and AB in G. From F draw a line parallel to BC or AD, meeting AB in H. From H draw a parallel to AEC, meeting BC at I. Then IE is perpendicular to FEG. Therefore, if a perpendicular is to be dropped on any given line F_1G_1 from any given point I_1, draw through E the line FG parallel to F_1G_1, erect EI perpendicular to FEG, and draw through I_1 the line I_1E_1 parallel to IE.

(*b*) By means of a straight-edge alone, when a fixed circle and its center are given in the plane.

If the given line AB is a diameter of the auxiliary circle, draw any chord CD parallel to AB. Draw the diameter DMD_1 and the chord CD_1. Then CD_1 is perpendicular to AB. Through the given point draw a parallel to CD_1.

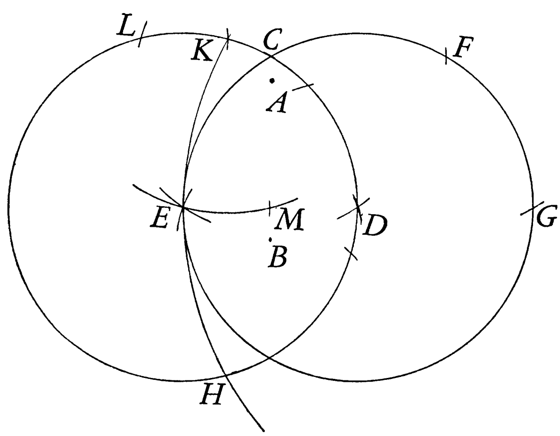
If the given line is a secant but not a diameter of the auxiliary circle, the intersection points being A and B, draw the diameters AA_1 and BB_1, and the chords AB_1 and BA_1. These last are perpendicular to AB and therefore parallel to each other. Then draw through the given point a parallel to these chords.

If the given line does not cut the auxiliary circle, draw any chord parallel to it, as DC_1. Then draw the diameters DD_1 and C_1C, and the chords CD and D_1C_1. These are perpendicular to DC_1 and therefore to the given line, and hence parallel to each other. Then through the given point draw a parallel to the chords CD and D_1C_1.

(This problem may also be solved in an interesting manner by means of harmonic properties.)

(*c*) By means of compasses alone (*i.e.*, find the point of intersection as well as one other point on the perpendicular).

Given A and B and a point D not on the line AB, find another point E such that DE is perpendicular to AB, and find the point M where AB and DE meet. With center A and radius AD draw an arc. With center B and radius BD draw a second arc meeting the first one at the desired point E. (This is, of course, a construction known to any reader.) Now bisect DE to find the desired point M as follows. With D as center and DE as radius describe a circle. Make $DE=EC=CF=FG$. (See figure.) Then $ECFG$ is a semicircumference. With center G and radius GE describe an arc. With E as center and radius ED describe a semicircumference, cutting this last arc at H and K, the semicircumference being $HDCKL$. With center K and radius KE draw an arc EM. Make EM equal to KL. The desired point M is thus seen to be the intersection of the last two arcs drawn.



The preceding examples give only a small sample of very easy constructions carried out with limited means. It is hoped that they will "taste like more" to some students, who may care to introduce this type of work to friends at Mathematics Club meetings.

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PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 481. *Proposed by J. A. Todd, University of Cambridge.*

Let

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

be two matrices of non-vanishing numbers, the elements of the second being the co-factors of the corresponding elements of the first. Prove that the relation

$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0 \quad \text{implies} \quad \begin{vmatrix} X_1^{-1} & Y_1^{-1} & Z_1^{-1} \\ X_2^{-1} & Y_2^{-1} & Z_2^{-1} \\ X_3^{-1} & Y_3^{-1} & Z_3^{-1} \end{vmatrix} = 0.$$

E 482. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find a four-digit square, other than 70^2, whose last two digits are unaltered when we change from the denary to the septenary scale.

E 483. *Proposed by N. A. Court, University of Oklahoma.*

Show that the four spheres having two points in common and each passing through a vertex and the foot of the corresponding altitude of a given orthocentric tetrahedron form a coaxal pencil.

E 484. *Proposed by David Segal, Kosow Huculski, Poland.*

Defining

$$\phi_n(x) = 1 - \binom{2n}{2} x^2 + \binom{2n}{4} x^4 - \cdots \pm x^{2n},$$

$$\psi_n(x) = \binom{2n}{1} x - \binom{2n}{3} x^3 + \cdots \mp \binom{2n}{2n-1} x^{2n-1},$$

prove that, if n is even,

$$\phi_n(3) = \pm 2^n \phi_n(2), \qquad \psi_n(3) = \pm 2^n \psi_n(2),$$

and that if n is odd,

$$\phi_n(3) = \pm 2^n\psi_n(2), \qquad \psi_n(3) = \pm 2^n\phi_n(2).$$

E 485. *Proposed by J. Goodfellow, West Rumney, N. H.*

Let AOB be an obtuse angle, A and B on a circle with center O. Take F and G on the minor arc AB, in directions perpendicular to OB and OA. Take D on the major arc AB so that AOD is an equilateral triangle. Take H on AD, and J on BD, so that HJ is equal and parallel to FG. Join FH, and produce to meet the circle again at K. Show that the arc AK is approximately one-third of the arc AB.

SOLUTIONS

E 443 [1940, 657]. *Proposed by N. A. Court, University of Oklahoma.*

(a) Two triangles, one inscribed in the other, are in perspective. Prove that on a parallel to the axis, the center of perspective trisects the intercept between any pair of corresponding sides. (*Educational Times Reprints*, vol. 4, 1903, p. 57, Question 15259.)

(b) Two tetrahedra, one inscribed in the other, are in perspective. Prove that on a line parallel to the plane of perspective, the center of perspective quadrisects the intercept between any pair of corresponding faces.

Solution (abridged) by L. M. Kelly, University of Missouri.

(a) The proof is precisely analogous to the following.

(b) We use general homogeneous coördinates, referred to the larger tetrahedron, with the center of perspective as "unit point" (1, 1, 1, 1), so that the inscribed tetrahedron has vertices

$$(0, 1, 1, 1), \quad (1, 0, 1, 1), \quad (1, 1, 0, 1), \quad (1, 1, 1, 0).$$

Corresponding faces, such as

$$x_1 + x_2 + x_3 + x_4 = 3x_4 \quad \text{and} \quad x_4 = 0,$$

meet on the plane of perspective

$$x_1 + x_2 + x_3 + x_4 = 0,$$

which is the tetrahedral polar of (1, 1, 1, 1). The coaxial plane through (1, 1. 1, 1) is

$$x_1 + x_2 + x_3 + x_4 = 4x_4.$$

The cross-ratio of these four planes is the same as that of the numbers 1/3, 0, ∞, 1/4, (namely, -3; see N. A. Court, *On the cevian tetrahedron*, this MONTHLY, vol. 43, 1936, p. 89). Hence, on a line parallel to the third plane, the fourth quadrisects the intercept between the first and second.

Also solved by the proposer.

E 444 [1940, 658]. *Proposed by Harry Goheen, Reed College, Portland, Oregon.*

Prove that there is no prime p such that $p^n+1=2^m$ if $n>1$, and that there is no prime p such that $p^n-1=2^m$ if $n>2$.

Solution by B. A. Hausmann, University of Detroit.

(*i*) Writing $p=2k+1$, we have

$$\begin{aligned}(p^n + 1)/2 &= \{(2k + 1)^n + 1\}/2 \\ &= 2^{n-1}k^n + n2^{n-2}k^{n-1} + \cdots + nk + 1.\end{aligned}$$

If n is even, this number is odd, and so cannot divide 2^m unless $k=0$. If n is odd and greater than 1, we have

$$(p^n + 1)/(p + 1) = p^{n-1} - p^{n-2} + \cdots + 1,$$

which is the sum of an odd number of odd numbers. Hence we cannot have $p^n+1=2^m$ with p and n greater than 1.

(*ii*) If n is odd and greater than 1, we have

$$(p^n - 1)/(p - 1) = p^{n-1} + p^{n-2} + \cdots + 1,$$

which is again odd. If n is even, we have

$$p^n - 1 = (p^{n/2} - 1)(p^{n/2} + 1).$$

But, by (*i*), p^r+1 cannot be a power of 2 if $r>1$. Hence we cannot have $p^n-1=2^m$ with $n>2$.

Also solved by E. P. Starke and P. R. Zilsel, who likewise observed that the restriction to primes is unnecessary.

E 445 [1940, 658]. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that if $0<m<n+2$, then

$$\sum_{r=0}^{m-1} (-1)^r \binom{n}{r}^{-1} = \frac{n+1}{n+2}\left\{1 - (-1)^m \binom{n+1}{m}^{-1}\right\}.$$

Solution by Louis Weisner, Hunter College of the City of New York.

From the identity

$$\sum_{r=0}^{m-1} x^{n-r}y^r = (x^{n+1} - x^{n-m+1}y^m)/(x - y),$$

we have, after substituting $y=-(1-x)$,

$$\sum_{r=0}^{m-1} (-1)^r x^{n-r}(1 - x)^r = x^{n+1} - (-1)^m x^{n-m+1}(1 - x)^m.$$

Observing that

$$\int_0^1 x^p(1-x)^q dx = \frac{p!q!}{(p+q+1)!} = \frac{1}{p+q+1}\binom{p+q}{q}^{-1}$$

for any non-negative integers p and q, we integrate the above identity termwise between 0 and 1, and obtain

$$\frac{1}{n+1}\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}^{-1} = \frac{1}{n+2} - \frac{(-1)^m}{n+2}\binom{n+1}{m}^{-1}.$$

Multiplying by $n+1$, we have the proposed relation.

Also solved (by induction over m) by Frances E. Crook, Nathan Newman, and E. P. Starke.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4002. *Proposed by F. A. Lewis, University of Alabama.*

Give an interpretation to the function that results from the Euler ϕ-function when the minus signs are changed to plus, namely $f(n)=n(1+1/p_1)(1+1/p_2)\cdots(1+1/p_k)$.

4003. *Proposed by G. W. Petrie, South Dakota State School of Mines.*

Three men have respectively l, m, and n coins which they match so that the odd man wins. In case all coins appear alike they repeat the throw. Find the average number of tosses required until one man is forced out of the game.

4004. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres, let (E) be the tetrahedron formed by their centers, and (F) the tetrahedron formed by the four polar planes, for these spheres, of their radical center U. Prove that (i) if (E) admits U for its orthocenter, the same holds for (F); (ii) conversely, if (F) admits U for orthocenter, the same holds for (E).

4005. *Proposed by V. Thébault, Le Mans, France.*

An arbitrary plane (P) cuts the planes of the faces of the given tetrahedron $ABCD$ in four straight lines. The four planes through these straight lines perpendicular each to the corresponding face determine a tetrahedron $A'B'C'D'$. Show that: (1) The straight lines AA', BB', CC', DD' are concurrent. (2) The point of concurrency I is the center of one of the spheres tangent to the four

planes symmetric to (P) with respect to the faces of $ABCD$. (3) The point I remains fixed when (P) moves parallel to itself.

3949 [1940, 182]. *Corrected. Proposed by P. Turán, Budapest, Hungary.*

Given the angles $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$ with the common initial line Ox, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{n(n+1)/2+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$, or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$.

SOLUTIONS

3942 [1940, 114]. *Proposed by H. E. Tester, Isleworth, Middlesex, England.*

A man is standing at the junction of two perpendicular cross-roads, and his dog, at a distance a from the junction along one of the roads, is watching him. At a given instant the man starts to walk with speed v along the other road, and the dog to run directly towards his master with speed $2v$. Determine the curve of pursuit.

Solution by H. A. Luther, A. and M. College of Texas.

Use cartesian coördinates, and let the initial position of the dog be the point $(0, a)$. Let the master start at the origin, and let his position at any time be $(x_1, 0)$, x_1 being positive. The coördinates giving the dog's position at any time are taken as x and y. The speed v of the master is assumed to be a variable quantity. Let ϕ be any function of y, except that (in part for simplicity's sake) we require it to be continuous and different from zero for values of y between 0 and a. We take the speed of the dog to be ϕ times v, since it develops that this change does not complicate the solution of the problem. For the problem as proposed, $\phi = 2$.

The dog's line of sight is tangent to his path. With this in mind, the following relationships are readily seen:

$$\dot{x}_1 = v, \tag{1}$$

$$\frac{dy}{dx} = \frac{y}{x - x_1}, \tag{2}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \phi v. \tag{3}$$

Solve equation (2) for x_1, and differentiate with respect to t. This gives

$$\dot{x}_1 = v = \dot{x} - \dot{y}\frac{dx}{dy} - y\frac{d}{dt}\left(\frac{dx}{dy}\right) = -y\dot{y}\frac{d^2x}{dy^2},$$

making use of $dx/dy = \dot{x}/\dot{y}$ and $d/dt = d/dy(dy/dt)$. Substitution in (3), together with division of both sides by $-\dot{y}$, gives

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \phi y \frac{d^2x}{dy^2}\cdot$$

This is readily integrated by use of the substitutions $u=dx/dy$ and $du/dy=d^2x/dy^2$ to yield

$$\frac{dx}{dy}=\sinh\left(\int_a^y \frac{dy}{\phi y}\right).$$

This in turn means

$$x=\int_a^y \sinh\left(\int_a^y \frac{dy}{\phi y}\right)dy.$$

It is perhaps not startling to find that the path of the dog is independent of v.

For the problem proposed, $\phi=2$. Hence its solution is

$$x=\frac{2a}{3}+(y-3a)\frac{\sqrt{y}}{3\sqrt{a}}\cdot$$

Solved also by M. W. Aylor, W. B. Brown, Ch'eng Ching, E. Comfort, H. H. Downing, B. E. Gatewood, T. R. Running, and the proposer.

Editorial Note. Some of the solvers gave the following references: Fine's *Calculus*, p. 295, where the result is given; Osgood's *Advanced Calculus*, p. 332, which gives a solution; the solution of E 387 [1940, 320] which gives several references to this MONTHLY and other sources.

3943 [1940, 114]. *Proposed by H. E. Tester, Isleworth, Middlesex, England.*

Express the integral

$$I=\int_0^3 dx\int_0^{(5-5x^2/9)^{1/2}} dy$$

by the use of the variables λ and μ defined by the equations

$$\lambda+\mu=[(x+2)^2+y^2]^{1/2},$$
$$\lambda-\mu=[(x-2)^2+y^2]^{1/2},$$

and thus verify that the value of the integral is $3\pi\sqrt{5}/4$.

Solution by Li Ou, Yenching University, Peking, China.

We are to find the area of a quadrant of the ellipse $b^2x^2+a^2y^2=a^2b^2$, where $a=3$ and $b=\sqrt{5}$, and thus $c=2$. The new variables 2λ and 2μ in the equations of transformation of the problem mean geometrically the sum and the difference of the two focal radii. It is easily seen from the graph that the limits for λ are from 2 to 3, and those for μ from 0 to 2. We solve for x and y, and thus obtain

$$(1)\qquad 2x=\lambda\mu,\qquad 4y^2=(\lambda^2-4)(4-\mu^2).$$

The Jacobian of (x, y) with respect to (λ, μ) can be computed directly to be numerically $(\lambda^2-\mu^2)/2y$. From (1), we therefore have

$$
\begin{aligned}
I &= \int_2^3 \int_0^2 (\lambda^2 - \mu^2)/\sqrt{(\lambda^2 - 4)(4 - \mu^2)}\, d\mu d\lambda \\
&= \frac{\pi}{2}\int_2^3 \lambda^2/\sqrt{\lambda^2 - 4}\, d\lambda - \pi\int_2^3 1/\sqrt{\lambda^2 - 4}\, d\lambda \qquad (2)\\
&= 3\pi\sqrt{5}/4.
\end{aligned}
$$

Solved also by the proposer.

3944 [1940, 114]. *Proposed by V. Thébault, Le Mans, France.*

The positive integral point-masses x, y, z are placed respectively at the vertices A, B, C of an equilateral triangle with sides of length a. Determine x, y, z, a so that the distances of the centroid of the three masses to the three vertices shall be integers.

Editorial Note. The proposer gave the following results:

$$GA = au/s, \qquad s = x + y + z, \qquad u^2 = y^2 + yz + z^2,$$

where GB and GC are obtained from the above by cyclic permutations. The numbers u, v, w must be integers, for example,

$$
\begin{aligned}
x &= 435, & y &= 4669, & z &= 1656,\\
u &= 5681, & v &= 1911, & w &= 4901.
\end{aligned}
$$

3947 [1940, 181]. *Proposed by N. A. Court, University of Oklahoma.*

If M, M' are two isogonal conjugate points for the tetrahedron $DABC$, S the projection of M upon the plane ABC, and S' the point common to the planes perpendicular to the lines $M'A$, $M'B$, $M'C$ at the points A, B, C, show that the line SS' and its three analogs PP', QQ', RR' have a point in common.

Solution by V. Thébault, Le Mans, France.

The projections P, Q, R, S of the point M on the planes of the faces BCD, CDA, DAB, ABC of the tetrahedron $DABC$ are the vertices of the pedal tetrahedron of M, while the points P', Q', R', S' are the vertices of the antipedal tetrahedron of M'. The straight lines AM', BM', CM', DM' are perpendicular to the planes QRS, RSP, SPQ, PRQ. The tetrahedrons $PQRS$ and $P'Q'R'S'$, whose corresponding faces are parallel, are therefore homothetic; and the straight lines PP', QQ', RR', SS' are concurrent in the homothetic center of the two tetrahedrons. This property is classic.

Solved also by the proposer in a similar manner.

Editorial Note. The locus of the points whose pedal tetrahedron with respect to $DABC$ is degenerate is a surface through the vertices of the latter tetrahedron; and in the problem above, no one of the two conjugate points M, M' with respect to $DABC$ lie on this surface. The proof of the classic property, using polar theory, is easy and essentially the same for three dimensions as for two. It would then seem easy to prove the theorem of the problem for n dimensions.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Guggenheim Fellowships for 1941–42 include the following awards to mathematicians: Dr. Richard Brauer of the University of Toronto, who will study at the University of Chicago; Dr. Jesse Douglas, Brooklyn; Dr. Deane Montgomery of Smith College, who will work at the Institute for Advanced Study; and Dr. Alfred Tarski, refugee mathematician from Poland, where he was professor of mathematics at the University of Warsaw.

W. G. Banks of Centenary College has been promoted to an assistant professorship.

J. J. Barron of St. Joseph College, Hartford, Connecticut, has been promoted to an associate professorship.

Associate Professor M. A. Basoco of the University of Nebraska was promoted February 1, 1941, to a professorship.

Brother Bernard Alfred of Manhattan College has been promoted to an assistant professorship.

Professor G. A. Bliss of the University of Chicago retired on September 30 with the title of Martin A. Ryerson Distinguished Service Professor Emeritus of Mathematics.

Assistant Professor J. G. Bowker of Middlebury College has been promoted to an associate professorship.

Assistant Professor Fannie W. Boyce of Wheaton College, Illinois, has been promoted to an associate professorship.

Dr. J. R. Britton of the University of Colorado has been promoted to an assistant professorship of engineering mathematics.

Assistant Professor B. L. Brown of Amherst College has been promoted to an associate professorship.

At Case School of Applied Science Assistant Professor O. E. Brown has been promoted to an associate professorship, and Dr. R. F. Rinehart to an assistant professorship.

Assistant Professor I. S. Carroll of Syracuse University has been promoted to an associate professorship.

Professor R. V. Churchill of the University of Michigan will be on leave of absence for the first semester 1941–42, in order to accept a visiting lectureship at the University of Wisconsin for that semester.

Dr. A. H. Clifford of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Miss Rachel Davison of Houghton College has been promoted to an associate professorship.

Assistant Professor E. D. Eaves of the University of Tennessee has been promoted to an associate professorship.

Associate Professor Nat Edmonson, Jr., of the A. and M. College of Texas has been promoted to a professorship.

Associate Professor P. D. Edwards of Ball State Teachers College has been promoted to a professorship.

Dr. Samuel Eilenberg of the University of Michigan has been promoted to an assistant professorship.

Dean L. P. Eisenhart of Princeton University has been appointed one of the directors of the National Science Fund of the National Academy of Sciences.

Professor G. C. Evans of the University of California and Professor J. R. Kline of the University of Pennsylvania have been elected to membership in the American Philosophical Society.

Professor C. H. Fischer of Wayne University has been appointed visiting assistant professor of mathematics at the University of Michigan.

Dr. C. H. Frick of Montana State College has been appointed professor of mathematics at Mary Washington College, Fredericksburg, Va.

Professor R. E. Gaines of the University of Richmond has retired as head of the department of mathematics after fifty-one years of service in this position. He will continue to be associated with the department. Associate Professor C. H. Wheeler, III, has been promoted to a professorship and made head of the department.

Assistant Professor F. C. Gentry of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Dr. Harriet M. Griffin of Brooklyn College was promoted in January 1941 to an assistant professorship.

Assistant Professor D. F. Gunder of the Colorado State College of A. and M. A. has been promoted to an associate professorship.

Dr. E. H. Hadlock of Cornell University has been appointed an assistant professor at Hastings College, Nebraska.

Dr. Mary E. Haller of the University of Washington has been promoted to an assistant professorship.

Dr. H. J. Hamilton of Pomona College has been promoted to an assistant professorship.

Associate Professor H. E. Hartig of the University of Minnesota has been made professor of communication engineering.

Assistant Professor M. A. Heaslet of San José State College has been promoted to an associate professorship.

Miss Gertrude Hendrix of Eastern Illinois State Teachers College has been promoted to an assistant professorship.

Dr. P. G. Hoel of the University of California at Los Angeles has been promoted to an assistant professorship.

C. W. Hook of Georgia School of Technology has been promoted to an assistant professorship.

Dr. M. Gweneth Humphreys of Sophie Newcomb College has been promoted to an assistant professorship.

Assistant Professor E. D. Jenkins of Eastern Kentucky State Teachers College has been promoted to an associate professorship.

Dr. R. N. Johanson of Bradley Polytechnic Institute has been promoted to an assistant professorship.

Assistant Professor D. C. Lewis of the University of New Hampshire has been promoted to an associate professorship.

Associate Professor G. R. Livingston of San Diego State College has been promoted to a professorship.

Assistant Professor Saunders Mac Lane of Harvard University has been promoted to an associate professorship.

Associate Professor W. G. McGavock of Davidson College has been promoted to a professorship.

Assistant Professor A. S. McMaster of the University of Colorado is on leave of absence for the year 1941–42 for study at Iowa State College.

Dr. Dorothy Manning of Wells College has resigned her position and has become the bride of Dr. M. F. Smiley of Lehigh University.

Assistant Professor E. W. Miller of the University of Michigan has been promoted to an associate professorship.

Dr. G. T. Miller of Purdue University has been promoted to an assistant professorship.

Dr. C. J. Nesbitt of the University of Michigan has been promoted to an assistant professorship.

Dr. E. D. Rainville of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor A. W. Recht of the University of Denver has been promoted to an associate professorship of mathematics and astronomy.

Associate Professor C. J. Rees of the University of Delaware has been made professor and head of the department of mathematics and astronomy.

Dr. M. A. Sadowsky of Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Arthur Sard of Queens College has been promoted to an assistant professorship.

Assistant Professor Nathan Schwid of the Texas College of Mines has been promoted to an associate professorship.

Assistant Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to an associate professorship.

Dr. Joseph Slepian of the Westinghouse Electric and Manufacturing Company and Professor T. Y. Thomas of the University of California at Los Angeles have been elected to membership in the National Academy of Sciences.

Dr. C. E. Smith, assistant professor of astronomy at San Diego State College, has been promoted to an associate professorship.

Associate Professor I. S. Sokolnikoff of the University of Wisconsin has been promoted to a professorship.

Associate Professor C. E. Springer of the University of Oklahoma has been promoted to a professorship.

Dr. Anna A. Stafford of the University of Utah has been promoted to an assistant professorship.

Associate Professor G. W. Starcher of Ohio University has been promoted to a professorship.

Assistant Professor W. R. Talbot of Lincoln University, Missouri, has been promoted to an associate professorship.

Dr. C. C. Torrance of Case School of Applied Science has been promoted to an assistant professorship.

Dr. J. W. Tukey of Princeton University has been promoted to an assistant professorship.

Dr. E. P. Vance of the University of Nevada has been promoted to an assistant professorship.

C. E. Van Orstrand, geophysicist in the U. S. Geological Survey, a charter member of the Mathematical Association, has retired.

Dr. G. B. Van Schaack of Michigan State College has been promoted to an assistant professorship.

Assistant Professor R. W. Veatch of the University of Tulsa has been promoted to an associate professorship.

Associate Professor J. A. Ward of Delta State Teachers College has been promoted to a professorship.

Assistant Professor E. D. Wells of the University of Pittsburgh, Erie Center, has been promoted to an associate professorship.

R. L. Wilson of the University of Wisconsin has been appointed to an assistant professorship in military science and tactics at the Alabama Polytechnic Institute.

The following appointments to instructorships are announced:
Fenn College, Cleveland, Ohio: D. H. Erkiletian, Jr.
Johns Hopkins University: Dr. J. D. Bankier, E. A. Coddington
University of Kansas City: D. E. Kibbey
University of Maryland: Dr. Martha H. Williams
Michigan State College: Dr. E. E. Blanche, Vladimir Morkovin
University of Michigan: Dr. C. J. Everett, Dr. C. J. Thorne
Northwestern University: L. L. Cronvich, J. F. Paydon, R. H. Stark
University of Pennsylvania: Dr. A. D. Wallace, Dr. P. M. Whitman
University of Richmond: E. S. Grable, F. B. Key
A. and M. College of Texas: W. B. Coleman
U. S. Naval Academy: W. H. Sears, Jr.
Yale University: J. G. Herriot

Dr. W. H. Butts, professor emeritus of mathematics at the University of Michigan, died June 25, 1941 in his eighty-fifth year.

O. C. Edwards, assistant professor of mechanical engineering at the University of Minnesota, died July 19, 1941.

H. L. Sweet, instructor in mathematics at Phillips Exeter Academy since 1910, died in Monrovia, Calif., on March 27 while on a year's sabbatical leave. He was an active member of the New England Association of Teachers of Mathematics, and was a charter member of the Mathematical Association.

SALE OF REPORT ON EDUCATION FOR SERVICE

Copies of the report of the Subcommittee on "Mathematical Education for Defense" (this MONTHLY, vol. 48, pp. 353–362) may be obtained while the supply lasts in lots of 25 for $1.25 by addressing Professor J. R. Kline, University of Pennsylvania, Philadelphia, Pennsylvania, or Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

CERTIFICATION OF MATHEMATICAL TRAINING FOR MILITARY SERVICE

W. L. Hart, University of Minnesota

Bulletin of Information by the War Preparedness Committee,
Subcommittee on Education for Service

In several branches of the Army and Navy, it is required of candidates for certain positions that they have had "*the equivalent of a year of college mathematics.*" Similar requirements differently phrased are sometimes used. College departments of mathematics are frequently called upon to certify as to the completion of the specified content. Hence, the following advice is presented with the desire to influence uniformity in this evaluation of mathematical training.

"The Subcommittee on Education for Service recommends that a student be certified as having had *the equivalent of a year of college mathematics* under either of the following conditions:

(*a*) The student has taken in college a year course in mathematics requiring at least two years of high school mathematics as a prerequisite.

(*b*) The student has satisfactorily passed courses containing algebra, trigonometry, and analytic geometry sufficient to meet the usual prerequisite for a first course in calculus.

Under (*b*), it is understood that some of the mathematics may have been taken in high school."

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

Allegheny Mountain, Pittsburgh, Pa., May 3; Washington, Pa., October 25.
Illinois, Peoria, May 9–10.
Indiana, Indianapolis, May 2–3.
Iowa, Indianola, April 25–26.
Kansas, Manhattan, April 4–5.
Kentucky, Richmond, April 26.
Louisiana-Mississippi, New Orleans, La., March 7–8.
Maryland-District of Columbia-Virginia, Annapolis, Md., May 10.
Michigan Ann Arbor, March 15; Detroit, November 15.
Minnesota, St. Joseph, May 10.
Missouri, Columbia, April 18.
Nebraska, Lincoln, May.
Northern California, San Francisco, January 25.
Ohio, Columbus, April 3.
Oklahoma, Tulsa, February 7.
Philadelphia, Swarthmore, November 29.
Rocky Mountain, Colorado Springs, April 18–19.
Southeastern, Chapel Hill, N. C., March 28–29.
Southern California, Redlands, March 8.
Southwestern, Lubbock, Tex., April 28–29.
Texas, Denton, April 4–5.
Upper New York State, Ithaca, May 3.
Wisconsin, Beloit, May 3.

Affiliated Organizations: The New England Association of Teachers of Mathematics, The National Council of Teachers of Mathematics.

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CONTENTS

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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EASTERN NEW MEXICO COLLEGE, Portales, New Mexico.

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W. D. CAIRNS, *Secretary-Treasurer*

THE EIGHTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The eighteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Tulane University, New Orleans, Louisiana, on Friday and Saturday, March 7–8, 1941.

The attendance was about eighty-five, including the following thirty-seven members of the Association: T. A. Bickerstaff, H. E. Buchanan, Annie M. H. Christensen, W. A. Cordrey, W. E. Cox, G. F. Cramer, D. S. Dearman, W. L. Duren, Virginia I. Felder, H. T. Fleddermann, Elizabeth Freas, F. C. Gentry, H. S. Kaltenborn, H. T. Karnes, Z. L. Loflin, Janet MacDonald, E. J. McShane, R. L. Menuet, B. E. Mitchell, I. C. Nichols, Irene A. Nolan, R. L. O'Quinn, W. V. Parker, J. S. Petersen, Jr., C. R. Pettis, F. A. Rickey, S. T. Sanders, H. F. Schroeder, P. C. Scott, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, Marelena White, R. C. Yates.

Sessions were held Friday afternoon and evening, and Saturday morning. Professor C. D. Smith, chairman of the Section, presided at the sessions on Friday afternoon and Saturday morning. The annual dinner was held on Friday evening with Professor G. F. Cramer presiding. The visiting speaker, Professor E. J. McShane of the University of Virginia, delivered an address at the dinner.

At the business session on Saturday morning it was decided to hold the meeting at Louisiana Polytechnic Institute, Ruston, Louisiana, in 1942. The following officers were chosen for 1941–42: Chairman, B. A. Tucker, Southeastern Louisiana College; Vice-Chairman for Louisiana, F. A. Rickey, Louisiana State University; Vice-Chairman for Mississippi, W. E. Cox, Mississippi State College; Secretary-Treasurer, W. V. Parker, Louisiana State University.

The following fourteen papers were presented:

1. "Curvature of an arc in a metric space" by Professor H. L. Smith, Louisiana State University.
2. "Harmonic properties of the tetrahedron" by Professor F. C. Gentry, Louisiana Polytechnic Institute.
3. "An extension of a concentration problem," by Professor F. A. Rickey, Louisiana State University.
4. "Remarks on mathematics in national defense" by Professor C. R. Pettis, Mississippi State College.
5. "Probability of detecting protozoa" by Professor J. F. Thomson, Tulane University.
6. "Some implications of calendar revision" by Professor T. A. Bickerstaff, University of Mississippi.
7. "Mathematics as a world power" by Professor V. B. Temple, Louisiana College.
8. "Research and teaching" by Professor E. J. McShane, University of Virginia.
9. "The trace of a projectile on an oblique screen" by Professor B. E. Mitchell and Nelson Nail, Millsaps College.

10. "Folds and creases" by Professor R. C. Yates, Louisiana State University.

11. "A plea for analytic mechanics" by Professor Alfred Hume, University of Mississippi, by title.

12. "Professional preparation of teachers of secondary mathematics" by Dr. H. T. Karnes, Louisiana State University.

13. "At the crossroads in secondary mathematics" by Professor P. K. Smith, Louisiana Polytechnic Institute.

14. "Generalized curves" by Professor E. J. McShane, University of Virginia.

Abstracts of some of the papers follow, numbered in accordance with their listing above:

1. Professor Smith proved two formulas for the Alt curvature $K(p_0)$ of an arc in a metric space at a point p_0 of it. These formulas are

$$K(p_0) = \lim \{24[S(p_0p) - \Delta(p_0p)]/\Delta(p_0p)^3\}^{1/2},$$
$$K(p_0) = \lim 12A_*(p_0p)/\Delta(p_0p)^3 = \lim 12A^*(p_0p)/\Delta(p_0p)^3,$$

where the limits are taken as p approaches p_0 on the arc, $S(p_0p)$ is the length of the sub-arc between p_0 and p, $\Delta(p_0p)$ is the distance from p_0 to p, and $A_*(p_0p)$, $A^*(p_0p)$ are two functions which reduce, in case the arc is in a euclidean plane, to the area between the arc and the chord p_0p.

2. By using homogeneous quadriplanar coördinates, Professor Gentry proved that a given tetrahedron, the tetahedron whose vertices are any point P and its harmonic conjugates with respect to the pairs of opposite edges of the original tetrahedron, and the tetrahedron whose vertices are the harmonic conjugates of P with respect to the vertices and the opposite faces of the given tetrahedron, are mutually self-polar. The vertices of the original tetrahedron and either of the others are the centers of eight spheres which touch the faces of the third tetrahedron.

3. A familiar problem in differential equations is that of determining the concentration of solution in a tank at a given time, assuming that the initial concentration is known and that water flows into the tank at a constant rate, an overflow outlet maintaining a constant volume in the tank. In his discussion, Professor Rickey assumed that a series of n such tanks are connected in order so that the overflow from the rth one runs into the $(r+1)$st one. He determined the concentration in the rth tank at any time and considered the effect upon the concentration in the nth tank as the number of tanks is increased indefinitely.

5. Professor Thomson applied the logistic function, $y_x = A/(1+e^{B-Cx})$ to a series of data taken in the department of bacteriology at Tulane University, using the method of least squares. He showed that the result is the probability that an individual is free of a particular protozoa after a certain number of examinations.

8. Professor McShane took issue with the prevalent idea that a research

worker is necessarily a poor teacher. He disputed this contention and supported the exact opposite view. If the word "research" be given a sufficiently inclusive meaning, he insisted that it can be argued that most good teachers of mathematics do research, for the lively interest needed to inspire students can hardly exist apart from the spirit of inquisitiveness which is the foundation of research.

9. A series of vertical electrically sensitized screens is used at some artillery proving grounds to determine the velocity, time of flight, range, lag, and drift of the projectile of a gun being tested. Professor Mitchell and Mr. Nail considered the problem when the screens are taken obliquely. The problem resolves itself into two parts. Part I considers the plane of projection fixed and the angle of projection variable (parameter). The trace turns out to be the join of the traces of the envelope of the trajectories. The second part considers the angle of projection fixed and the plane of projection variable. A surface is thereby generated for which the same toric paraboloid is suggested. The trace on a given screen is a quartic which has an axis of symmetry and for special cases reduces to two conic sections. The non-degenerate quartic is related rather intimately with the class of curves known as cartesian ovals.

10. Professor Yates presented a set of three postulates to establish the equivalence between the constructions possible by creasing paper and those of straight-edge and compasses. By admitting a process of knotting strips of paper with parallel edges, creasing may be enlarged to include problems of a general quartic nature. Models of regular polygons up to the octagon were displayed.

12. Dr. Karnes presented the results of a questionnaire study made in 1940 dealing with the preparation of prospective teachers of secondary mathematics, and concluded with a program of studies based upon these results. Secondary teachers were defined on three levels: junior high school, senior high school, and junior college. The data used were collected from 633 individuals distributed among the following groups: state superintendents of education, state high school supervisors of secondary education, administrators of secondary education, college teachers of secondary education, college teachers of educational psychology, heads of college departments of mathematics, and secondary teachers of mathematics. Of these 633 individuals, 198 had taught a course in "The Curriculum" or had served on a curriculum committee.

14. The English mathematician L. C. Young invented a class of things which he called generalized curves, and which includes all curves (in the usual sense of the word) as well as other new entities. Professor McShane showed that the theory of the calculus of variations can be extended at least in part to this new class, with certain advantages. He also showed that it is possible to introduce a concept of distance between generalized curves in such a way that every integral of the calculus of variations depends continuously on the generalized curve, while on the other hand every bounded infinite sequence of generalized curves contains a convergent sub-sequence. This led readily to existence theorems. He stated that for large classes of problems it can be shown that the minimizing

generalized curve is in fact a curve in the ordinary sense, so that new existence theorems are obtained for Bolza problems, isoperimetric problems, *etc.*

W. V. PARKER, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at the North Texas State Teachers College in Denton, Texas, on Friday afternoon, April 4, and Saturday morning, April 5, 1941.

Among the fifty-two people attending the meeting were the following twenty-five members of the Association: B. T. Adams, J. D. Bankier, A. A. Blumberg, H. E. Bray, Myrtle C. Brown, W. D. Cairns, J. V. Cooke, Alice C. Dean, Nat Edmonson, Jr., H. J. Ettlinger, E. H. Hanson, E. A. Hazlewood, G. B. Huff, B. C. Moore, E. D. Mouzon, Jr., M. E. Mullings, C. A. Murray, W. L. Porter, P. K. Rees, W. A. Rees, C. R. Sherer, F. W. Sparks, F. E. Ulrich, R. S. Underwood, C. T. York.

The Texas Section was unusually fortunate in having as its guest of honor during this meeting the National Secretary of the Association, Professor W. D. Cairns.

During the business session on Friday afternoon, Professor E. H. Hanson of North Texas State Teachers College and Professor R. S. Underwood of Texas Technological College were elected, respectively, Chairman and Vice-Chairman for the 1942 meeting. Those present voted to accept the invitation of Texas Technological College to hold the 1942 meeting at that institution.

The following papers were read:

1. "On the regularization of sequences and infinitely differentiable functions" by Professor S. Mandelbrojt, Rice Institute, introduced by Professor Bray.
2. "Quasi-analyticity and singularities of analytic functions" by Dr. F. E. Ulrich, Rice Institute.
3. "Some properties of quasi-linear oscillators" by Professor H. J. Ettlinger, University of Texas.
4. "The relation of mathematics to the practice of navigation" by A. A. Blumberg, A. and M. College of Texas.
5. "Some aspects of point set theory" by Dr. F. B. Jones, University of Texas, introduced by Professor Ettlinger.
6. "On compact unicoherent continua" by Miss Harlan C. Miller, University of Texas, introduced by Professor Ettlinger.
7. "A topological generalization of Schwarz's lemma" by George Piranian, Rice Institute, introduced by Professor Bray.
8. "Arithmetical continued fractions" by J. D. Bankier, Rice Institute.
9. Report and discussion of the work of Professor C. A. Murray's committee. This committee was appointed to study the reasons for the excessive number of failures in mathematics among college students.

Abstracts for some of these papers follow, the numbers corresponding to the numbers in the list of titles:

2. The classical fundamental theorem of quasi-analytic classes of functions is as follows: If $f(x)$ is an infinitely differentiable function on an interval (a, b) and belongs to class $C_{(M_n)}: |f^{(n)}(x)| < k^n M_n$, k a positive constant, $n \geqq 1$, for $a \leqq x \leqq b$, a necessary and sufficient condition that $f(x) \equiv 0$ when $f^{(n)}(a) = 0$, $n \geqq 0$, is that

$$\int_1^\infty \frac{\log T(r)dr}{r^2} = \infty,$$

where $T(r) = \max_{n \geqq 1} r^n / M_n$. In the present paper it was only assumed that $f^{(k)}(a) = 0$, $k \neq \lambda_n$, where (λ_n) is an infinite sequence of positive integers with $\lim_{n\to\infty} \lambda_n/n = G$. Then conditions on the sequences (M_n) and (λ_n) were sought which would assure that $f(x) \equiv 0$. Examples show that some restriction on $f(x)$ is also necessary. The following two cases were considered: (1) $\overline{\lim}_{n\to\infty} \sqrt{|f^{(n)}(a)|} = R < \infty$; (2) $\overline{\lim}_{n\to\infty} \sqrt[n]{|f^{(n)}(a)|} = \infty$, and certain results obtained. The complete results will be published soon in a paper by Mandelbrojt and Ulrich.

3. Professor Ettlinger discussed the properties of the solutions of a pair of first order differential equations of the type $y_i' = a_{i1}(y_1, y_2, t)y_1 + a_{i2}(y_1, y_2, t)y_2$, $(i = 1, 2)$, which may be called quasi-linear. These equations represent certain important non-linear types, such as the simple pendulum, a vibrating bead on a stretched massless string, *etc.* By use of polar coördinates, under suitable restrictions on the coefficients a_{ij}, separation and oscillation theorems are obtained for $y_1(t)$ and $y_2(t)$.

4. Mr. Blumberg discussed methods of determining lines of position and the construction of navigational charts.

6. Miss Miller took her material for this paper from two papers: one as covered by abstract no. 121, *Bulletin of the American Mathematical Society*, vol. 45, no. 3; the other, *Concerning compact unicoherent continua* read by title at the Chicago meeting of the Society, April 11–12, 1941, and whose abstract will appear in an early issue of the *Bulletin*.

7. Mr. Piranian gave an extension of Schwarz's lemma to certain transformations of a space into itself. The space is any convex metrized Kuratowski space with the property that each point of it lies on one of the surfaces of a system of concentric compact spheres. The transformations are required to keep the center of this system fixed and to belong to some normal family N which includes a non-trivial continuous group of rotations. It is shown that all transformations in N that are not rotations are nilpotent.

8. Mr. Bankier pointed out that certain properties of regular continued fractions are not dependent on their "regular" character since they are shared by continued fractions with positive, rational elements. In particular, it was shown that Galois's theorem holds for pure periodic continued fractions of the latter class, and it was noted that the complete quotients of the conjugate expansions have the same form as those which occur in the regular case.

NAT EDMONSON, JR., *Secretary*

THE 1941 MEETING OF THE MISSOURI SECTION

The 1941 meeting of the Missouri Section of the Mathematical Association of America was held at the University of Missouri, Columbia, Missouri, on Friday, April 18. The meeting was presided over by the Secretary, Professor J. H. Butchart.

Among the thirty-seven persons attending the meeting were the following thirteen members of the Association: L. M. Blumenthal, J. H. Butchart, W. L. Graves, Nola Anderson Haynes, G. H. Jamison, W. B. Jason, L. M. Kelly, R. J. Michel, M. E. Shanks, G. E. Wahlin, J. V. Wehausen, W. D. A. Westfall.

At the business session the following officers were elected for the coming year: Chairman, R. J. Michel, Cape Girardeau State Teachers College; Secretary-Treasurer, M. E. Shanks, University of Missouri.

The following papers were presented:

1. "Special homeomorphisms" by Dr. G. E. Schweigert, University of Missouri, introduced by Professor Blumenthal.
2. "Isogonal conjugates as foci of tangent conics and quadrics" by Professor J. H. Butchart, William Woods College.
3. "A nowhere differentiable arc" by Professor L. M. Blumenthal, University of Missouri.
4. "A new derivation of the basic formulas of trigonometry" by Professor Herman Betz, University of Missouri, introduced by the Secretary.
5. "Remarks on Cauchy's integral formula" by Dr. M. E. Shanks, University of Missouri.

Abstracts of the papers follow, numbered as in the list above:

1. Dr. Schweigert used the functions $y=x^n$ for $0 \leqq x \leqq 1$ to illustrate the concept of a homeomorphism $f(A)=B$, where the A and B spaces (compact and metric) were represented by the unit intervals on the x- and y-axes. Identification of the axes led to the type $f(A)=A$ and to the definition of $x+f(x)+f^2(x) + \cdots +f^n(x)=x$ as a finite orbit. Some examples, built from circles and line segments, were sketched geometrically and were used to illustrate cases of the action of pointwise periodic (each point has a finite orbit) and periodic (no point with more than N points) homeomorphisms. The work of W. L. Ayres, Deane Montgomery, and D. W. Hall was mentioned.

2. Professor Butchart called attention to the theorem that the lines from an exterior point to the foci of a conic make equal angles with the tangents to the conic and the consequence that isogonal conjugate points in a triangle can be regarded as foci of an inscribed conic. Following N. A. Court, generalizations were made to 3-space.

3. If x, y are two non-negative real numbers not exceeding 1, define distance xy as $|x-y|^{1/2}$. A quite elementary proof was given by Professor Blumenthal to show that the arc (homeomorph of a segment) obtained by imposing this metric on the closed unit interval is congruently imbeddable in Hilbert space and *has at no point a regressive or a progressive tangent.*

4. Professor Betz showed that the basic formulas of trigonometry can be derived by applying a few of the simplest notions of analytic geometry to the unit circle.

5. Dr. Shanks discussed conditions under which a function $f(z)$, analytic in the interior of a rectifiable Jordan curve J, may be given by the Cauchy integral formula applied to the boundary values assumed by $f(z)$ on J. Two types of conditions were considered: (1) conditions on the function interior to J, and (2) conditions on the boundary values of the function. The case where J is the unit circle was considered separately, and necessary and sufficient conditions of types (1) and (2) were given. In the general case, necessary and sufficient conditions of type (2) only were known. Some sufficient conditions of type (1) were stated.

J. H. BUTCHART, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirtieth regular meeting of the Iowa Section of the Mathematical Association of America was held at Simpson College, Indianola, Iowa, on Friday and Saturday, April 25–26, 1941, in conjunction with the fifty-fifth annual meeting of the Iowa Academy of Science. Mr. Fred Robertson, chairman of the Section, presided. He was relieved by Professor L. E. Ward for part of the session on Friday afternoon.

The attendance was about forty-two, including the following twenty-six members of the Association: J. W. Beach, F. A. Brandner, J. O. Chellevold, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, Marian E. Daniells, C. W. Emmons, Cornelius Gouwens, Gertrude A. Herr, L. A. Knowler, O. C. Kreider, A. T. Lonseth, R. B. McClenon, J. V. McKelvey, Mrs. J. V. McKelvey, I. F. Neff, E. N. Oberg, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, H. C. Trimble, Henry Van Engen, L. E. Ward, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the Section elected for 1941–42 are as follows: Chairman, O. C. Kreider, Ellsworth Junior College; Vice-Chairman, N. B. Conkwright, State University of Iowa; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

The committee appointed last year to make a study of Survey Courses made a preliminary report. The committee went on record as favoring the idea of a general course in mathematics for college freshmen, and that it should not be considered as a terminal course but a course to make friends for mathematics among the students. The movement towards introductory courses seems to be retarded by the lack of desirable text-books. The committee was continued for another year.

A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the host, Simpson College and the Department of Mathematics, was adopted at the business meeting. Professor E. W. Chittenden presented a paper *A new proof of a theorem of Kuratowski*

on the Academy program; this will be published in the *Proceedings* of the Iowa Academy of Science. The following fifteen papers were read on the Association program:

1. "On the convergence of series involving Hermite functions" by Professor E. N. Oberg, State University of Iowa.

2. "Extensions of the Plateau problem" by Dr. A. T. Lonseth, Iowa State College.

3. "Non-negative functional transformations" by Professor Erich Rothe, William Penn College, introduced by the Secretary.

4. "A problem in the distribution of serial correlation for a general lag" by R. L. Anderson, Iowa State College, introduced by the Secretary.

5. "An approximate solution for infinite corner-loaded plate on an elastic foundation" by C. J. Thorne, Iowa State College, introduced by the Secretary.

6. "Orthogonal functions for frequency distributions" by W. M. Stone, Iowa State College, introduced by the Secretary.

7. "The scientific work of Vito Volterra" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

8. "An analysis of the relationships between the thinking process of secondary school mathematics and that of life problems" by Dr. H. C. Trimble, Iowa State Teachers College.

9. "Curriculum practices in Iowa public junior colleges" by Professor O. C. Kreider, Ellsworth Junior College.

10. "The independence of operators on a lattice" by F. A. Haight, State University of Iowa, introduced by Professor Chittenden.

11. "Some notes on Euler's integral calculus" by Professor R. B. McClenon, Grinnell College.

12. "On a certain concept of cryptography" by R. L. Smith, State University of Iowa, introduced by Professor Chittenden.

13. "Remarks on exterior ballistics" by Professor L. E. Ward, State University of Iowa.

14. "Some elementary problems in interior ballistics" by Professor E. R. Smith, Iowa State College.

15. "Properties of a generalized operator" by Fred Robertson, Iowa State College.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Oberg showed that if $\phi_0, \phi_1, \cdots, \phi_n$ is a closed set of orthonormal functions increasing with n, and if

$$\sum_{i=0}^{\infty}\left(\lambda_i \int_a^b f\phi_i dx\right)^2 \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\phi_i(x)\phi_i(t)}{\lambda_i}$$

are convergent, the latter on $a \leqq x$, $t \leqq b$, then the series $\sum_{i=0}^{\infty} a_i\phi_i$, $a_i = \int_a^b f\phi_i dx$, converges uniformly to $f(x)$ on $a \leqq x \leqq b$. This theorem was used to obtain criteria for the convergence of series involving Hermite's polynomials.

2. Dr. Lonseth told of some of the more recent investigations by himself and others in the problem of Plateau, particularly the various generalizations: (1) to more complicated boundary curves, surfaces of higher topological structure, and surfaces which have parts of their boundaries free to move on prescribed surfaces; (2) to least-area problems in non-euclidean space.

3. Professor Rothe considered a space E whose points $\bar{x}=x(t)$ are functions of the variable t, and called the point $\bar{x}$ non-negative either if $x(t)\geqq 0$ for all t of the domain in which $x(t)$ is defined, or if all Fourier coefficients of $x(t)$ are non-negative. In either case, under certain assumptions on E, he proved a theorem concerning the existence of a positive eigen-value for non-linear completely continuous transformations $\overline{F}(\bar{x})$, mapping non-negative points into non-negative points. These theorems are applications of a general eigen-value theorem in certain abstract linear spaces E. The proof of this general theorem is based on a fixed-point theorem for the mapping on itself of a "convex" set s lying on a sphere of E. This latter theorem in turn is proved by using stereographic projection in E and thus mapping s into a convex set lying in a "plane" space for which a fixed-point theorem of Schauder is applicable.

4. Mr. Anderson considered a set of n observations, $x_1, x_2, \cdots, x_n$, which are normally and independently distributed with mean 0 and variance 1, and defined the serial correlation coefficient for lag 1 to be $R_1=C_1/V$, where $C_1=x_1x_2+x_2x_3+\cdots+x_nx_1$ and $V=\sum_{i=1}^{n}x_i^2$. It was shown that C_1 is distributed as $\sum_{i=1}^{n}\lambda_i u_i$, where u_i is distributed as χ^2 with one degree of freedom and the λ's are solutions of the equation

$$F_{n,1}(\lambda) = \prod_{i=1}^{n}\left[\lambda_i - \cos\frac{2\pi i}{n}\right] = 0.$$

The distribution of R_1 is formed by integrating out the u's and V from the simultaneous distribution of C_1 and V. For the general lag L, the C_L are distributed as C_1 except that the λ's are now solutions of the equation

$$F_{n,L}(\lambda) = \prod_{i=1}^{n}\left[\lambda_i - \cos\frac{2\pi iL}{n}\right] = 0.$$

Several relationships concerning $F_{n,L}(\lambda)$ were found.

5. Mr. Thorne obtained an approximate solution to his thin plate theory problem by the use of a general functional method. The solution contains the values for sixteen constants of the expansion function and a convergence factor, and is a function of Poisson's ratio, resistance modulus of the sub-grade, and plate constants.

6. Mr. Stone made a study of the well known Hermitian functions in the discrete variable $x=1, 2, 3, \cdots$, bringing out a number of interesting analogies between this and the continuous case. Finite difference equations were set up to facilitate the construction of numerical tables used in curve fitting to discrete data. The new functions were found to be especially appropriate for the smoothing of frequency curves and approximation by polynomials.

7. This paper appears in this issue of the MONTHLY.

8. Dr. Trimble discussed a growing concern for the possible contribution of the study of mathematics to training in clear thinking. Enthusiasm for this educational objective is accompanied by some confusion as to its implications. Professional mathematicians have an interest in this situation and can help to guide the movements which it is stimulating. This paper presented a definition of thinking by relating thinking to problem solving. Some differences between "problems" and mathematical "exercises" were pointed out. Recommendations were based upon five objectives which professional mathematicians are assumed to share. An emphasis upon "problems" in mathematics was suggested. Some of the practical limitations of teaching mathematics from the "problem" standpoint were pointed out. The responsibilities of professional mathematicians in this regard were again stressed.

9. Professor Kreider gave the conclusions he could draw from a questionnaire sent to the mathematics teachers in the public junior colleges of Iowa on the course content in mathematics. The course work seems to follow the corresponding work of the regular four-year college courses. A study was also made of the needs for other mathematics than pre-professional mathematics and for survey courses in mathematics.

10. Mr. Haight considered lattices obeying the following three postulates: (1) $d(A+B)=dA+dB$, (2) $d^2A \leqq dA$, (3) $d(0)=0$. With the operation d undefined, and c representing the taking of complements, a study was made of nine basic operators, dA, AdA, $AcdA$, kA (maximum dense-in-itself sub-set of A), $AckA$, $AcdcA$, $AdcA$, $A+dA$, and $AdcA+cAdA$. The dependence or independence of 162 operators of the form $u\,v$ and $u\,c\,v$, where u and v represent one of the nine basic operations, was studied.

11. Professor McClenon gave some extracts from Euler's integral calculus, and pointed out some ways in which they influenced later developments of mathematics. Incidentally, some of them offer suggestions for present day college courses in calculus.

12. Mr. Smith proceeded along a line similar to that developed by L. S. Hill, this MONTHLY, June–July, 1929, and demonstrated a procedure for enciphering and deciphering messages by the use of a bioperational alphabet composed of 29 letters.

13. Professor Ward gave a brief sketch of the various methods of computing the trajectory of a particle in air, including the use of the Mayevski zones, the Siacci approximations, and the method of numerical integration. Several works and tables on exterior ballistics were exhibited.

14. Professor Smith presented a number of problems suitable for exercises in algebra, analytic geometry, calculus, theory of equations, and differential equations involving the language and some of the principles of interior ballistics.

15. Mr. Robertson defined a generalized operator and gave a summary of some of its operational and algebraic properties such as the differential, difference, and integral equations which it satisfies. The paper is being published in the Iowa State College *Journal of Science*.

CORNELIUS GOUWENS, *Secretary*

THE FIFTH ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The fifth annual meeting of the Southwestern Section of the Mathematical Association of America was held at Texas Technological College, Lubbock, Texas, on April 28-29, 1941, in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science.

The attendance was seventy-five, including the following eighteen members of the Association: E. T. Bell, J. W. Branson, M. W. Fleck, E. A. Hazlewood, E. R. Heineman, H. D. Larsen, Roy MacKay, W. A. McLaughlin, Lida B. May, C. V. Newsom, E. J. Purcell, F. W. Sparks, P. M. Swingle, E. L. Thompson, R. S. Underwood, R. K. Wakerling, William Wallis, G. A. Whetstone.

At the business meeting the following officers were elected for next year: Chairman, Roy MacKay, Eastern New Mexico College; Vice-Chairman, L. E. Mehlenbacher, Arizona State Teachers College, Flagstaff; Secretary, H. D. Larsen, University of New Mexico. It was voted to hold the 1942 meeting at New Mexico State College in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science. A vote of thanks was extended to Texas Technological College for its generous hospitality.

The Section was honored to have Professor E. T. Bell of the California Institute of Technology as visiting speaker. He spoke on *Diophantine analysis* before a special audience of seventy-five mathematicians and students following the regular session on Tuesday morning. The success of the meeting was due in large measure to the address by Professor Bell.

Professor Bell began his address with a brief historical review of Diophantine analysis, emphasizing the rôle of certain classic problems in determining current interests in the subject. He then outlined in some detail the second of the main divisions of additive and multiplicative analysis, indicating how an application of the theory of algebraic numbers to multiplicative systems induces complete solutions of certain types of additive systems. Approachable unsolved problems, suitable for research, were pointed out.

Professor Roy MacKay, vice-chairman of the Section, presided over the Monday afternoon session, and Professor E. J. Purcell, chairman of the Section, presided over the Tuesday morning session. The annual luncheon was held on Tuesday, with Professor R. S. Underwood in charge of the arrangements.

The following papers were presented during the two sessions:

1. "The expansion of a certain determinant" by Professor E. R. Heineman, Texas Technological College.

2. "Some notes on systems of planes in S_r" by Dr. R. K. Wakerling, Texas Technological College.

3. "An Euler plane" by Professor Roy MacKay, Eastern New Mexico College.

4. "Note on transformations in self-dual lattices" by Dr. F. D. Rigby, Texas Technological College, introduced by the Secretary.

5. "A three-space representation of equations in four, five, and six variables" by Professor R. S. Underwood, Texas Technological College.

6. "A classification of cubic surfaces by means of their representations on a plane" by William Wallis, Texas Technological College.

7. "A note on scales of notation" by Professor H. D. Larsen, University of New Mexico.

8. "What mathematics shall we teach in the freshman engineering classes?" by L. C. Christianson, Texas Technological College, introduced by the Secretary.

9. "Monoidal involutions in S_n" by Professor E. J. Purcell, University of Arizona.

10. "Note on duality in point-set theory" by Dr. P. W. Gilbert, Texas Technological College, introduced by the Secretary.

11. "The mathematical foundation of flash and sound ranging" by Major G. B. Drummond, New Mexico School of Mines, introduced by the Secretary.

12. "Some mathematical phases of aeronautics" by Lida B. May, Texas Technological College.

13. "Observations concerning the domain of mathematics" by Professor P. M. Swingle, New Mexico State College.

14. "Decomposition of rational fractions into partial fractions" by Professor A. W. Boldyreff, University of Arizona, read by E. J. Purcell.

Abstracts of some of these papers follow, the numbers corresponding to those in the list of titles:

1. Professor Heineman presented a special method of expansion for the determinant whose $(i+1)$th row is

$$\left| a^{nl_i}, a^{(n-1)l_i}b^{l_i}, a^{(n-2)l_i}b^{2l_i}, \cdots, b^{nl_i} \right|,$$

where $i=0, 1, 2, \cdots, n-1$, and $l_{i+1}>l_i$. The value of the determinant was expressed in terms of $(b-a)$, ab, and the symmetric functions $F_r=\sum_{i=0}^{i=r}a^{r-i}b^i$.

2. Dr. Wakerling showed that the system of ∞^{3r} hypersurfaces V_{r-1}^3 in S_r through a common $(r-2)$-space twice may be regarded as the representation of a variety W_r^{2r+1} in S_{3r}. A plane of S_r represents on W_r^{2r+1} a surface F_2^5 which is rational and ruled, and therefore is contained in a 6-space S_6. Consequently there are in S_{3r} $\infty^{3(r-2)}$ S_6's corresponding to the planes of S_r. Through a general point of S_{3r} passes one and only one such S_6. In S_{3r} a fixed $3(r-2)$-space is met by the S_6's in points which are in one-to-one reciprocal correspondence with the planes of S_r. This correspondence may be used to study systems of planes in S_r.

3. Professor MacKay extended to the plane the notion of an Euler line of a simplex.

4. Dr. Rigby showed that, if a lattice S is self-dual by virtue of the dual automorphism $x \rightarrow x'$, then the semi-complement x' defined by this correspondence can be regarded as analogous to $-x$ in ordinary algebra. This fact suggested

that, if g represents an unknown operator, a solution of the equation $g^2(x)=x'$ is to be found in the product lattice S^2. This was found to be the case, and the problem was generalized to that of studying other transformations in the product lattice.

5. Professor Underwood proposed a pictorial device whose element is the line segment AB. For an equation in six variables, the variables x, y, and z represent the ordinary rectangular coördinates of the point A, while u, v, and w denote the coördinates of B with respect to A. The geometric counterpart of an equation was shown to be a solid figure composed of line segments. Various equations with their representations were discussed.

6. Mr. Wallis used a Cremona cubic transformation to set up an involution by which a plane Π goes into a cubic surface F^3. The plane Π may be considered as a map of F^3 with the six points A_i in which Π is met by the Jacobian sextic curve of the transformation as fundamental points. Plane sections of F^3 are represented by cubic curves of the net through the points A_i. There are at most twenty-seven straight lines on this surface. These lines arise from the six points A_i, their joins two by two, and the conics through A_i five at a time. Salmon has classified cubic surfaces into twenty-three distinct types. It was the object of Mr. Wallis to determine the specializations required in the above transformation to give rise to some of these special forms of the surface.

7. Professor Larsen showed that a number system in the scale of $1+i$, where i is a rate of interest, could be used to express the fundamental functions of compound interest. In particular, $(1+i)^n=100\cdots 00$, $(1+i)^{-n}=.000\cdots 01$, $s_{\overline{n}|}=111\cdots 11$, and $a_{\overline{n}|}=.111\cdots 11$. These relationships were used to establish in simple fashion several theorems in the mathematics of finance.

8. Mr. Christianson reported on some observations which he had obtained from several sources. He believed that more time should be spent on the fundamental operations, especially on those involving fractions, and that definitions should be emphasized.

10. Dr. Gilbert showed that a set of dual propositions results if the sense of inclusion in the axioms for point-set theory is reversed; *i.e.*, write "contains" instead of "is contained in." These dual propositions can be proved true from the original axioms by using the complement relation. Some of the fundamental properties and theorems of point-set theory were dualized. In particular, it was noted that if a set has any property, then its complement has the dual property.

11. Because the national emergency prevented Major Drummond from presenting his paper in person, the abstract was read by the Secretary. He considered the question of locating invisible hostile artillery, a problem approaching solution at the close of the World War and now solved by the application of a hyperbola as the locus of a point which moves so that the difference of its distances from two fixed points remains a constant. The difference of the time of arrival of a sound wave at two microphones is converted to a distance which defines one branch of a hyperbola, the microphones being located at the foci. A second hyperbola is defined by another pair of microphones. The origin of the

sound wave is the intersection of the two hyperbolas. In actual practice, the hyperbolas are not plotted, but it is assumed that the source of the sound wave lies on the asymptotes. An asymptote correction is applied to compensate for the error of this assumption. A second correction compensates for the varying time of travel of the sound wave due to the velocity and direction of the wind. A third correction compensates for variations in the time of travel of the sound wave due to departure of the temperature of the air from standard. The problem admits of graphical solution, but requires personnel of high intelligence and excellent training.

12. Miss May gave a summary of the developments in aeronautics during the past few years.

13. Professor Swingle made an abstraction of several fundamental definitions of topology, and discussed the possible domain of application.

14. Professor Boldyreff considered the existence and uniqueness of decomposition of a rational fraction into partial fractions, and the explicit formulas for the numerators of the partial fractions for all cases. The numerical properties of the coefficients were investigated in connection with the numerators of partial fractions corresponding to repeated prime quadratic factors.

H. D. LARSEN, *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE INDIANA SECTION

The eighteenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 2 and 3, 1941, at Butler University, Indianapolis, Indiana.

Seventy-five registered at the meetings, including the following thirty-five members of the Association: W. C. Arnold, Emil Artin, Max Astrachan, Juna Lutz Beal, I. W. Burr, W. W. Denton, R. H. Downing, W. E. Edington, P. D. Edwards, B. C. Getchell, E. L. Godfrey, G. H. Graves, W. R. Hardman, H. H. Hartzler, Cora B. Hennel, H. K. Hughes, M. W. Keller, W. C. Krathwohl, Cornelius Lanczos, D. A. Lehman, Florence Long, H. A. Meyer, C. N. Moore, P. M. Pepper, J. C. Polley, D. H. Porter, C. K. Robbins, L. S. Shively, D. R. Shreve, W. O. Shriner, Anna K. Suter, M. S. Webster, Agnes E. Wells, F. J. Weyl, H. E. Wolfe.

At the business meeting on Saturday the following officers were elected for next year: Chairman, P. D. Edwards, Ball State Teachers College; Vice-Chairman, J. C. Polley, Wabash College; Secretary, M. W. Keller, Purdue University. On account of the increased number of papers being presented it was voted that the Indiana Section of the Association should hold two meetings per year. The spring meetings will be continued and a second meeting will be held jointly with the Mathematics Section of the Indiana Academy of Science. The first joint meeting with the Indiana Academy will be held at DePauw University in November, 1941.

At the annual dinner on Friday evening Professor Beal of Butler University served as toastmaster and introduced Dr. D. S. Robinson, president of Butler University, who welcomed the visitors. Dr. Robinson paid tribute to the importance of the study of mathematics to the student of philosophy, in which field he achieved national prominence before taking over his duties as president of Butler University.

Following the dinner the first session of the Section was held, at which time Professor C. N. Moore of the University of Cincinnati was guest speaker. His subject was "On the interdependence of pure and applied mathematics." Professor Moore pointed out that the history of mathematics reveals many instances in which the methods needed for the solution of an applied problem had been developed far in advance of the need, in the course of the natural growth of mathematical theory. Likewise, the study of an applied problem has frequently raised questions which stimulated extensive developments in the field of pure mathematics. This mutual relationship between the pure and applied branches of the subject was illustrated by means of various special cases of particular importance.

At the two sessions on Saturday the following program was presented:

1. "And gladly teach" by Professor Cora B. Hennel, Indiana University, retiring chairman of the Indiana Section.

2. "Predicting class quality on the basis of orientation tests" by Professor W. C. Krathwohl, Illinois Institute of Technology, by invitation.

3. "Further findings from the diagnostic testing program" by Dr. M. W. Keller and Dr. D. R. Shreve, Purdue University.

4. "A report of pre-college mathematics by correspondence" by Dr. D. R. Shreve and Dr. M. W. Keller, Purdue University.

5. "The appeal of useful mathematics" by Professor Emeritus D. A. Lehman, Goshen College.

6. "After sectionizing; what?" by Professor Max Astrachan and Professor I. W. Burr, Antioch College.

7. "History of mathematics in Indiana" by Professor W. E. Edington, DePauw University.

8. "Fundamental properties of the Gamma function" by Professor Emil Artin, Indiana University.

9. "The motion of a particle in a Riemannian world" by Professor Cornelius Lanczos, Purdue University.

10. "The value of the p-adic logarithm" by David Gilbarg, Indiana University, introduced by Professor Artin.

11. "A locus related to the Euler line" by K. W. Crain, Purdue University, introduced by Professor Graves.

12. "Value distribution of ring meromorphic functions" by Dr. F. J. Weyl, Indiana University.

13. "Automorphisms of a simple algebra" by Dr. G. W. Whaples, Indiana University, introduced by Professor Artin.

Abstracts of papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Hennel discussed the responsibility of mathematics teachers for rendering various types of service. Teachers must serve as investigators, contributing to the development of the subject in its pure and applied phases; as historians, recording and evaluating subject-matter and writing biographies; as teachers, instructing students in the different branches of mathematics; as writers of text-books; and as faculty members, working toward the all-around development of students. Special emphasis was placed on the importance of the work as teacher. Too often promotions in the college field are based on the work in the first two fields with the result that there is neglect of the primary purpose for which the undergraduate school is organized.

2. Professor Krathwohl presented some results of investigations carried out at the Illinois Institute of Technology. At the time of the first meeting, it is possible by means of such tests as the Iowa Placement Mathematics Aptitude Examination and the American Council Psychological Examination to predict the quality of a freshman class in mathematics. Because standards and types of students vary in different colleges, the constants involved in the computation have to be computed separately for each institution. The advantage of such a prediction is that if an instructor has a weak class, he knows he must work much harder on fundamentals. If he has an unusually good class, he can use this fine opportunity to enrich the content of the course.

3. Dr. Keller presented a second progress report on the diagnostic testing program which has been inaugurated at Purdue University. Some of the findings from the results of two years of testing were given. In addition, the ability of students to perform the fundamental operations with exponents and radicals as revealed by the revised tests which were given in 1940–1941 was discussed briefly.

4. Dr. Shreve presented a report on the results of an experimental review course given by correspondence to 230 students planning to enter Purdue University. The study shows (1) an analysis of the types of errors prevalent among entering students, (2) a discussion of the opportunities for remedial work by the university before the student enters, and (3) a report on the noticeable achievements of the course. The study indicates that the university can, by pre-college training, prepare students to compete with superior students with equal success in a single course. Following the study of the results of this course, Purdue University now offers this pre-college course by correspondence as a regular summer project.

5. Professor Lehman discussed a large number of applications of elementary mathematics which are not ordinarily found in elementary texts. These included applications to surveying problems and problems in astronomy, as well as the more familiar problems commonly found in text-books.

6. Professor Astrachan described the Antioch program in mathematics as developed by himself and Professor Burr. There is a horizontal sectionizing on

the basis of high school records and placement tests. Candidates for the Bachelor of Science degree are given the usual training in algebra, trigonometry, analytics, and calculus. Emphasis is primarily on skills and applications. The courses taken by candidates for the Bachelor of Arts degree are planned to be of a more cultural nature. They include, among other things, certain skills useful in many fields and in everyday intelligent living. Advanced courses included some aspects of modern mathematics. Procedure in most courses is on a laboratory basis. Achievement is measured by a system of quizzes which test the mastery of all material as it is covered.

7. Professor Edington discussed the growth of instruction in mathematics in Indiana. The Territory of Indiana was organized in 1800 and the Territorial Legislature, following special action by congress in 1804, passed acts in 1806 and 1807 leading to the incorporation of Vincennes University. These acts required, among other things, the instruction of the youth in mathematics. During the next twenty-five years Indiana University, Hanover, and Wabash Colleges were founded and instruction in algebra, geometry, navigation, and surveying was offered. By 1850 trigonometry and analytic geometry were regularly offered in several of the colleges, and some work in fluxions was given. Following the Civil War the growth of colleges in number, enrollment, and curriculum offerings was more rapid. The first M.A. degree in mathematics granted within the state was conferred on Joseph Swain by Indiana University in 1885. The first Ph.D. in mathematics was granted by Purdue University to James Byrnie Shaw in 1897, but no other Ph.D. degree in mathematics was granted within the state until 1912 when Miss Cora B. Hennel received this degree from Indiana University. Following are the names of the more prominent early mathematicians of Indiana: Bishop Matthew Simpson, John Steele Thomson, John H. Harney, J. Harrison Thomson, John S. Hougham, Emerson E. White, Frank L. Morse, Moses C. Stevens, Erastus Test, John L. Campbell, John P. D. John, Henry T. Eddy, Joseph Swain, Clarence A. Waldo, Arthur S. Hathaway, and Robert J. Aley.

8. Professor Artin showed that the fundamental properties of the Gamma function are derived in a very simple manner if the function be defined as a logarithmic-convex solution of its functional equation. The main reason for the simplicity of the proofs is that the logarithmic-convex functions form a family that is closed under addition, multiplication, and the taking of limits. With very little formal manipulation of symbols Professor Artin was able to obtain all the usual forms which are used to define the Gamma function.

9. Professor Lanczos discussed the motion of a particle in a Riemannian field. Linear differential equations satisfy the principle of superposition; two separate solutions can be superposed on each other without any disturbance. Hence, linear differential equations cannot account for the fact that a particle is put in motion by the action of a superposed external field. The field equations of relativity, based on Riemannian geometry, are non-linear. Thus the possibility is given that here the dynamics of a particle may be understood as a con-

sequence of the field equations. Indeed, the laws of motion can be derived in the form of integral relations based on the Gaussian integral transformation. The equations of motion come out in the classical Newtonian form: 1. The time rate of change of the momentum is equal to the moving force. 2. The momentum is equal to the total mass times the velocity of the center of the mass. The "moving force" can be transformed into a boundary integral, extended over the surface of the particle. The resulting law of motion does not coincide necessarily with the customarily assumed law of the geodesic line.

10. Mr. Gilbarg defined the p-adic absolute value on the rational numbers in the following way: If $m/n=(m'/n')\,p^{+n}$, then the absolute value of $m/n=p^{-n}$. By means of this sort of absolute value, it is possible to define convergence. In particular, the convergence of the logarithmic series can be discussed from this point of view. The problem of the values taken on by the logarithm function in the p-adic domain was considered.

11. Mr. Crain employed the analytic method to establish the following results: (1) If a circle is cut by a straight line in two points A and B, the locus of the circumcenters of the triangle PAB, where P is any point on the given circle, is a point circle, and the locus of the orthocenters is a circle. (2) The locus of any point, which divides in a constant ratio the line joining the circumcenter and the orthocenter, is a circle. (3) Each member of this family of circles is tangent to two lines which intersect at the circumcenter. (4) Considering only the members of this family which form an unlimited chain of tangent circles, and starting with the circle determined by the orthocenters, their radii taken in decreasing order may be summed.

12. Dr. Weyl's paper was concerned with a generalization of R. Nevanlinna's now classical results about the distribution of meromorphic functions (R. Nevanlinna, *Eindeutige Analytische Funktionen*, Julius Springer, 1936). The principal aim of this theory is the characterization of the class of functions, one-valued on a given Riemann surface F, in terms of the distribution of those places where any one of them assumes given values. If F is the doubly punctured sphere, the corresponding class of functions is called ring-meromorphic. If F is open, one is forced to exhaust it by means of a sequence of ever-expanding regions. For the principal estimates of the classical theory it is furthermore imperative that the exhausting regions exhibit rotational symmetry. How to do this in the classical case, where F is the open euclidean plane, is evident. But the sphere, punctured at the south and north poles, also permits an exhaustion by rotationally symmetric regions. On this basis the classical procedure as well as its results can be reproduced, throwing into sharp relief their dependence on the above symmetry.

13. Dr. Whaples gave a new, simplified proof of the theorem that two isomorphic simple sub-algebras of a simple algebra are connected by an inner automorphism.

P. D. EDWARDS, *Secretary*

THE SECOND MEETING OF THE UPPER NEW YORK STATE SECTION

The second meeting of the Upper New York State Section of the Mathematical Association of America was held at Cornell University, Ithaca, New York, on Saturday, May 3, 1941. Sessions were held both in the morning and in the afternoon. Professor B. C. Patterson of Hamilton College presided at the morning session and Professor Joseph Seidlin of Alfred University at the afternoon session. Tea was served to members and guests with Mrs. W. B. Carver in charge. Professor W. B. Carver of Cornell University presided at the dinner which concluded the program.

The attendance was about ninety, including the following thirty-seven members of the Association: E. B. Allen, Harry Birchenough, F. J. H. Burkett, C. L. Buxton, A. D. Campbell, I. S. Carroll, W. M. Carruth, W. B. Carver, J. H. Curtiss, F. F. Decker, H. A. DoBell, W. H. Durfee, W. W. Flexner, H. M. Gehman, E. H. Hadlock, H. N. Hubbs, W. A. Hurwitz, B. W. Jones, Caroline A. Lester, L. L. Lowenstein, Harriet F. Montague, C. W. Munshower, Abba V. Newton, E. R. Ott, B. C. Patterson, Theresa L. Podmele, L. R. Polan, R. W. Price, J. F. Randolph, M. A. Scheier, Wladimir Seidel, Joseph Seidlin, R. E. Street, Mary C. Suffa, A. K. Waltz, J. F. Wardwell, A. E. Whitford.

Professor H. M. Gehman of the University of Buffalo, chairman of the Section, presided at the business meeting which preceded the afternoon session. The following officers were elected for the year 1941–1942: Chairman, A. D. Campbell, Syracuse University; Vice-Chairman, D. S. Morse, Union College; Secretary-Treasurer, C. W. Munshower, Colgate University. The invitation to hold the next meeting at the University of Rochester was accepted.

The following papers were presented:

1. "The potential function in electro-dynamics" by Dr. R. E. Street, Rensselaer Polytechnic Institute.
2. "A course on the significance of mathematics" by Professor Harriet F. Montague, University of Buffalo.
3. "Some applications of mathematics to aerial mapping" by Professor E. F. Church, Syracuse University, introduced by Professor Campbell.
4. "Report on a survey of the mathematical preparations for defense" by Professor E. B. Allen, Rensselaer Polytechnic Institute.
5. "An extension of Hill's methods in cryptography" by Professor B. W. Jones, Cornell University.
6. "Quadrilaterals inscribed and circumscribed to curves of the third and second orders, respectively, with a common line of symmetry" by the Reverend M. A. Scheier, St. Bonaventure College.
7. "Class interval assumptions in frequency distributions" by Professor H. A. DoBell, New York State College for Teachers.
8. "Generating functions in the theory of statistics" by Professor J. H. Curtiss, Cornell University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Street discussed a special case of a theorem of Helmholtz in which the close connection between the electromagnetic equations of Maxwell and the force equation of Lorentz is brought out. If it is required that the force vector be a function only of the position and velocity vectors, and be derivable from a potential function in the form of Lagrange's equations, then the Lorentz force is the most general force allowed. There is a very simple relation between the potential function of the electromagnetic field and the potential function of the force field in such a case.

2. The "Significance of Mathematics" is a course offered at the University of Buffalo primarily for students outside the field of mathematics. Non-technical lectures on important concepts in higher mathematics are presented, emphasizing methods of procedure and relationship to other fields of knowledge. Professor Montague surveyed the content of the lectures and quoted statistics on the students enrolled in the course with reference to mathematical preparation, college course, and class.

3. Professor Church cited a number of applications of mathematics to problems in photogrammetry or aerial photographic surveying.

4. The mathematical societies have appointed committees to assist in stimulating departments of mathematics in colleges and individual mathematicians to contribute to the national defense effort. The survey reported by Professor Allen gave information on what the departments of mathematics and the mathematicians of the colleges of upper New York State are actually doing to further the national defense program. The information was gathered by a committee consisting of E. B. Allen, Wladimir Seidel, E. R. Ott, B. C. Patterson, and A. K. Waltz. Questionnaires were sent to the departments of mathematics concerned, and tables showing the information were included in the committee report.

5. In two articles in this MONTHLY (1929 and 1931) L. S. Hill described a method of cryptography in which the letters of a message were transformed by a key matrix into another set. The matrices used in practice are 3 by 3 matrices. Professor Jones considered a method by which the matrix is varied throughout the message. For such a method a 2 by 2 matrix is thought to be sufficient. In this connection, the multiplicative group of non-singular matrices (mod p) was considered.

6. The Reverend Scheier discussed a relation between points on a cubic determined by tangents to a conic. By imposing certain conditions on the cubic and conic (which may be done without loss of generality), a certain symmetric (3, 3) correspondence is obtained involving four projective constants. It was shown that this correspondence leads to one self-symmetric quadrilateral and to two other quadrilaterals not self-symmetric (but symmetric to each other) such that their vertices lie on the cubic and their sides are tangent to the conic. Furthermore, the conditions of reality were given for the self-symmetric case. There was also deduced a theorem proving the constancy of the double ratio of a certain set of four points on the rational cuspidal quartic with a line of symmetry.

7. Professor DoBell emphasized the importance of proper class boundaries in a frequency distribution to suit the assumptions concerning the manner of approximation of the original scores.

8. This paper appeared in the June–July, 1941, issue of this MONTHLY.

C. W. MUNSHOWER, *Secretary*

THE NINTH ANNUAL MEETING OF THE WISCONSIN SECTION

The ninth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Beloit College, Beloit, on Saturday, May 3, 1941. The meeting was presided over by Chairman W. W. Bigelow of Beloit College.

Dean Conwell of Beloit College read a telegram from the secretary, Professor G. A. Parkinson, extending good wishes for a successful meeting.

There were fifty-five present, including the following twenty-one members of the Association: R. H. Bardell, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, H. H. Conwell, H. P. Evans, Cornelius Gouwens, R. C. Huffer, J. F. Kenney, Elizabeth E. Knight, R. E. Langer, H. W. March, Morris Marden, Sister Mary Felice, E. A. Nordhaus, R. E. Norris, H. P. Pettit, Irene Price, Fred Robertson, P. L. Trump, J. I. Vass.

The Section was honored by having Professor Cornelius Gouwens of Iowa, who is Regional Governor for this district, and Fred Robertson of Iowa State College as guests.

Sessions were held in the morning and in the afternoon, with a luncheon in Chapin Hall. Immediately following the luncheon the members were shown moving pictures of the Solar Prominences. Mrs. Beckwith was instrumental in securing the film, which was made at the McMath Observatory in Michigan.

At 2:30 P.M. the business meeting was held, at which the following officers were elected: Chairman, Irene Price, Oshkosh State Teachers College; Secretary-Treasurer, P. L. Trump, University of Wisconsin; Program Committee, R. H. Bardell, University of Wisconsin Extension Division, Chairman, and R. E. Norris, Milwaukee State Teachers College.

Appreciation for the hospitality extended to the Section by Beloit College was expressed by rising vote.

At the morning meeting the following papers were presented:

1. "On Descartes's rule of signs and its extensions" by Professor Morris Marden, University of Wisconsin Extension Division.

2. "Applications of mathematics in radio" by Professor R. D. Spangler, Physics, LaCrosse State Teachers College, introduced by Mr. Bigelow.

3. "Applications of mathematics to engineering" by A. D. Freas, U. S. Forest Products Laboratory, Madison, introduced by Professor March.

The afternoon session was devoted to a panel discussion on "Contents of secondary mathematics." Professor H. P. Evans of the University of Wisconsin presided. The discussion was divided as follows:

a. "Contents of secondary mathematics" led by Irene Eldredge, West Division High School, Milwaukee, introduced by Professor Evans.

b. "Secondary school preparation for college mathematics" led by W. W. Hart, Kenilworth, Illinois.

c. "Curriculum problems in a high school located in a university community" by R. O. Christopherson, Assistant Principal, West Side High School, Madison, introduced by Professor Evans.

The afternoon session was well attended by high school teachers, and there was lively participation in the discussion of the above-mentioned topics.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Marden discussed sequences of functions $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$ which have the property that, for all choices of real constants c_j, the number of zeros of the linear combination $c_0\phi_0(x)+c_1\phi_1(x)+ \cdots +c_n\phi_n(x)$ on a given interval $a<x<b$ does not exceed the number of variations of sign in the sequence $c_0, c_1, \cdots, c_n$. Examples of such sequences are $\phi_k(x)=x^k$ as given in the Descartes's rule of signs; $\phi_k(x)=e^{\lambda_k x}$ with $\lambda_k<\lambda_{k+1}$ as given by Laguerre; $\phi_k(x)=(x-\zeta_1)(x-\zeta_2)\cdots(x-\zeta_k)$ as given by Runge; and the orthogonal polynomials as given by Laguerre, Obreschkoff, and Marden. He also discussed: (1) the necessary and sufficient conditions, as given by Pólya and Szegö, for any sequence of n times continuously differentiable functions $\phi_j(x)$ to have the above property; (2) Schoenberg's applications of these conditions to polynomial sequences $\phi_k(x)=a_{k0}+a_{k1}x+ \cdots +a_{kk}x^k$; (3) Schoenberg's results on the comparison of the various rules of signs; and (4) the extensions of the rules of signs to complex zeros and complex c_j, as given by Obreschkoff, Marden, and Schoenberg.

2. Professor Spangler gave some illustrations of the development of formulas very useful or essential in radio. The crowning illustration in this connection, and a classic one, was Maxwell's prediction of electromagnetic waves.

3. Mr. Freas called attention to the fact that all of the various branches of engineering employ mathematics to a considerable degree. These applications run the gamut from the simplest forms of mathematics to the more highly complicated forms. For instance, most engineers find it necessary in their daily work to solve equations or sets of simultaneous equations, or to apply the principles of geometry and trigonometry, and so on. More exceptionally, they are called on to solve differential equations, or to use infinite series of various kinds, or to solve problems which lead to some of the less common functions. In addition to these direct applications of mathematics, engineers are constantly using principles which have been developed by mathematical procedures. From this it may be seen that a knowledge of mathematics, particularly of the fundamentals, constitutes one of the most useful tools of the engineer.

R. H. BARDELL, *Acting Secretary*

In closing I would pay tribute to the integrity of this scientist. At the time when the fascist oath was exacted of all teachers, Volterra realized that a promise of allegiance to the regime of Mussolini would violate the pledge he had made on becoming Senator of the kingdom. With sorrow he left the University which he had so greatly served and honored.

THE FACTORIZATION OF CERTAIN SECOND ORDER POLYNOMIAL DIFFERENTIAL OPERATORS

E. D. RAINVILLE, University of Michigan

1. Introduction. We shall consider differential operators which are polynomials in a variable x and in $D\equiv d/dx$. We are concerned with the factorization, if possible, of such operators into factors which are operators of the same type. Our aim is the actual determination of such factors, not the exhibiting of criteria for reducibility. The problem is solved here, in the above sense, only for a restricted set of polynomial differential operators. A previous paper on Riccati equations furnishes the tool for our present attack. It will be seen that one reasonable line of approach to the general problem is an extension of the results of that paper.

In the study of the algebraic properties of linear differential operators, Blumberg's thesis* may well be used as a starting-point. Most writers on the reducibility of such operators concern themselves with coefficients which are rational† functions of x. The results are naturally vastly different from those in the problem considered here. F. H. Miller‡ has discussed a variation of reducibility in which the coefficients of the operator are also rational in a parameter.

With the understanding that in this paper we speak only of polynomial differential operators, we shall use the terms divisor, proper divisor, and trivial operator or divisor in the usual sense. If A, B, C, are operators such that $A=BC$, then B and C are *divisors* of A and A is a *multiple* of B and of C. At times we are more precise and speak of right and left divisors and multiples. For instance, B is a *left divisor* of A and A is a *right multiple* of B. An operator, or divisor, is said to be *trivial* if, and only if, it is independent of both x and D. A *proper divisor* of an operator is a divisor which is neither trivial nor a trivial multiple of the original operator. Again we may speak of right and of left proper divisors.

2. A class of second order operators. Consider the operator

$$y=D^2+a_1(x)D+a_0(x),$$

where a_1 and a_0 are polynomials in x. If y has a proper divisor, then evidently y may be written

* H. Blumberg, Über algebraische Eigenschaften von linearen homogenen Differentialausdrücken, Göttingen, 1912.

† See remarks on page 9 of F. H. Miller's thesis, Reducible and irreducible linear differential equations, Columbia, 1932.

‡ See Miller, *loc. cit.*, also for further references on the question of reducibility.

$$y = (D + b)(D + c),$$

where b and c are polynomials in x. But

$$(D + b)(D + c) = D^2 + (b + c)D + bc + c',$$

in which primes denote differentiation with respect to x.

We now see that b and c must satisfy the equations

$$b + c = a_1,$$
$$bc + c' = a_0.$$

From these equations it follows at once that c must be a solution of

$$(1) \qquad c' = a_0 - a_1 c + c^2,$$

which we shall call the Riccati equation associated with the operator y.

The problem of the determination of polynomial solutions of this type of Riccati equation has already been treated.* We need a definition and a theorem from that paper. First, if $P(x)$ is a polynomial of even degree, then by $[\sqrt{P(x)}]$ we mean the polynomial part of the expansion of $\sqrt{P(x)}$ in descending powers of x. In order to insure uniqueness of the result we choose from the two possible coefficients of the highest power of x in $\sqrt{P(x)}$ that coefficient with the smaller positive amplitude. Secondly, from Theorem II of that paper we merely change notation to state that:

If in (1) a_1 and a_0 are polynomials in x and if the degree of

$$(2) \qquad \Delta = a_1^2 - 4a_0 + 2a_1'$$

is even, then no polynomial other than

$$(3) \qquad c = \tfrac{1}{2}(a_1 \pm T),$$

where $T = [\sqrt{\Delta}]$, may be a solution of equation (1). If the degree of Δ is odd, then there is no polynomial solution of the equation.

We see now that there may be at most two right proper divisors of y and that if there is one, its polynomial $c(x)$ must be given by one of the right members of equation (3). We have proved the following:

THEOREM 1. *The polynomial differential operator* $y = D^2 + a_1(x)D + a_0(x)$ *has* (*a*) *no proper divisors, if the degree of* $\Delta = a_1^2 - 4a_0 + 2a_1'$ *is odd, and*

(*b*) *no proper divisors other than* $y_1 = D + \frac{1}{2}a_1 + \frac{1}{2}[\sqrt{\Delta}]$ *or* $y_2 = D + \frac{1}{2}a_1 - \frac{1}{2}[\sqrt{\Delta}]$, *if the degree of* Δ *is even.*

We are using the fact that $b + c = a_1$, which shows that, if either y_1 or y_2 is a right divisor of y, then the other is the corresponding left divisor of y. It should be noted that in part (*b*) of the theorem it is not implied that y_1 or y_2 does

* E. D. Rainville, Necessary conditions for polynomial solutions of certain Riccati equations, this MONTHLY, vol. 43, 1936, pp. 473–476.

divide y. The theorem states that no other polynomial differential operator is a proper divisor of y. In practice we need only form the products y_1y_2 and y_2y_1 to settle completely the questions of reducibility and factorization in our sense.

Example 1. The operator $y=D^2+xD+x^3$ has no proper divisors because $\Delta=x^2-4x^3+2$ is of odd degree.

Example 2. The operator $y=D^2+xD-x+1$ has no proper divisors because the only "trial divisors" according to Theorem 1 are $y_1=D+x+1$ and $y_2=D-1$, from which

$$y_1y_2 = D^2 + xD - x - 1 \neq y,$$
$$y_2y_1 = D^2 + xD - x \neq y.$$

Example 3. The operator $y=D^2+D-x^2-x-1$ has the one factorization $y=(D+x+1)(D-x)$. Here we have $y_1=D+x+1$, $y_2=D-x$, $y_1y_2=y$, but $y_2y_1=y+2\neq y$.

In closing this section we remark that, from Theorem III of the Riccati paper, we may conclude that the operator $y=D^2+a_1D+a_0$ has two distinct proper factorizations if, and only if, Δ is a non-zero constant.

3. The general second order operator. In attacking the problem of the factorization of $y=a_2(x)D^2+a_1(x)D+a_0(x)$, we find that placing

$$a_2D^2 + a_1D + a_0 = (b_1D + b_0)(c_1D + c_0)$$

leads to the system of equations

$$b_1c_1 = a_2,$$
$$b_1c_1' + b_1c_0 + b_0c_1 = a_1,$$
$$b_1c_0' + b_0c_0 = a_0.$$

Here again primes denote differentiation with respect to x and we are searching for polynomial solutions b_0, b_1, c_0, c_1.

It is easy to show that (b_1c_0) must be a polynomial solution of the Riccati equation

$$a_2u' = a_0a_2 + (a_2' - a_1)u + u^2.$$

Again we have reduced the problem to the determination of polynomial solutions of a Riccati equation, but this time it is not an equation of the simple type used in section 2. The presence of the non-constant coefficient of u' has, for example, removed the *a priori* limitation to two polynomial solutions.

Finally we note that the solution of the problem of this section will carry with it the solution of the problem of factorization of differential operators of any order, but with coefficients only quadratic polynomials in x. The reduction of the one problem to the other is accomplished by a variation on a classical device involving the use of the Laplace transformation and the adjoint operator. This tool will be used in a later paper on factorization of operators.

FUNDAMENTAL CONCEPTS OF THE THEORY OF PROBABILITY*

A. H. COPELAND, University of Michigan

1. Introduction. In the foundations of the theory of probability there are two principal schools, the *a priori* and the statistical. These schools are of course further subdivided by differences of opinion within the ranks of each, but the division still serves as a useful classification when one attempts to study the controversial issues relating to the theory of probability.

The distinction between the two points of view can perhaps best be brought out by imagining that an experiment, such as the throwing of a die, is to be performed. An adherent of the *a priori* theory will interest himself in the character of the die, its geometrical symmetry, the location of its center of mass, its moments and principal axes of inertia, or the character of its faces as regards friction with the air or friction with the surface upon which it is to land. When he has satisfied himself with regard to some or all of these characteristics, his interest in the die ceases. The outcome of the experiment which is to be performed is not important to the *a priori* theory.

An adherent of the statistical theory may concern himself with the characteristics of the die and may also formulate hypotheses concerning the probabilities of its respective faces. He will, however, regard these probabilities as purely hypothetical until they have been tested by experiment. That is, the outcome of the tossing of the die *is* important to the statistical theory of probability. Moreover, it is essential that this experiment be performed not once but a large number of times.

2. A single trial. This requirement has been the subject of an attack on the theory. It has been contended that the statistical theory thereby excludes an important class of events, namely, those which permit only a single trial. Let us investigate what relationship is possible between the probability of an event and a single trial of that event. As regards the outcome, only two things are of importance to the theory of probability—success and failure. What relation is possible between the probability of an event and these alternatives? Can we assert that certain probabilities are invariably associated with success and the others invariably associated with failure? Such an apportionment of probabilities is universally rejected. Of course we might state that certain probabilities are likely to be or probably are associated with success; the others, with failure. It is, however, agreed that the phrases "likely to be" and "probably are" both mean "there is a probability greater than a half that." We now arrive at the statement that there is a probability greater than a half that certain probabilities are associated with success and others with failure. We can next ask whether a probability greater than a half is necessarily associated with success, and I suppose we

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* An address given at the invitation of the Program Committee at the Iowa meeting of the American Mathematical Society, November 27, 1937.

can answer that it probably is. This leads us to conclude that there is a probability greater than a half that there is a probability greater than a half that certain probabilities are associated with success. We can continue this vicious regression as long as we like, but there is a probability greater than a half that we will not be much enlightened by so doing. The fact is, probability can have no important bearing on the result of a single trial.

I repeat a statement which I previously made and with which some of you may have disagreed. The *a priori* theory is not concerned with the outcome of events. Certainly this is true for the case of a single trial.

3. Many trials. Next let us consider the situation with regard to a number of trials. Suppose, for example, we make a thousand trials of an event with the probability 1/2 and ask whether the number of successes will lie between 400 and 600. The occurrence of between 400 and 600 successes is a single event. It merely succeeds or fails. We can, in fact, compute the probability of this event in terms of the probability 1/2. The same can be said for any given probability, for any given number of trials, or for any given distribution of successes and failures. Thus we have returned to the same difficulty which we met in the case of a single trial.

Perhaps we have been following a will-o'-the-wisp. Perhaps we should admit that probability can have no bearing upon the results of trials. But let us see what this admission does to the application of the theory. An experimental physicist makes a series of experiments. He regards the mean as the value of the quantity which he is measuring. He then computes the probable error of his result. Shall we say that this probable error has no bearing on the actual error? Life insurance companies compile elaborate mortality tables. Shall we say that these tables have no bearing on the mortality rate, and that the relation between these tables and the insurance premiums charged is purely fictitious? A telephone engineer computes the probability that a given number of telephone calls will be made over a given trunk line within a certain interval of time. Shall we conclude that this probability has nothing to do with the number of calls which actually will be made?

4. A purely mathematical theory. Concerning these questions there is a point of view which enjoys considerable popularity today. Namely, that probability may be regarded as a purely mathematical theory and that it is not necessary for a mathematical theory to have applications in order to be important. Probability can then be taken as an undefined term in a postulational system. I heartily agree with the claim that a mathematical theory does not necessarily have to be applied in order to be important. Moreover, I believe that a postulational treatment of probability is highly desirable. There exists today a very excellent system of postulates in which probability is regarded as an undefined term. This system is due to Kolmogoroff. Unfortunately, however, the point concerning a mathematical theory does not tell the whole story. In the first place, there is scarcely a paper or a book written on the theory of probability

which does not mention some application to experiment. Hence if probability has no important bearing on the results of experiment, then we are failing to maintain our intellectual integrity when we keep up the pretense. In the second place, there are not many of us who would like to see the theory completely deprived of all its applications. In the third place, those applying the theory of probability seem to be getting away with it. If we inquire how it happens that probability is successfully applied, it is at least conceivable that we may be led to questions which are both philosophically and mathematically interesting.

5. The statistical theory. Let us see what the statistical theory has to offer as a solution of our difficulty. The fundamental assumption of the statistical theory is that the probability is the limit approached by the success ratio as the number of trials is indefinitely increased. This assumption has frequently been given the following rough interpretation: the success ratio will be approximately equal to the probability, provided a large number of trials has been made. But what is a large number? Is 1000 a large number? We have already seen that the probability cannot determine any limits to the success ratio in 1000 trials. The same is true for 1,000,000 or 1,000,000,000 trials. To answer this objection it will be convenient to express the statistical assumption in terms of the usual mathematical definition of a limit. Let r_n be the number of successes in the first n trials of a given event with probability p. Let $s_n = r_n/n$ be the success ratio. Then given a positive number ϵ, there exists a number N such that

$$| s_n - p | < \epsilon \quad \text{whenever} \quad n > N.$$

The degree of approximation is represented by ϵ and we shall say that the number of trials is large or at least sufficiently large for the given degree of approximation if it exceeds N. The condition which we have written down need not be interpreted in terms of success ratios and probabilities. It can be interpreted as the condition that an arbitrary sequence s_n approach an arbitrary limit p. We cannot be assured that if ϵ is say 1/3, then the nth term will differ from the limit of the sequence by less than ϵ provided n exceeds 1,000 or 1,000,000 or even 1,000,000,000. The same is true when s_n is a success ratio and p the corresponding probability.

The following objection has been raised against the statistical assumption—namely, that we do not know that any limit whatsoever will be approached. Even less do we know that the limit will be what we think it ought to be. On this account, certain suggestions have been made in recent years as to how to weaken the statistical assumption. It seems to me that in adopting this point of view we are being unnecessarily timid and that the attempt to weaken the statistical assumption is a trend in the wrong direction. The reason for my attitude is the following. I think I am safe in claiming that we shall never learn how to make and record an infinite number of trials of an event, and to compute the corresponding success ratios. Therefore if the statistical assumption is false, we shall never know it. It seems that in general the more nearly a theory approximates

incontrovertibility, the more nearly does it approximate triviality. Since the statistical assumption cannot be contradicted by physical experiment, it is already too close to the trivial. Nevertheless, this assumption has some force and if we build a theory around it, we can to a certain extent reinforce it.

6. Weakening the statistical assumption. We shall now discuss suggestions for weakening the statistical assumption. One suggestion is the following. We consider the set of limit points of the success ratios. It is assumed that only one of these limit points is invariant under the operation of certain selections by which sub-sequences of trials are obtained. This invariant limit point is the probability. If it is true that success ratios can behave in this manner when more than one limit point of the original sequence of trials exists, then the assumption is considerably weaker than the statistical assumption. This weakened assumption might lead to many interesting mathematical investigations, but as a basis for the theory of probability it has the doubtful advantage of weakening an assumption which is already too weak.

A second method of weakening the statistical assumption is obtained by considering sets of trials instead of sequences of trials. Two possibilities are open. First, given ϵ, there exists an N such that for *every* set of at least N trials the success ratio differs from the probability by less than ϵ. Second, there exists *some* set of at least N trials for which this inequality holds. Let m be a positive integer such that $1/m<\epsilon$, and let us make a sufficient number of trials of an event so that we obtain at least m successes and at least m failures. We can then select a set of these trials in such a manner that the corresponding success ratio can be made to approximate any given number in the interval from 0 to 1 to within ϵ. It follows that the first assumption, which contains the word "every," is untenable. The second assumption, which contains the word "some," can be accepted but it is so weak that it is trivial.

Perhaps it may be objected that I am not being fair, and that in this theory it is intended that the trials be selected at random without any knowledge of whether they are successes or failures. I am willing to accept this revision, but I must point out that it does not make the slightest bit of difference in the argument since any selection is possible even though this selection be made at random.

A third mode of weakening the statistical assumption is to reject it altogether and replace it by nothing. We have already seen the consequences of this procedure. I mention this third mode chiefly since I believe that in certain cases the statistical assumption has been rejected because of a feeling that it was too weak. I could appreciate this attitude if something stronger had been suggested.

7. The causal theory. A new theory, called the causal theory of probability, has attained some popularity recently. Although this theory does not require any modification of the statistical assumption, there has nevertheless been some tendency to replace the statistical theory by the causal theory. Roughly, this new theory is the following. It is observed that many of the probability problems

are related to mechanical devices. Thus we start with a given mechanical system and a given set of initial conditions. By means of the equations of motion we can calculate the position in the phase space which the system will occupy after a time t. Hence, the equations of motion furnish us with a one-parameter group of transformations on the phase space, the parameter being the time. By means of these transformations a point representing the initial conditions of the system is carried into a point representing the phase position of the system after a time t. Furthermore, any given probability distribution for the initial conditions can be transformed into a new distribution giving the probability that the system will occupy a given phase position at a given time t. In general as t becomes infinite, a definite limiting distribution is approached which is essentially independent of the distribution for the initial conditions. This limiting invariant distribution assigns a number to each region of the phase space. The number assigned to a given region is defined as the probability that the system will occupy a phase position within this region after a sufficiently long time. I might mention that other physical parameters can be used in place of the time, but the time furnishes a good illustration of the way this theory is formulated.

It is interesting to investigate to what extent the causal theory can stand on its own feet. If we strip the theory of its probability connotation, we obtain the following. Let $\Pi_0(E)$ be an additive non-negative function such that $\Pi_0(S)=1$, where S is the given phase space and E is any sub-set of S. The equations of motion furnish us with a transformation which carries $\Pi_0(E)$ into a function $\Pi_t(E)$ after a time t. Allowing t to become infinite we obtain, in general, a definite limiting function $\Pi(E)$ which is essentially independent of $\Pi_0(E)$. This is undoubtedly an interesting and important mathematical result. But what is its physical importance? If we define $\Pi(E)$ as the probability that the system will occupy a phase position which is an element of E and if we apply the statistical assumption, then and not until then does the causal theory of probability give us a basis for physical predictions. When we apply the statistical assumption to the causal theory of probability, we can obtain some relation between the number $\Pi(E)$ and the question as to whether or not the system will occupy a phase position within the set E.

8. A large number. We have noted that up to the present the statistical theory gives us the least trivial known relationship between probabilities and the results of trials. I have also mentioned that the statistical assumption can be reinforced to some extent, but I shall not take the time to go through the theory whereby this is accomplished. One result is given in the following formula for a large number:

$$N = Kp(1 - p)/\epsilon^2,$$

where p is the probability of the event in question and ϵ is the desired degree of approximation. The number K appearing in this formula is to some extent arbitrary and is to be settled by agreement. If K is as small as 1, the formula will be unsatisfactory. The choice of 100 for K is generous, while 16 is a fairly satis-

factory choice. For definiteness, let us set $K=16$. Then we will seldom fail to obtain an approximation as good as ϵ provided we make at least N trials of the event and provided our determination of the probability is nearly correct. If we should fail to obtain this degree of approximation, we should be very skeptical of our determination of the probability. It should be noted, however, that the above formula is merely a criterion for skepticism. It is not and cannot be a fundamental assumption in the theory of probability. In fact, such an assumption would necessarily lead to inconsistencies no matter what our choice might be for the number K. In this connection, there is one contingency which we must face. That is, it may happen that the respective success ratios from the Nth on will oscillate to such an extent that no determination of the probability can possibly lead to the desired degree of approximation. Would such an event cast doubt upon the statistical assumption? Let us observe that this event has a finite probability. Thus it is not inconsistent with the statistical theory that such an event should occur. It is rather a consequence of the theory that it must occur at some time or other. Under such circumstances, our criterion for skepticism tells us that we must be skeptical of all probabilities until more conclusive experiments are performed.

9. Systems of events. If we accept the statistical assumption, there are a number of questions which demand immediate investigation. For example, the independence of trials. First I should say a word or two about independence in general. If events are independent, then the probability of their conjunction is obtained by multiplying together their respective probabilities. Hence, independence is immediately related to the results of trials by means of the statistical assumption. The case of the independence of trials, however, offers an additional difficulty. It is customary to associate with each trial that probability which is associated with the whole sequence of trials. Thus, apparently, we are dealing with events which are capable of only a single trial—namely, the trials themselves or the conjunctions thereof. Unless this difficulty is overcome, the statistical theory is almost a complete failure since the concept of independence of trials is fundamental to the theory of probability. In order to resolve this difficulty, let us consider first a simple case. Let us attempt to prescribe a repeatable experiment which will enable us to test the probability of the conjunction of two trials. A natural mode of experimentation is the following. If we make $2n$ trials of some event, we obtain thereby n pairs of trials. The first pair consists of the first two trials; the second pair, of the second two trials; *etc.* Each pair then constitutes a single trial which succeeds if and only if both members of the pair succeed. We should expect the success ratio thus obtained to approach the limit p^2 if the original event has the probability p and if the trials are independent. That is, the probability of the conjunction should be p^2. But of what two events is this the conjunction? There is also a simple and natural answer to this question. One is the event whose trials are the odd-numbered trials of the original event. The trials of the other event are the even-

numbered trials. In general, if we wish to test the probability of n successes in n consecutive trials of an event x, we form the conjunction of events $x_1, x_2, \cdots, x_n$ defined as follows. The kth trial of the event x will be a trial of the event x_r provided

$$k \equiv r (\mathrm{mod}\ n).$$

The conjunctive event will consist of groups of n trials each. In each group we form the conjunction of the n trials. Each group thus becomes a single trial of the conjunctive event. This method of interpreting the independence of trials is, I think, natural. In fact, Reichenbach and I have both been led to this device entirely independently. There are many alternative devices, but they seem more artificial and less useful. If the independence of trials is defined in the above manner, then the question arises as to whether the assumption of such independence is free from contradiction. However, as some of you probably know, there is no inconsistency.

It will be observed that the trials of the event x_r are obtained by selecting a sub-sequence of the event x. We assume that the probability of the event x_r is the same as the probability of x. That is, probability is invariant under the operation of such a selection. This raises the question as to what selections should possess this invariant quality. R. von Mises requires that probability shall be invariant under the operation of all selections which are made in accordance with any mathematical law provided that law is independent of the occurrence or non-occurrence of the trials selected. This assumption has been questioned both as to its meaning and as to its consistency, but I shall not go into these considerations here. The assumption does seem to imply a non-denumerable infinity of selections under which the probability must be invariant. The following facts may be noted. It is necessary to place some restrictions on the system of selections. It is possible to obtain a non-denumerable system. It is also possible to obtain a denumerable system of selections possessing the property that invariance under the denumerable system cannot be distinguished by experiment from invariance under the non-denumerable system.

Other questions concern sets of events. How general can these sets be? What relations can exist among the elements of the sets—for example, independence or specified dependence or mutual exclusiveness? What combinations can be formed of the elements of these sets—say, by conjunction or disjunction? Can we assume that the elements of the sets and such combinations constitute events whose trials are independent or events whose trials have certain specified dependences? Some of these questions have been settled, but a number of them remain unanswered.

10. Postulational basis. I shall note one other important consideration. Is it possible to construct a strictly postulational system which gives an accurate picture of the relation between probabilities and the results of trials? I have mentioned two characteristics which the desired system should possess. First, that it should give an accurate picture of the relation between probability and ex-

periment. This means that something closely related to the statistical assumption should appear in the system. Secondly, it is to be a *strictly* postulational system. By this I mean that the system should satisfy the modern requirements of postulational technique. For a long time I have wondered how to state these requirements precisely. It finally occurred to me that the criterion was simple. Namely, a system is strictly postulational if and only if it can be translated completely into the symbolism of *Principia Mathematica*. Without such a system of postulates, the statistical theory of probability is not a completely mathematical theory. Moreover, there is a certain vagueness as to what the assumptions are. I might mention that I have recently constructed a system which I believe does possess these desired characteristics. It will be published in the *American Journal of Mathematics*.

11. Questions. I shall now return to another question which was raised earlier in this paper. The question concerns probabilities associated with events which are capable of but a single trial. A friend of mine once constructed an example in which the result of a single trial was of vital importance. A man had been captured by gangsters who were undecided as to whether they should kill him. They finally agreed that the occurrence or non-occurrence of a seven on a single throw with a pair of dice should decide the issue. The man was given a choice as to whether the occurrence of seven should spell life or death, and was even allowed to examine the dice to convince himself that they were not loaded. He reasoned that the probability was 5/6 that the seven would not occur, and hence decided that the occurrence of seven should signify death. Was his choice a wise one? Most of us would say that it was. But why? We have already seen that there can be no dependence of the result of a single trial upon the probability. I need not reproduce the vicious regression in order to convince you that there would be no logical gain in arguing that he made the choice which was most likely to lead to a favorable outcome. At least there is no logical gain unless we frankly take this to be a definition of "the wise choice." There is, however, some advantage in making this definition. We can investigate its consequences and see whether they coincide with what we mean by "a wise choice."

Let us assume that there are two groups of people and that all members of each group are presented with the above predicament. In the first group, the wise choice is a universal law of conduct. In the second group, the unwise choice is universally made. We can say that if the groups are large enough, the first group will fare better than the second. We cannot say of any individual in the first group that he will fare better than a given individual of the second group. I think, however, that most of us would prefer to belong to the first group. We have a prejudice in favor of Kant's categorical imperative, at least under such circumstances. We would decide upon the wise choice as a universal law of conduct.

Two further examples will be useful in this connection. The first concerns the familiar life-on-Mars paradox. Let us imagine that I have been able to determine

the correct probability that there is life on Mars. It turns out that it is not 1/2, but 5/6. What conclusion can you draw as to whether there is life on Mars? I shall leave this question with you without further discussion.

In the second example, I have a die which is loaded so that one face is certain to turn up, but I do not tell you which face this is. I am going to make a large number of throws of this die. What is the probability that I will get a three on the first throw of the series? Since your ignorance is equally distributed among the six faces of the die, you might conclude that the answer was 1/6. Assuming this to be correct, what has it to do with the result of the first trial? And what has it to do with the results of all the trials?

I shall also consider a variation of this problem. I have six dice in an urn. One is loaded so that the ace always turns up; another, so that the deuce; another, the three; *etc.* A die is drawn at random from the urn and thrown, and then returned. This experiment is repeated many times. What is the probability of getting a three on a given trial? Should this problem be treated in the same manner as the preceding one? Again I shall leave these questions with you without further discussion.

In conclusion, I will state that the rejection of the statistical assumption entails an obligation either to replace this assumption by something else which will serve the purpose, or to admit that the theory of probability is not concerned with the results of trials. If the latter course is chosen, then the phrase "the probability that an event will occur" is misleading.

SOME PROPERTIES OF THE TRIANGLE

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1. Introduction. Given arbitrarily a triangle $A_1A_2A_3$, let the triangles $B_1A_2A_3$, *etc.*,† of fixed shapes be constructed on its sides. Then it is a known theorem that if $\triangle B_1A_2A_3$, *etc.*, are all similar to an isosceles triangle with 120° vertex angle, $\triangle B_1B_2B_3$ is always equilateral, no matter what triangle $A_1A_2A_3$ is given. Is the converse also true? That is, if $\triangle B_1B_2B_3$ is always equilateral, no matter what triangle $A_1A_2A_3$ is given, the shapes of $\triangle B_1A_2A_3$, *etc.*, remaining fixed, then are the triangles $B_1A_2A_3$, *etc.*, necessarily all similar to an isosceles triangle with 120° vertex angle? This question does not seem to have been answered. In this paper, I solve a few problems of this type, yielding some interesting results. The instrument employed is the so-called complex coördinates;‡ the method used will be explained in the next few paragraphs.

A (real) point in the plane can be represented by a single complex number, which is called the complex coördinate of the point. For convenience points will be denoted by capital letters and their coördinates by the corresponding small

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* The author is a Chinese "Indemnity Funds" Student.

† Throughout this paper an expression followed by "*etc.*" means the totality of this expression and of two similar ones obtained from it by permuting cyclically the subscripts 1, 2, 3.

‡ See, for example, Morley and Morley, Inversive Geometry.

letters. Thus, whenever we speak of a point, say P, it is understood that the coördinate of P is p; the only exception is the origin, which has the coördinate zero. Numbers which are definitely known to be real will be denoted by Greek letters.

With the help of a parametric complex number r_1, a point B_1 may be adjoined to two arbitrarily given points A_2, A_3 by the following equation:

$$b_1 = r_1' a_2 + r_1 a_3, \tag{1.1}$$

where

$$r_1' + r_1 = 1. \tag{1.2}$$

In what follows it is assumed, unless stated otherwise, that different capital letters denote distinct points, so that $r_1, r_1' \neq 0$ or 1. Expressing r_1, r_1' in terms of a_2, a_3, b_1, we have

$$\begin{gathered} r_1 = (b_1 - a_2)/(a_3 - a_2), \qquad r_1' = (b_1 - a_3)/(a_2 - a_3), \\ r_1/r_1' = -(b_1 - a_2)/(b_1 - a_3). \end{gathered} \tag{1.3}$$

Any of these equations shows that the parameter r_1 determines the shape of $\triangle B_1A_2A_3$, and conversely.

With this means of adjoining a point to any two points of a figure at our disposal, it is now clear that the problem mentioned at the beginning of the paper can be solved by carrying out the following procedures:

(*i*) Adjoin to the sides A_2A_3, *etc.*, of a triangle $A_1A_2A_3$ the points B_1, *etc.*, the parametric complex numbers being r_1, *etc.*

(*ii*) Set up the condition $f(a_1, a_2, a_3, r_1, r_2, r_3) = 0$ for $\triangle B_1B_2B_3$ to be equilateral, and equate to zero the coefficients of the a's in it, thus obtaining three equations in the r's alone.

(*iii*) If the last set of equations are consistent, obtain geometrically (or obtain and then interpret geometrically) its most general solution for the r's.

In the following two sections analogous processes are carried out, with the respective requirements that the triangle $B_1B_2B_3$ is similar to a given triangle $C_1C_2C_3$ and that the lines A_1B_1, *etc.*, are concurrent.

2. Some properties of a triangle. To the sides A_2A_3, *etc.*, of a triangle $A_1A_2A_3$ let the points B_1, *etc.*, be adjoined by the parameters r_1, *etc.*; thus

$$b_1 = r_1' a_2 + r_1 a_3, \quad etc., \tag{2.1}$$

$$r_1' + r_1 = 1, \quad etc. \tag{2.2}$$

Now the condition that the triangle $B_1B_2B_3$ be similar to a given triangle $C_1C_2C_3$ is

$$\begin{vmatrix} r_1' a_2 + r_1 a_3 & c_1 & 1 \\ r_2' a_3 + r_2 a_1 & c_2 & 1 \\ r_3' a_1 + r_3 a_2 & c_3 & 1 \end{vmatrix} = 0. \tag{2.3}$$

Equating to zero the coefficients of a_1, a_2, a_3 in (2.3), we obtain

$$r_2(c_3 - c_1) + r_3'(c_1 - c_2) = 0, \quad etc. \tag{2.4}$$

These can be written, because of (2.2),

$$r_2(c_3 - c_1) - r_3(c_1 - c_2) = - c_1 + c_2, \quad etc.$$

The solutions for the r's are now readily seen to be

$$r_2(c_3 - c_1) = d - c_1, \quad etc., \tag{2.5}$$

where d is an arbitrary complex number. To find the geometric meanings of equations (2.5), we express the r's in terms of the a's and b's by means of (1.3) and their analogous equations. We obtain

$$(d - c_1)/(c_3 - c_1) = (b_2 - a_3)/(a_1 - a_3), \quad etc., \tag{2.6}$$

which are the conditions that the triangles $B_1A_2A_3$, *etc.*, are (directly) similar to the triangles DC_3C_2, *etc.* Hence we have the following:

THEOREM 2.1. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, etc., of fixed shapes are constructed, then in order that the triangle $B_1B_2B_3$ may always be similar to a given triangle $C_1C_2C_3$, no matter what triangle $A_1A_2A_3$ is chosen, it is necessary and sufficient that a point D exists such that the triangles $B_1A_2A_3$, etc., are similar to the triangles DC_3C_2, etc.*

When $\triangle C_1C_2C_3$ is equilateral and $\triangle DC_3C_2$, *etc.*, are all similar, then it is geometrically evident that D must be the incenter of $\triangle C_1C_2C_3$. This gives an answer to the question we asked in §1.

COROLLARY 2.1 a. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, similar triangles $B_1A_2A_3$, etc., of fixed shapes are constructed, then in order that the triangle $B_1B_2B_3$ may be equilateral, no matter what triangle $A_1A_2A_3$ is chosen, it is necessary and sufficient that the triangles $B_1A_2A_3$, etc., are isosceles with* 120° *vertex angles at the B's.*

It will be observed that the similar isosceles triangles $B_1A_2A_3$, *etc.*, are constructed all outward or all inward according as the equilateral triangle $C_1C_2C_3$ is described in the same or opposite sense as $\triangle A_1A_2A_3$.

When C_1, C_2, C_3 are three points on a line, we have the following:

COROLLARY 2.1 b. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, etc., of fixed shapes are constructed, then in order that the points B_1, B_2, B_3 may always be collinear, no matter what triangle $A_1A_2A_3$ is chosen, it is necessary and sufficient that $\measuredangle B_2A_3A_1 = \measuredangle B_3A_2A_1$, etc.*

Here $\measuredangle B_2A_3A_1$ denotes the directed angle* from B_2A_3 to A_3A_1, *i.e.*, the angle through which the line B_2A_3, taken as a whole, must be rotated about A_3 in the

* R. A. Johnson, Modern Geometry, §16.

positive direction in order to coincide with A_3A_1. From this definition, directed angles are equivalent when they differ by π.

Finally, if C_1, C_2, C_3, D are four points on a line, Theorem 2.1 gives the theorem of Menelaus.

3. Some properties of a triangle (continued). We now consider the condition that the lines A_1B_1, *etc.*, be concurrent, the coördinates of B_1, *etc.*, being given by equations (2.1). The equations of the lines A_1B_1, *etc.*, are

$$(\bar{a}_1 - \bar{b}_1)x - (a_1 - b_1)\bar{x} + a_1\bar{b}_1 - \bar{a}_1 b_1 = 0, \quad etc.,$$

where, as well as in what follows, conjugate complex numbers are indicated by bars; thus, for example, $\bar{a}_1$ is the conjugate of a_1. Since we are dealing with a similitude property, we may suppose without loss of generality that

$$a_1 = a, \qquad a_2 = 0, \qquad a_3 = 1. \tag{3.1}$$

Then

$$b_1 = r_1, \qquad b_2 = r_2' + r_2 a, \qquad b_3 = r_3' a, \tag{3.2}$$

and the condition for the concurrence of the lines A_1B_1, *etc.*, is

$$\begin{vmatrix} -\bar{r}_1 + \bar{a} & -r_1 + a & -\bar{a}r_1 + a\bar{r}_1 \\ \bar{r}_2' + \bar{r}_2\bar{a} & r_2' + r_2 a & 0 \\ 1 - \bar{r}_3'\bar{a} & 1 - r_3' a & \bar{r}_3'\bar{a} - r_3' a \end{vmatrix} = 0. \tag{3.3}$$

Equating to zero the coefficients of $a^2\bar{a}$, a^2, $a\bar{a}$, and a, we obtain

$$\begin{aligned} \bar{r}_1(\bar{r}_2 r_3' - r_2\bar{r}_3') + r_3'(r_2 - \bar{r}_2) &= 0, \\ \bar{r}_1(\bar{r}_2' r_3' + r_2) - r_3'(\bar{r}_1 r_2 + \bar{r}_2') &= 0, \\ r_1(\bar{r}_2' r_3' + r_2) - \bar{r}_3'(\bar{r}_1 r_2 + \bar{r}_2') + \bar{r}_1(r_2'\bar{r}_3' + \bar{r}_2) - r_3'(r_1\bar{r}_2 + r_2') &= 0, \\ \bar{r}_1(\bar{r}_2' - r_2') + r_3'(\bar{r}_1 r_2' - r_1\bar{r}_2') &= 0. \end{aligned} \tag{3.4}$$

The equations obtained by equating to zero the coefficients of $\bar{a}^2 a$, $\bar{a}^2$, $\bar{a}$ are the conjugates of $(3.4)_1$, $(3.4)_2$, $(3.4)_4$,* and are therefore included in (3.4).

We shall now solve the equations (3.4), noting that $r_1, r_2, r_3 \neq 0$ or 1. If one of the r's is real, it follows from $(3.4)_1$ and $(3.4)_4$ that the other two r's are also real. In this case, equations $(3.4)_1$ and $(3.4)_4$ are identically satisfied, while equations $(3.4)_2$ and $(3.4)_3$ both reduce to

$$r_1 r_2 r_3 = r_1' r_2' r_3'.$$

Geometrically we have that B_1, *etc.*, lie respectively on A_2A_3, *etc.*, and

$$\frac{B_1A_2}{B_1A_3}\,\frac{B_2A_3}{B_2A_1}\,\frac{B_3A_1}{B_3A_2} = -1.$$

* Throughout this paper we adopt the convention that by $(3.4)_1$ we mean the first equation or the first set of equations in (3.4), and by (3.5) the group of equations consisting of the equations $(3.5)_1$, $(3.5)_2$, $\cdots$.

Let us now suppose that none of the r's is real. Then from $(3.4)_1$, we have

$$(3.5)_3 \qquad r_3' = \rho_3 \bar{r}_1,$$

where, as we have pointed out, a Greek letter always denotes a real number. In virtue of this, $(3.4)_3$ is a consequence of $(3.4)_2$, and $(3.4)_1$ and $(3.4)_4$ become, respectively,

$$\bar{r}_1\bar{r}_2 - r_1r_2 + (r_2 - \bar{r}_2) = 0, \qquad \bar{r}_2' - r_2' + (r_3'r_2' - \bar{r}_3'\bar{r}_2') = 0,$$

i.e., $r_2r_1' - \bar{r}_2\bar{r}_1' = 0$, $-\bar{r}_2'\bar{r}_3 + r_2'r_3 = 0$, which are equivalent to

$$(3.5)_1 \qquad r_1' = \rho_1\bar{r}_2,$$

$$(3.5)_2 \qquad r_2' = \rho_2\bar{r}_3.$$

Writing $(3.4)_2$ in the form

$$\bar{r}_1r_2(1 - r_3') - (1 - \bar{r}_1)\bar{r}_2'r_3' = 0,$$

i.e., $\bar{r}_1r_2r_3 - \bar{r}_1'\bar{r}_2'r_3' = 0$, we have, because of (3.5),

$$(3.6) \qquad \rho_1\rho_2\rho_3 = 1.$$

Now if we eliminate $\bar{r}_1$ and r_2 from the equations

$$r_3 + \rho_3\bar{r}_1 = 1, \qquad \bar{r}_1 + \rho_1 r_2 = 1, \qquad r_2 + \rho_2\bar{r}_3 = 1,$$

which are equivalent to (3.5), we get

$$r_3 + \rho_1\rho_2\rho_3\bar{r}_3 = 1 - \rho_3 + \rho_1\rho_3.$$

Since the ρ's are real but the r's are not, we must have (3.6). Thus (3.6) is only a consequence of (3.5), which is therefore equivalent to (3.4).

Expressing the r's in terms of the a's and b's, we can write (3.5) in the form

$$(3.7) \qquad (a_2 - b_3)/(a_2 - a_1) = \rho_3(\bar{a}_2 - \bar{b}_1)/(\bar{a}_2 - \bar{a}_3), \quad etc.$$

These are the conditions that

$$(3.8) \qquad \measuredangle A_1A_2B_3 + \measuredangle A_3A_2B_1 = 0, \quad etc.$$

Hence we have the following:

THEOREM 3.1. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, etc., of fixed shapes are constructed, then in order that the lines A_1B_1, etc., may always be concurrent, no matter what triangle $A_1A_2A_3$ is chosen, it is necessary and sufficient that either of the following conditions is satisfied:*

(*i*) *B_1, etc., lie on A_2A_3, etc., such that*

$$\frac{B_1A_2}{B_1A_3}\,\frac{B_2A_3}{B_2A_1}\,\frac{B_3A_1}{B_3A_2} = -1;$$

(*ii*) $\measuredangle A_3A_1B_2 + \measuredangle A_2A_1B_3 = 0$, *etc.*

We note that (*i*) gives the well known theorem of Ceva, and that the sufficiency of (*ii*) is also known.*

In order that in the figure of Theorem 2.1 the lines A_1B_1, *etc.*, may also be concurrent, it is necessary and sufficient, by Theorem 3.1, that $\sphericalangle C_1C_3D + \sphericalangle C_1C_2D = 0$, *etc.*; *i.e.*, that D is the orthocenter of triangle $C_1C_2C_3$. Hence we have the following:

THEOREM 3.2. *If, on the sides of an arbitrarily chosen triangle $A_1A_2A_3$, triangles $B_1A_2A_3$, etc., of fixed shapes are constructed, then in order that the triangle $B_1B_2B_3$ may always be similar to a given triangle $C_1C_2C_3$ and the lines A_1B_1, etc., concurrent, no matter what triangle $A_1A_2A_3$ is chosen, it is necessary and sufficient that*

$$\sphericalangle A_3A_1B_2 = \sphericalangle B_3A_1A_2 = \tfrac{1}{2}\pi - \sphericalangle C_2C_1C_3, \quad etc.$$

THE GEOMETRY OF THE TRIANGLE IN THE KASNER PLANE

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1. Introduction. The aim of this paper is to advance a symmetric notation for the study of the triangle in the Kasner plane. A new definition of perpendicularity, namely quasi-perpendicularity, will be given, and new points of the triangle and their properties will be discussed.

The geometry of the Kasner plane depends on the following definitions:

A *point* is an ordered pair of complex numbers.

The *distance* d_{12} from the point $P_1(x_1, y_1)$ to the point $P_2(x_2, y_2)$, where $y_1 \neq y_2$, is $(x_2 - x_1)^2/(y_2 - y_1)$.

A *line* is a linear equation $Ax + By + C = 0$ with complex coefficients, where A and B are not both zero. A line is said to be a *zero line* if $A = 0$, and an *infinite line* if $B = 0$.

A linear equation is said to be a *general line* if both A and B are not zero.

The *slope* of the general line $y = px + q$, $(p \neq 0)$, is p.

Lines are *parallel* if they have no points in common. Clearly, all zero lines are parallel, all infinite lines are parallel, and two general lines are parallel if their slopes are equal.

The *angle* θ_{12} from the line L_1 to the line L_2 is p_2/p_1, where p_i is the slope of L_i, $(i = 1, 2)$, and $p_2 \neq p_1$.

A *circle* is the locus of all points at a given distance r from a given point (x_0, y_0), together with the point (x_0, y_0). The equation of the circle is $(x - x_0)^2 = r(y - y_0)$.

Rigid motion: The distance d_{12} and the angle θ_{12} as defined above are invariant under the three-parameter group of transformations

$$(1) \qquad G_3: \quad x' = mx + h, \qquad y' = m^2y + k,$$

which may be called the group of rigid motions of the Kasner plane.

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* Johnson, Modern Geometry, §356.

Similitude transformations: The four-parameter group of transformations

$$G_4:\quad x' = mx + h, \qquad y' = ny + k, \tag{2}$$

leave angles invariant, but distances are multiplied by m^2/n.

Kasner has used this plane extensively in his study of the horn angle. Consider a horn set, the set of all curves tangent to each other at a point P. Each curve C may be represented by a point (x, y) in the Kasner plane, where x is the curvature of C at P, and y is the derivative of the curvature at P with respect to the arc length of the curve. The distance between two points becomes the well known Kasner measure of the horn angle.*

We shall study in this paper the general triangle, that is, the triangle all of whose sides are general lines. Since G_3, the group of rigid motions, includes translations $(m=1)$, any three vertices of a general triangle may be transformed into points lying on a circle through the origin. We now employ a similitude transformation of G_4, $x'=x$, $y'=Ry$, where R is the radius of this circumscribed circle. Thus we have transformed the original vertices to points lying on the unit circle, $y=x^2$.

To study the triangle, we consider then the three points $A_i(t_i, t_i^2)$, $(i=1, 2, 3)$. Since we are limiting ourselves to the study of general triangles, we assume throughout this paper that

$$(t_2^2 - t_3^2)(t_3^2 - t_1^2)(t_1^2 - t_2^2) \neq 0. \tag{3}$$

The famous points of the triangle have rational coördinates, and thus are expressible in the symmetric functions of t,

$$\begin{aligned} s_1 &= t_1 + t_2 + t_3, \\ s_2 &= t_2t_3 + t_3t_1 + t_1t_2, \qquad s^2 = s_1^2 - 2s_2 = t_1^2 + t_2^2 + t_3^2. \\ s_3 &= t_1t_2t_3, \end{aligned}$$

A slight computation yields the following results: $O(0, 0)$, the circumcenter, the center of the circumscribed circle; $I(-s_1, -s_2)$, the incenter, the center of the circle tangent to the three sides of the triangle; $M(s_1/3, s^2/3)$, the median point, the intersection of the medians of the triangle; $F(\frac{1}{2}s_1, \frac{1}{2}s^2)$, the Feuerbach point, the center of the nine-point circle (the circle through the midpoints of the sides of the triangle).

The sides of the triangle are given by $y=(s_1-t_i)x-s_3/t_i$, $(i=1, 2, 3)$. The Euler line OMF is given by $s^2x=s_1y$.

The three famous circles of the triangle are given by the following equations: $x^2=y$, the circumcircle; $4(y+s_2)=(x+s_1)^2$, the inscribed circle; $(2x-s_1)^2 = -2y+s^2$, the nine-point circle.

All of Kasner's and De Cicco's theorems concerning the above may now be easily verified.†

* Kasner, Conformal geometry, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, vol. 2, 1912.

† John De Cicco, Analog of nine-point circle in the Kasner plane, this MONTHLY, vol. 46, 1939, pp. 627–634.

2. Perpendicularity. Consider a point $Q(x_0, y_0)$ and a general line L, $y=px+r$. Let a variable line through Q have slope P. Then the distance from Q to L along this line is

$$D(P) = \frac{y_0 - px_0 - r}{P(p - P)}. \tag{4}$$

By choosing P close enough to either 0 or p, $D(P)$ can be made as small as we please. Thus there is no shortest distance from a point to a line. Hence, the usual method of defining perpendicularity fails.

Kasner has done the very next best thing; he has defined the one relative minimum of $D(P)$, $P=\frac{1}{2}p$, as the slope of the perpendicular from the point Q to the line L. Perpendicularity is not a symmetric relation. The line L_1 is perpendicular to L_2 if $\theta_{12}=\frac{1}{2}$. Line L_2 is then said to be *anti-perpendicular* to L_1. We see that $\theta_{21}=2$.*

We consider now a third type of perpendicularity based on symmetry.

DEFINITION. *The non-parallel lines L_i, $(i=1, 2)$, are said to be quasi-perpendicular to each other if $\theta_{12}=\theta_{21}$.*

Since $\theta_{12}=p_2/p_1$, where p_i is the slope of L_i, two lines are quasi-perpendicular if and only if their slopes are negatives of each other.

THEOREM 1. *The quasi-perpendicular distance from a point to a line is equal to the anti-perpendicular distance, and to one-eighth of the perpendicular distance from the point to the line.*

From now on we deal only with quasi-perpendicularity.

THEOREM 2. *The three altitudes of a triangle meet in a point H, called the orthocenter of the given triangle. Its coördinates are $(-s_1, s_1^2)$.*

THEOREM 3. *The quasi-perpendicular bisectors of the sides of a triangle meet in a point Q, called the perpencenter of the given triangle. Its coördinates are $(s_1, -s_2)$.*

THEOREM 4. *The orthocenter of a triangle lies on the circumcircle, while its perpencenter lies on the nine-point circle.*

THEOREM 5. *The angles formed by the altitudes of a triangle at its orthocenter are equal, respectively, to the angles of the given triangle.*

3. Angle bisectors. DEFINITION. *A line L_3 is called a bisector of the angle θ_{12} between lines L_1 and L_2 if $\theta_{13}=\theta_{32}$.*

If p_i is the slope of L_i, $(i=1, 2, 3)$, then the condition is $p_3/p_1=p_2/p_3$ or $p_3^2=p_1p_2$. Clearly, any angle has two bisectors. Furthermore, they are quasi-perpendicular to each other.

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* Edward Kasner, Fundamental theorems of trihornometry, Science, vol. 88, pp. 480–483, Theorem 17.

Let $q_i = \sqrt{s_1 - t_i}$, taking the sign of each q_i arbitrarily, but fixed, $(i = 1, 2, 3)$.

DEFINITION. *The internal bisectors of the angles of a triangle are the lines through the respective vertices whose slopes are* $-q_2q_3$, $-q_3q_1$, $-q_1q_2$, *respectively.*

DEFINITION. *The external bisectors of the angles of a triangle are the lines through the respective vertices whose slopes are* q_2q_3, q_3q_1, q_1q_2, *respectively.*

THEOREM 6. *The three interior angle bisectors of a given triangle are concurrent at a point* J, *called the bisector point of the given triangle.*

THEOREM 7. *Any two exterior angle bisectors and the third interior angle bisector are concurrent at points* J_i, $(i = 1, 2, 3)$, *called the exbisector points of the given triangle. Furthermore, the interior and exterior bisectors through any vertex of the triangle are quasi-perpendicular.*

The coördinates of the bisector points are irrational in t_i; they could, however, be expressed rationally in terms of M_i, $(i = 1, 2, 3)$. We have

$$
\begin{array}{lll}
q_i = \sqrt{s_1 - t_i} \text{ (sign fixed arbitrarily)}, & t_1 = \frac{1}{2}[M^2 - 2q_i^2], & \\
M_1 = q_1 + q_2 + q_3, & M^2 = q_1^2 + q_2^2 + q_3^2, & \\
M_2 = q_2q_3 + q_3q_1 + q_1q_2, & p_i = M_1 - 2q_i, & (i = 1, 2, 3). \\
M_3 = q_1q_2q_3, & &
\end{array}
$$

With the above notation the coördinates of the bisector points may be computed, with the following results: $J(s_1 + M_2, -s_2 - M_1M_3)$, the bisector point of the triangle; $J_i(\frac{1}{2}p_i^2, -s_2 + M_3p_i)$, the exbisector points of the triangle.

There is no geometrical distinction between internal and external bisectors, or between the point J and the points J_i, for they depend on the signs of q_i, $(i = 1, 2, 3)$, which were chosen arbitrarily. Thus, all the bisector points have the same geometrical properties. This is not surprising, since angle is an ordered pair of lines, not rays, in the Kasner plane. There is no exterior angle of a given angle. Two lines form only one angle in a given direction.

If the vertices of the given triangle are real, we have two cases to consider: (1) the slopes of the sides all have the same sign, the bisectors at all vertices are real, and all bisector points are real; or (2) there are two slopes with one sign and one with the other, the bisectors at one vertex are real and at the other two imaginary, and all bisector points are imaginary.

Since the four bisector points have the same geometric properties, any circle through three of them contains the fourth. This circle is called the bisector circle of the triangle, and its center C is called the center point of the triangle.

THEOREM 8. *The equation of the bisector circle of a given triangle is rational in* t_i, *and is given by* $x^2 - 2s_1x + 2y + s_2 = 0$. *Its radius is* -2 *times the radius of the circumcircle, and its center* C, *is given by* $[s_1, \frac{1}{2}(s_1^2 - s_2)]$.

4. Miscellaneous theorems. DEFINITION. *The parallel triangle of a given triangle is the triangle whose sides are the parallels to sides of the given triangle drawn from the opposite vertices. Its circumcircle is called the parallel circle of the triangle.*

THEOREM 9. *Both a given triangle and its parallel triangle have the same medians and median point.*

THEOREM 10. *The parallel circle of a given triangle is given by* $x^2-2s_1x+2y-s^2-s_2=0$. *Its center is given by* $P(s_1, s^2)$, *and its radius is equal to that of the bisector circle* (-2 times that of the circumcircle).

THEOREM 11. *The parallel circle and the bisector circle of a given triangle are congruent, and have no points in common. Their radii are equal, and their centers lie on a zero line,* $x=s_1$.

DEFINITION. *The tangent triangle of a given triangle is the triangle whose sides are the tangents to the circumcircle at the vertices of the given triangle. The tangent circle of a triangle is the circumcircle of its tangent triangle. Its center* T' *is called the tangent point of the given triangle.*

THEOREM 12. *The tangent circle of a given triangle is given by* $4x^2-2s_1x-y+s_2=0$. *The tangent point is given by* $T'(s_1/4,\ s_2-s_1^2/4)$. *The radius of the tangent circle is one-fourth of the radius of the circumcircle.*

DEFINITION. *A coaxal system of circles is the set of all circles passing through two fixed points, real or imaginary.*

THEOREM 13. *The bisector circle, nine-point circle, tangent circle, and circumcircle are members of the same coaxal system of circles. Furthermore, if the vertices of the triangle are real, the fixed points of the coaxal system are real* (for then $s^2>s_2$).

There are quite a few pairs of points collinear with M, the median point, whose line segment is trisected at M. De Cicco has shown that the medians of a triangle are trisected at M, and that on the Euler line, the line segment joining the circumcenter and the Feuerbach point is trisected at M.*

THEOREM 14. *The parallel point and the circumcenter of a triangle are collinear with the median point on the Euler line, and their line segment is trisected at the median point.*

THEOREM 15. *The orthocenter and perpencenter of a triangle are collinear with the median point, and their line segment is trisected by it.*

THEOREM 16. *The incenter and center point of a triangle are collinear with the median point, and their line segment is trisected by it.*

Finally, some points lie on zero or infinite lines.

THEOREM 17. *The incenter and the orthocenter of a triangle lie on a zero line,* $x=-s_1$. *The perpencenter and the parallel point of a triangle lie on a zero line,* $x=s_1$. *The median point and the common point of the inscribed and nine-point circles lie on a zero line,* $x=s_1/3$. *Finally, the perpencenter and the incenter lie on the infinite line,* $y=-s_2$.

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* De Cicco, *loc. cit.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS OF FOURTH ORDER

R. A. FERTIG, Burlingame High School

Given:

$$a_i x + b_i y + c_i z + d_i w = e_i, \qquad (i = 1, 2, 3, 4).$$

Solution:

1. Graph

$$a_i x + b_i y = e_i, \qquad a_i x + d_i y = e_i, \qquad c_i x + b_i y = e_i,$$

and designate these lines by α_i, β_i, and γ_i, respectively.

2. Let A_{ij} be the intersection of α_i and α_j, B_{ij} of β_i and β_j, and C_{ij} of γ_i and γ_j. Actually we need only three sets of these involving all four subscripts, such as 12, 23, and 34, but others can be used as checks.

3. Let V_{ij} be the projection of B_{ij} vertically onto the x-axis, and H_{ij} the projection of C_{ij} horizontally onto the y-axis.

4. Let D_{ijk} be the intersection of the lines $A_{ij}V_{ij}$ and $A_{jk}V_{jk}$, and E_{ijk} the intersection of the lines $A_{ij}H_{ij}$ and $A_{jk}H_{jk}$.

5. Then all the lines $D_{ijk}E_{ijk}$ pass through the point (x, y), and z and w can easily be obtained.

Note by the Editor. Fertig's solution can be checked algebraically, but it is also interesting to consider its geometric significance. Each of the equations $a_ix+b_iy+c_iz+d_iw=e_i$ determines a hyperplane P_i in four-dimensional space with coördinates x, y, z, w. Here α_i is the intersection of P_i with the xy-plane; β_i is its intersection with the xw-plane if we replace the y in the equation by a w; and with a similar interpretation γ_i is its intersection with the yz-plane. Therefore A_{ij} is the intersection of the xy-plane with the plane of intersection of P_i and P_j, and similarly for B_{ij} and C_{ij}. Now V_{ij} is the projection on the xy-plane of B_{ij} (regarded as a point of the xw-plane), and H_{ij} is a similar projection of C_{ij}. We then see that the line $A_{ij}V_{ij}$ is the projection on the xy-plane of the line in which the plane of intersection of P_i and P_j intersects the xyw-hyperplane, and from this it follows that D_{ijk} is the projection on the xy-plane of the point of intersection of the xyw-hyperplane and the line common to P_i, P_j, and P_k. Similarly, E_{ijk} is the projection of the point of intersection of this line with the xyz-hyperplane, and so $D_{ijk}E_{ijk}$ is the projection of the line itself and hence passes through (x, y).

The application of a similar process to the case of three equations in three unknowns leads to the standard method used in descriptive geometry for constructing the intersection of three planes.

BIBLIOGRAPHY ON CYCLIC PROPERTIES OF MIQUEL POLYGONS

V. THÉBAULT, Tennie, Sarthe, France

1. A. Boutin was undoubtedly the first person to consider the polygon (P^n), the nth polar polygon of a point M with respect to a polygon (P) of n sides. He proved the following theorem (*Jour. de Math. élém.*, (4) I, p. 95):

THEOREM 1. *Polygon (P^n) is similar to (P).*

This result was extended recently by J. Dieudonné (*Mathesis*, 1935, p. 215, and 1937, p. 381):

THEOREM 2. *Polygon (P^n) is obtainable from (P) by a similitude from M.*

Finally, we have added (*Annales de la Société scientifique de Bruxelles*, vol. 59, sérié I, p. 347):

THEOREM 3. *The ratio of similitude of the polygons (P) and (P^n) is equal to the ratio of the product of the distances of M from the vertices of (P) to the product of its distances from the sides of (P).*

2. For the case in which (P) is a triangle, several other results can be cited: Bernès (*Jour. de Math. élém.*, (3) II, p. 109); A. Boutin (*loc. cit.*); R. Goormaghtigh (*Mathesis*, 1934, p. 435); V. Thébault (*loc. cit.*, pp. 347–354).

3. These remarks are not meant to detract from the credit due to B. M. Stewart, who considered the general polygon (P^n) isopolar of angle θ_1 with respect to (P). The bibliographical information above complements that given by Stewart at the end of his interesting article (this MONTHLY, vol. 47, 1940, p. 464).

ON THE SOLUTIONS OF A CERTAIN CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

J. D. MANCILL, University of Alabama

We shall obtain the most general solution of the partial differential equation

$$\left(\sum_{i=1}^{n} x_i\partial/\partial x_i\right)^k F(x) = mF(x), \tag{1}$$

where k is a positive integer and m is any constant. We have written $F(x)$ for $F(x_1, \cdots, x_n)$, and throughout this paper all necessary continuity and differentiablity properties of $F(x)$ will be tacitly assumed.

We shall define the function

$$V(\lambda) = F(u), \qquad u_i = \lambda x_i,$$

and impose the condition that it satisfy equation (1). By successive differentiation with respect to λ, we find that

$$V^{(k)} = \left(\sum_{i=1}^{n} x_i\partial/\partial u_i\right)^k F(u).$$

On multiplying both members of this equation by λ^k, the right member becomes the left member of equation (1) for the function $F(u)$. Therefore, by imposing the condition that the function $F(u)$ satisfy the equation (1), we find that $V(\lambda)$ must satisfy the differential equation

$$\lambda^k V^{(k)} - mV = 0. \tag{2}$$

In order to obtain the most general solution of equation (2), we shall let

$$V = \lambda^r$$

and determine those values of r for which this is a solution of equation (2). This gives the condition that r must satisfy the equation

$$r(r-1)(r-2)\cdots(r-k+1) - m = 0. \tag{3}$$

We shall denote the distinct roots of this equation by r_j, ($j=1, 2, \cdots, p$). The solutions of (2) are known to be linear combinations of the k solutions

$$\lambda^{r_j}(\log \lambda)^s, \qquad (s = 0, 1, \cdots, t_j - 1),$$

where r_j is a t_j-fold root of (3). The most general solution of (2) takes the form

$$\sum_{j,s} G_{j,s}(x)\lambda^{r_j}(\log \lambda)^s. \tag{4}$$

However, our problem requires that the function $V(\lambda)$ defined in (4) shall be identically equal to a function $F(u)$ of the variables $u_i=\lambda x_i$. This requires that the replacement in (4) of λ by $z\lambda$ and x_i by x_i/z must leave this function unchanged. That is,

$$\sum_{j,s} \left[G_{j,s}(x/z)z^{r_j}\lambda^{r_j}(\log \lambda + \log z)^s - G_{j,s}(x)\lambda^{r_j}(\log \lambda)^s\right] = 0.$$

Since this is an identity in λ we must have

$$\sum_{s} \left[G_{j,s}(x/z)z^{r_j}(\log \lambda + \log z)^s - G_{j,s}(x)(\log \lambda)^s\right] = 0$$

for each j. And since it is an identity in z, it follows that the terms involving $\log z$ cannot appear. The condition therefore reduces to

$$G_{j,0}(x/z)z^{r_j} = G_{j,0}(x); \qquad G_{j,s}(x) = 0, \qquad s > 0;$$

that is, $G_{j,0}(x)$ is homogeneous of degree r_j. On setting $\lambda=1$, we see that the *most general solution of the differential equation* (1) *has the form*

$$F(x) = \sum_{j=1}^{p} F_j(x),$$

where the functions $F_j(x)$ are homogeneous of degree r_j but are otherwise arbitrary, the r_j being the distinct roots of equation (3).

ON THE REPRESENTATION OF NUMBERS AS SUMS OF FIGURATE NUMBERS*

W. GOLUBEW, Kuvshinowo, U.S.S.R.

In 1772 J. A. Euler† stated that at least $a+2n-2$ terms are necessary to express every number as a sum of figurate numbers

$$1,\ n+a,\ (n+1)(n+2a)/1\cdot 2,\ (n+1)(n+2)(n+3a)/1\cdot 2\cdot 3,\ \cdots . \tag{1}$$

About the same time Beguelin‡ arrived at the erroneous conclusion that $a+2n-2$ summands actually suffice. The smallest example of the failure of this theorem is afforded by the number 64 which is the sum of no fewer than 8 numbers selected from the sequence

$$1,\ 5,\ 15,\ 35,\ 70,\ \cdots,\ \binom{4+k}{k},\ \cdots$$

in which $a=1$, and $n=4$. In fact, the representation

$$64 = 4\cdot 1 + 2\cdot 5 + 1\cdot 15 + 1\cdot 35$$

uses the least possible number of summands.

It is the purpose of this note to point out that in general for all n, and $a=1$ there exist numbers

$$\begin{aligned} N_n = n &+ \left[\frac{n}{2}\right]\binom{n+1}{1} + \left[\frac{n}{3}\right]\binom{n+2}{2} + \cdots \\ &+ \left[\frac{n}{n-1}\right]\binom{2n-2}{n-2} + \binom{2n-1}{n-1} \end{aligned} \tag{2}$$

which are sums of no fewer than $\sum_{k=1}^{n}[n/k]$ figurate numbers of the form

$$1,\ \binom{n+1}{1},\ \binom{n+2}{2},\ \binom{n+3}{3},\ \cdots . \tag{3}$$

It is obvious from the definition of N_n that $\sum_{k=1}^{n}[n/k]$ terms suffice to represent N_n. In order to prove that $\sum_{k=1}^{n}[n/k]$ is the least possible number of summands, it is sufficient to show that no partial sum in the representation (2) of N_n can be replaced by a larger figurate number of the form (3), so as to reduce the number of summands required. To do this we establish by induction the following inequality:

$$n + \left[\frac{n}{2}\right]\binom{n+1}{1} + \left[\frac{n}{3}\right]\binom{n+2}{2} + \cdots + \left[\frac{n}{k}\right]\binom{n+k-1}{k-1} < \binom{n+k}{k}$$

* Translated from the Russian by Emma Lehmer.

† Opera Postuma, vol. 1, pp. 203–204.

‡ Nouv. Mem. Acad. Sc. Berlin, 1772, 1774, p. 411.

for a fixed n and for all $k<n$. In the first place the inequality is obviously true for $k=1$, since $n<n+1$. If we now assume that it is true for any $k<n$, we can show that it is also true for $k+1$, for by assumption

$$\left\{n+\left[\frac{n}{2}\right]\binom{n+1}{1}+\cdots+\left[\frac{n}{k}\right]\binom{n+k-1}{k-1}\right\}+\left[\frac{n}{k+1}\right]\binom{n+k}{k}$$
$$<\binom{n+k}{k}+\left[\frac{n}{k+1}\right]\binom{n+k}{k}$$
$$<\binom{n+k}{k}\frac{n+k+1}{k+1}=\binom{n+k+1}{k+1}.$$

Hence at least $\sum_{k=1}^{n}[n/k]$ terms are necessary to represent the numbers N_n; but* $\sum_{k=1}^{n}[n/k]>n\log n$, and therefore exceeds $2n-1$ for all $n\geqq 4$. This disproves Beguelin's statement for all $n\geqq 4$, and $a=1$.

Translator's Note. A similar result can be shown to hold for any fixed a and for all n if we choose for N_n the numbers

$$N_n=n+a-1+\left[\frac{n}{2}\left(1+\frac{a-1}{n+a}\right)\right]\frac{n+a}{1}$$
$$+\left[\frac{n}{3}\left(1+\frac{a-1}{n+2a}\right)\right]\binom{n+1}{1}\frac{n+2a}{2}+\cdots$$
$$+\left[\frac{n}{n}\left(1+\frac{a-1}{n+(n-1)a}\right)\right]\binom{2n-2}{n-2}\frac{n+(n-1)a}{n-1}.$$

The numbers N_n are then the sums of no fewer than

$$\sum_{k=1}^{n}\left[\frac{n}{k}\left(1+\frac{a-1}{n+(k-1)a}\right)\right]$$

general figurate numbers (1). Hence, in general, for any fixed a the number of general figurate numbers of order n necessary to represent every number cannot be given by a linear function of n.

THE INTEGRAL OF POWERS OF THE SECANT AND COSECANT

W. W. Burton and W. G. Miller, Clemson College

The methods presented in this paper are applicable for any integral power of the secant and cosecant, although for even powers the standard procedure is probably shorter. The method given for integrating products of integral powers of the secant and cosecant applies only when both powers are even or both odd.

One of the chief values of these methods is that, in the teaching of elementary calculus, they permit the integration of odd powers of the secant and cosecant

* Transactions of the American Mathematical Society, vol. 46, 1939, p. 365.

to be discussed with other similar trigonometric integrals. Hitherto these integrals, in most texts, cannot be evaluated until integration by parts, by partial fractions, or by reduction formulas have been considered. Also, the results of the methods herein submitted are, in general, of simpler form, particularly for high powers and for the evaluation of a definite integral.

These formulas were presented to calculus classes at Clemson College and it was found that, with the aid of a Pascal's triangle for determining the coefficients of the binomial expansion, these troublesome forms reduced to an oral exercise.

I.
$$\begin{aligned}\int \csc^n x\,dx &= \int \frac{dx}{\sin^n x}\\ &= \int \frac{dx}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^n}\\ &= \frac{1}{2^{n-1}}\int \frac{dy}{\sin^n y\cos^n y}\\ &= 2^{-n+1}\int \tan^{-n} y\sec^{2n} y\,dy\\ &= 2^{-n+1}\int \tan^{-n} y(1+\tan^2 y)^{n-1}\sec^2 y\,dy,\end{aligned}$$

where $y=\frac{1}{2}x$.

II.
$$\int \sec^n x\,dx = -\int \csc^n z\,dz,$$

where $z=\frac{1}{2}\pi - x$. Hence

$$\int \sec^n x\,dx = -2^{-n+1}\int \tan^{-n} y(1+\tan^2 y)^{n-1}\sec^2 y\,dy,$$

where $y=\frac{1}{4}\pi-\frac{1}{2}x$.

III.
$$\begin{aligned}\int \sec^m x\csc^n x\,dx &= \int \tan^{-n} x\sec^{m+n} x\,dx\\ &= \int \tan^{-n} x(1+\tan^2 x)^{(m+n-2)/2}\sec^2 x\,dx.\end{aligned}$$

This formula is applicable when m and n are both even or both odd integers.

A comparison of the work involved in computing $\int \sec^5 x\,dx$ by using Formula II and by the usual process of integrating by parts or by partial fractions will show how much labor these formulas can save.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

A New System of Reckoning which turns at 8 instead of the usual turning at the number 10 whereby everything respecting coinage, weights, dimensions, and measures, can be reckoned many times more easily than in the ordinary way. By Emanuel Swedberg. (Translated from a photostat copy of the original Swedish ms., now preserved in the Royal Library, Stockholm, by Alfred Acton.) Philadelphia, Pa., Swedenborg Scientific Association, 1941. 34 pages. $0.60.

Elementary Logic. By W. V. O. Quine. Boston, Ginn and Company, 1941. 6+170 pages. $2.25.

Engineering Drawing. By D. E. Hobart. Boston, D. C. Heath and Company, 1941. 7+430 pages. $2.75.

The Elements of Statistics. By E. B. Mode. New York, Prentice-Hall, Inc., 1941. 16+378 pages. $3.50.

REVIEWS

A Survey of Methods of Apportionment in Congress. By E. V. Huntington. (Senate Document, No. 304.) Washington, D. C., Government Printing Office, 1940. 41 pages. $0.10.

This document contains a survey of all known methods of apportionment with emphasis upon such arguments as favor the author's method of equal proportions, which uses percentage errors as criteria of fairness. There are few scientific facts brought forth which are not contained in earlier papers by the author and others, to which fairly complete references are given. Numerous hypothetical examples are included to show the results of the various methods under the conditions assumed. Results of application to the 1940 census are included in an "Addendum" issued by the author as Appendix 2 to the original paper.

F. W. OWENS

Introduction to the Theory of Equations. By N. B. Conkwright. Boston, Ginn and Co., 1941. 8+214 pages. $2.00.

The book varies in topics discussed from the usual text in the theory of equations by the omission of constructions with ruler and compasses and the addition of a chapter on the Graeffe method.

It is to be regretted that in several of the more difficult proofs the author seems to feel that a vague, sketchy proof is preferable to a precise, careful one. For instance, in the proof of Descartes's rule of signs the author once refers to "the type of argument employed on" previous pages including an exercise, and

further on he hides part of his proof behind the maddening phrase "it is obvious." He defines a reciprocal equation in terms of its coefficients rather than its roots, and his argument that all reciprocal equations can be reduced to one of even degree and of the first class is obscure; a complete proof can be given in little more space and would be more easily understood. The proofs of certain properties of determinants are at times similarly sketchy.

The definitions and theorems are clearly and accurately stated, almost without exception. The only inaccuracy noted occurs in the statement of DeMoivre's theorem which is given for "the n-th power of any complex number," without specifying whether n can be anything but a positive integer; the chief clue is that the proof given holds only for positive integral values of n.

There is a good list of problems. The proof of Descartes's rule of signs has the makings of a considerable improvement over the usual one. The author's method of introduction of topics is uniformly good. And the format is excellent.

B. W. JONES

Elementary Functions and Applications. Revised edition. By A. S. Gale and C. W. Watkeys. New York, Henry Holt and Company, 1941. 21+409 pages. $2.25.

The first edition of this book appeared in 1920. It was reviewed in this MONTHLY, vol. 27, 1920, pp. 473–474.

The order of topics has been changed so as to unite parts of various topics distributed through the original text. There are new chapters for the treatment of statistical methods, measurement, and curve fitting. The derivative of each function is discussed as it is introduced rather than in a separate chapter on differentiation. New material includes a brief section on conics, one on effective rate of interest and discount, and a section on spherical trigonometry; the last includes but a minimum of technique to solve basic problems in astronomy and navigation.

VIRGIL SNYDER

Fundamental Mathematics. By D. C. Harkin. New York, Prentice-Hall, Inc., 1941. 15+434 pages. $3.00.

This book has been written as a text for use in the "cultural" and "orientation" courses offered by many colleges and universities. The author's intention is to offer the "principles of mathematics in outline without specializing in technical difficulties, and still get enough technique in the process to handle the simpler parts with some ease and confidence." Of all the various types of books in mathematics, this one resembles most nearly a unified text; but even there the resemblance is not great. The five divisions of the book have the following headings: (1) Dealing with counting; (2) Marks and their meanings; (3) Algebra; (4) Geometry; and (5) Wonder-working calculus. The approximate number of pages devoted to arithmetic and algebra, trigonometry, analytical geometry,

and calculus are 230, 15, 15, and 115, respectively. Many topics such as the following, which are not found ordinarily in a freshman text, are included in this one: calendars; finite geometries; group theory; arithmetic series of higher order; and Fourier series. An unusually large amount of historical background has been given. A large number of topics has been included; in some cases they have been explained elaborately, and in other cases the facts have been stated without proof. It is evident that the author has done an unusual amount of original work; teachers of mathematics will find the book to be an unusually interesting one for reference in case they do not adopt it as a text.

The type is good and the format is attractive. Several lists of problems have been included.

E. R. OTT

Elementary Mathematical Concepts. By J. H. Zant and A. H. Diamond. Minneapolis, Burgess Publishing Co., 1941. 125 pages. $1.50.

This spiral bound, photo-offset text contains material "designed for a freshman course for non-science students. The aim of the book is to present certain topics of mathematics from the historical and logical point of view." The first seventy pages, concerned with number notions and operations, fall into two parts. Chapters I and II attempt some discussion, largely through example, of the difficult problem of "the origin of number ideas," and more familiar aspects of number mysticism, combined with concrete historical material on number symbols and notations, their relative advantages and shortcomings. Chapters III, IV, and V present rational operations with numbers, rational fractions, and exponents; the motivation here can be indicated clearly by two quotations: "$2+2=4$ and $2+3=5$ are true by virtue of the *meanings we give to the symbols* 2, 3, 4, 5 and $+$." "It is desirable and convenient to keep the laws the same when we extend our system to include new symbols and number forms."

The next thirty pages, devoted to three chapters on Stocks and Bonds, Insurance, Compound Interest and Annuities, discuss the terms used, the meaning and need of insurance, *etc.*, with illustrative formulas in the simplest cases, culminating in the compound interest formula and tables, with applications. The authors reconcile the inclusion of this material in a treatment of mathematical concepts from the historical and logical point of view by some brief historical remarks on stocks and interest, and a remark on the close connection with exponents.

In the short chapter on Early History of Geometry, the paragraphs on Thales, Pythagoras, and Euclid include some exploratory statements about the nature of logical proof, in which "we first agree to accept certain statements without argument; these are called *assumptions.* · · · These assumptions, sometimes called *postulates*, usually agree with our experience, that is, they look like they ought to be true, but it is not necessary for them to be so." Apparently this chapter was to be followed by a chapter on non-euclidean geometry, referred to in the Euclid discussion and again in the summary. "The discovery of

non-euclidean geometry opened the way to modern postulational thinking by showing that it is not the *truth* of the postulates which is important, but the use of accepted postulates in a logical way that produces an acceptable argument." This reviewer, thoroughly in sympathy with a historical and logical approach, cannot help feeling that a principal aim in such an approach should be precisely a clarification of the use of such words as "true," in contradistinction to strictly mathematical vocabulary. In the chapters on number, a general statement such as "the *truth* of a statement depends on what you know in the beginning, or rather on what you accept to be a logical train of thought,"—rather dangerous for the untrained student—is later safeguarded by direct and definite restatement, following actual manipulation with symbols. The chapter on geometry needs just such safeguards, perhaps through actual manipulation of a small set of postulates, rather than by discussion of non-euclidean theories, where the terms have strong intuitive associations, and the full postulational basis cannot be laid.

The text ends with a short chapter on units of measurement. Most of the chapters include lists of references accessible to beginning students.

MARGUERITE LEHR

Tables of Natural Logarithms. Volume I. Logarithms of the Integers from 1 to 50,000. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York; A. N. Lowan, Technical Director. Conducted under the sponsorship of the National Bureau of Standards. New York, 1941. 18+501 pages. $2.00.

The present volume is the first of four on tables of natural logarithms. In the range and interval of the argument, that is, in the number of significant figures in the argument, and also in the number of decimal places of the logarithms—namely, 16—these tables supersede all other existing tables of natural logarithms. They are not intended to take the place of tables of common logarithms as a tool in computation, but rather to furnish values of natural logarithms where directly needed.

A feature of the tables is the extreme accuracy of the entries, all of which were subjected to various severe tests in manuscript. The actual computation was performed with the aid of the Tafula Wolframi, as given in the *Thesaurus Logarithmorum* by Vega, in 1794. These account for all integers to 2200, and primes to 10,009. For larger integers an infinite series is employed, the first term of which is sufficient to insure the correctness to 20 places of decimals. By independent computation, 13 errors in the Wolfram tables were detected and corrected.

Volume II of the present series is to contain the natural logarithms of integers from 50,000 to 100,000.

The present volume gives detailed instruction for direct and inverse interpolation to obtain logarithms of other numbers.

A table of constants includes $p \log 10$, $(p=1, 2, 3, \cdots, 10)$, the numbers e, π, and Euler's constant, together with their natural logarithms, all to 16 places of decimals.

The tables of natural logarithms are arranged in two columns of fifty entries each per page, in blocks of five, separated by double spacing. This provides a clear, open page that is not crowded. Each page was first typewritten, and then reproduced by the photographic process used in the earlier volumes.

VIRGIL SNYDER

Elementary Logic. By W. V. O. Quine. Boston, Ginn and Company, 1941. 6+170 pages. $2.25.

Here we have a book that is absolutely required reading for any teacher of a course in logic. Two out of the four chapters of the book should be required reading for any student of logic, particularly in a first course. Nevertheless, the reviewer cannot recommend the book as a text for a logic course. Before the end of the review, the reviewer will try to defend this peculiar attitude.

For purposes of the present book, Quine takes elementary logic as consisting of the propositional calculus and the theory of quantification. The first two chapters deal with the first topic, and the last two with the second. The discussion of each topic is separated into two parts (which take up, respectively, the two chapters allotted to each topic), namely a discussion of how to translate English sentences into the formal symbolism, and a discussion of the technique of manipulating the formal symbolism.

Quine's discussion of how to translate English sentences into the formal symbolism (Chapters 1 and 3) is superb. The first hurdle that any student of logic must clear is the tendency to let his habits of speech influence his habits of thought. Quine considers a really extensive set of English idioms, which he translates into the symbolism. Quine continually makes it clear that for most purposes of daily living, where brevity and picturesqueness of speech are preferable to precision, it would be foolish to abandon idioms for symbolism. Nevertheless, Quine is unsparing in his criticism of the lack of precision of the idioms, and at all times brings the contrasting precision of the symbolism into prominence. The word collections "pretty little girls' camp" and "No Third Term Rally," which Quine quotes as evidence of the necessity for parentheses, afforded the reviewer especial pleasure.

The symbolism which is used consists of the dot for "and," the curl for "not," and the $\exists x$ for "there exists." The preceding sentence is to be taken literally. Not once does Quine introduce an abbreviation into his symbolism. The familiar notations for "or," "implies," "is equivalent to," "for all x," *etc.*, never appear. One would think that without abbreviations, the technical development would become unbearably complex. It does not. Quine has done an incredible piece of work in manipulating the formalism without use of abbreviations or excessive complication, and has certainly made a contribution to our knowledge of the

axiomatics of formal logic. However, the reviewer feels that this innovation in technique is out of place in an elementary text. Because of the absence of "or" and the universal quantifier, such basic rules as the duality theorem (indispensable in indirect proofs) cannot even be hinted at. Presumably Quine would justify these omissions by saying that he intends his new technique to replace the old. However, the reviewer does not believe that the new technique will effect a notable abridgement in the labor of carrying out logical proofs, and the old technique is certainly much closer to non-symbolic modes of reasoning. For these reasons, we believe that the old technique will continue to be the standard one, and hence is the one that should be taught to beginners. However, the reviewer may be wrong in this. At any rate the book contains much material of great pedagogical value, and we shall end the review as we began it, by saying that every prospective teacher of logic should read Quine's book.

BARKLEY ROSSER

British Association Mathematical Tables. Volume VIII. Number-Divisor Tables. Designed and in part prepared by J. W. L. Glaisher. Cambridge, University Press; New York, Macmillan Company, 1940. 10+100 pages. $4.25.

The book contains four tables, as follows:

Table I gives, for each positive integer n not exceeding 10,000, (a) its canonical factorization into powers of primes (a dash indicating a prime value of n), (b) the number $\phi(n)$ of numbers not exceeding n and prime to n, often called the totient of n or Euler's ϕ-function, (c) the number $\nu(n)$ of divisors of n (including 1 and n), and (d) the sum $\sigma(n)$ of these divisors.

Table II gives, for each possible value of $\phi(n)$ not exceeding 2500, all those n for which $\phi(n)$ has this value, including the few cases for which $n>10{,}000$; that is to say, all solutions x of the equation $\phi(x)=m$, where $m\leqq 2500$, are given.

Table III gives, for each possible value of $\nu(n)$, (greatest value 64), all those n not exceeding 10,000 for which $\nu(n)$ has this value; that is, all solutions $x\leqq 10{,}000$ are given of the equation $\nu(x)=m$. The number of these solutions for each value of m is given in parentheses. Those solutions which are divisible by no square (except 1) are indicated in heavy type. Thus under $m=2$ is the list of primes up to 10,000. Entries in heavy type under $m=4$ are the products of two distinct primes, under $m=8$ the products of three distinct primes, and so on.

Table IV gives, for each possible value of $\sigma(n)$ not exceeding 10,000, all those n for which $\sigma(n)$ has this value; that is, all solutions x of the equation $\sigma(x)=m$, where $m\leqq 10{,}000$, are given.

In Table III the heavy type is so very nearly the same as the light type that they are very hard to distinguish. It is unfortunate that radically distinct fonts were not used for Table III.

BARKLEY ROSSER

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

PLANNING THE YEAR'S PROGRAM

How to have a better mathematics club during the coming year is the problem which often confronts the advisers and club officers soon after the opening of the college year. The *New York Zeta* chapter of *Pi Mu Epsilon* at *Columbia University* felt that an active group of members was needed. Therefore, the following letter was sent to a select group of students by Leon Henkin, chapter director:

"You have been selected for interest and ability in mathematics; your record thus far suggests the possibility of your membership in the Columbia chapter of Pi Mu Epsilon, National Honorary Mathematics Fraternity.

"The function of this society is to foster mathematical activity among the undergraduate student body. It accomplishes this largely through regular meetings, to which you are cordially invited. These meetings feature problem-solving and reading of prepared papers by fraternity members and others.

"The first meeting of the new term will be held on Wednesday, October 9, at 7:45 in Room 203 Hamilton Hall. We hope you will be able to attend. If you find this impossible, but would like to take part in our work in the future, please leave your name together with a list of your free evenings, with our faculty adviser, Professor Siceloff, in 206 Hamilton."

About 25 newcomers responded, 17 of whom were ultimately inducted into the society during the year because of their interest and participation in the work of the club.

Two clubs function at many colleges. One of these organizations is open to all students interested in the study of mathematics. The other is an honorary group, sometimes affiliated with one of the national organizations—*Pi Mu Epsilon* or *Kappa Mu Epsilon*. *Hunter College*, *Brooklyn College*, *St. Lawrence University*, *Montclair State Teachers College*, *Kansas State College at Manhattan*, and *Kansas State Teachers College of Emporia* are among the schools reporting an active year in two clubs.

At least eight program meetings a year are planned by most clubs. There are groups which meet as often as twenty times. This requires outlining the program and completing arrangements for the entire year within the first weeks of the fall semester. Printed cards listing the dates, speakers, and their topics are often distributed to all members and given a prominent place on all mathematics bulletin boards. Such cards have been used effectively by *The Shuttlesworth Mathematical Society* of the *University of Saskatchewan*, the *Mathematics Club* of *Brown University*, and *Delta X* of the *University of Kansas City*.

Topics for papers can be announced so as to attract a larger audience. A hasty review of club reports reveals the following: Dog did catch the rabbit, by Robert Morse of *Pi Mu Epsilon* at the *University of Arkansas*; The unnecessity of the ruler, by Ruth O'Donnell, and Pigs is pigs (a discussion of three standard puzzle problems of the Diophantine type), by Everett Yowell, both of the *Mathematics Club* at the *University of Cincinnati*; The spectroheliokinematograph, by Dr. J. Y. Stephens of *Pi Mu Epsilon* at *Washington University*; Mathematical tit-tat-toe, by David Dickinson of *Pi Delta Theta* at the *University of Denver*; and Solitaire on a checkerboard, by B. M. Stewart of *Pi Mu Epsilon* at *Michigan State College.*

Demonstrations of mechanical instruments may be held at several meetings. The *University of Nebraska* chapter of *Pi Mu Epsilon* devoted most of its meetings to such discussions during the past year. Included were the Rigge curve-tracing machine, the harmonic analyzer, and O. C. Collins's explanation of his astronomical projector which he called "A poor man's planetarium." Stuart Wadsworth and John Boudiette of *Pi Mu Epsilon* at *St. Lawrence University* built a mechanical differentiator based on references found in Lipka's *Graphical and Mechanical Computation*, p. 255. Another excellent source, including an excellent bibliography on the subject, is the paper by D. R. Hartree on *The mechanical integration of differential equations* in the Mathematical Gazette, October, 1938.

Open meetings featuring popular mathematical subjects for the entire student body or for the general public serve a real purpose and are reported as highly successful by a number of clubs. Joint programs with clubs from neighboring schools bring together students of similar interests. Such a plan has been used by the various clubs in the area of Greater Boston. Seven groups have met twice during the year to discuss mathematical problems and share experiences. Michigan state colleges held an all-day meeting at Ypsilanti in April, with members of each club contributing a paper to the program.

Mathematical contests attract the interest annually of a large number of students. Competitions may be in the form of a contest between two classes in calculus, or a friendly challenge issued by one club to that at a neighboring college, or participation in the William Lowell Putnam Mathematical Competition held in the spring of each year. The splendid records of teams from *Brooklyn College*, one of the youngest colleges in America, can be attributed to the semi-annual contests sponsored for the past ten years by the department of mathematics, as well as to the participation in the annual *Pi Mu Epsilon Intercollegiate Mathematical Contest* in Metropolitan New York. Certain clubs encourage their members to consider selected problems in the Problems and Solutions department of the MONTHLY as well as other mathematical publications. A list of 16 such problems was handed out to members of the *Case Mathematics Club* at the beginning of last year. Discussion of the solutions to these problems was used to fill in at meetings when waiting for the speaker, as well as at the close of the program.

Plans for a mathematical publication may be started early in the school year. Many high school mathematics clubs sponsor such publications, and a number of college students have had experience with such work while still attending the secondary schools. An interesting high school magazine, *Mathematics Student*, is published semi-annually by the *Brooklyn Technical High School*. Sample copies of recent issues may be obtained by addressing a request to Mr. Morris Cohen, chairman of the department of mathematics. A new publication to reach us this year is *The Mathematical Angle* which appeared in four issues and was prepared by the *Regis College Mathematics Club* of Weston, Massachusetts. Publications usually contain short student articles or reviews on interesting mathematical topics, challenging contest problems, reviews of recent mathematics books, and a record of the year's activities of the department and club.

Models of surfaces were made by members of *Pi Mu Epsilon* at *Ohio State University* with the coöperation of the department of ceramics. String models by Dr. Saul Pollock in California and by Dr. Robin Robinson of Dartmouth College have been used at many club meetings. The article on mathematical models in the 14th edition of the *Encyclopedia Britannica* is a good source, as are the four University of Illinois Bulletins on *Mathematical Models* by Professor Arnold Emch. A model for the class in integral calculus is discussed by Professor E. A. Whitman of Carnegie Institute of Technology in the MONTHLY for January, 1941.

Articles on popular mathematical subjects have appeared in the literary publications of some colleges. Margaret Kennedy and Bobbe Powers of *Kappa Mu Epsilon* at *Mount St. Scholastica College* of Atchison, Kansas, during the past year published such articles under the titles *Tens or twelves*, and *As a matter of calculation*. Some of the literary journals in the country have, during the last five or six years, published such popular articles as *All figured out*, by H. J. Fitzgerald, Harpers Magazine, October, 1936; and the following from the Atlantic Monthly: *Through a glass darkly*, by Stephen Leacock, July, 1936; *Revolving numbers*, by F. E. Andrews, February, 1935; *Genius and stupidity*, by E. T. Bell, March, 1937; and *Rules for making pi digestible*, July, 1935.

A mathematical movie was planned and filmed by a group at *Vassar College* Further information may be found in an article by Professor Grace M. Hopper in this department of the MONTHLY for October, 1940. Script writers who have an interest in mathematics will be glad to coöperate, and camera enthusiasts are always glad to be called on for technical advice and production work.

Broadcasting on a mathematical theme is another possibility. The Association of Teachers of Mathematics of New York City has sponsored weekly broadcasts for several years over radio station WNYC. A copy of a booklet including the material for twenty-two such broadcasts may be secured by sending 35 cents to Dr. Nathan Lazar, The Bronx High School of Science, New York City. The National Council of Teachers of Mathematics recently appointed a

radio committee of which Mr. A. Brown Miller of the Cleveland Schools is chairman. Your college mathematics department or club might sponsor a fifteen minute broadcast over the college or local radio station.

Three requests we should like to make of every club and its adviser.

1. Will you send us an outline of your plans for the year as early in November as possible.

2. Will you keep us informed, as soon as the information is available, of outstanding programs and speakers, dates for intercollegiate contests and meetings, radio programs, conventions, *etc.* In the case of broadcasts and outstanding meetings, it might be possible to call these to the attention of other interested persons through this department of the MONTHLY or by addressing a special mimeographed notice to those departments and clubs which sent us their annual report last spring, as well as to others requesting such information.

3. Will the secretary or faculty adviser of each organization keep a full and complete account of all activities for the entire year, including a summary of each paper given at meetings, as well as references used by each of the speakers in preparing his talk. Each spring we request such reports and many times replies are received stating that more accurate statements are not available because the adviser or secretary did not know that such facts would be of interest to others or to us.

P.S. There are still a few clubs whose reports for the year 1940–41 were not sent in before July 1. It will be possible to include such late statements in this department if mailed to us not later than November 30th.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 486. *Proposed by J. M. Andreas, Pasadena, California.*

A quadrilateral $ABCD$ has a right angle at A. The angles at B and C are bisected by the diagonals BD and CA. Is the quadrilateral necessarily a square?

E 487. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Prove that if the orthocenter of a triangle is conjugate to the three vertices with regard to the incircle and two of the excircles, respectively, then these three circles touch the respective sides of the orthic triangle, and conversely.

E 488. *Proposed by D. H. Browne, Buffalo, N. Y.*

The factorial $u_k = k!$, "sub-factorial" $v_k = k!\sum_0^k(-1)^r/r!$, and "super-factorial" $w_k = k!\sum_0^k 1/r!$, may be defined by the recurrence formulas

$$u_0 = v_0 = w_0 = 1,\ u_k = ku_{k-1},\ v_k = kv_{k-1} + (-1)^k,\ w_k = kw_{k-1} + 1.$$

Show that $\Delta^n w_0 = u_n$, $\Delta^n u_0 = v_n$.

E 489. *Proposed by Howard Eves, Lindy's Lake, N. J.*

Let A_0, A_m, A_h be the areas of the lower base, midsection, and upper base of a prismatoid. If $A_h = A_0$, prove that

(1) sections equidistant from the midsection are equal in area;
(2) the midsection bisects the volume of the prismatoid;
(3) if $A_m = A_0$, all sections have the same area;
(4) if $A_m \neq A_0$, A_m is the maximum or minimum section.

E 490. *Proposed by J. F. Kenney, University of Wisconsin at Milwaukee.*

In a gambling game, a player is permitted to deal ten cards from a bridge deck (which has been thoroughly shuffled) and wins if, at any stage of the dealing, the number on a card is the same as the number of cards dealt. (Face cards are assigned the number 0.) Find the probability that the dealer will win.

SOLUTIONS

E 446 [1940, 708]. *Proposed by L. S. Johnston, University of Detroit.*

In an exercise on the piano, n notes, $k_1, k_2, \cdots, k_n$, are played in the order $k_1, k_2, \cdots, k_{n-1}, k_n, k_{n-1}, \cdots, k_2, k_1, k_2, \cdots$, in measures of m. If k_1 is the first note of the first measure, in what measure will the jth note be k_i?

Solution by Nathan Newman, Bronx, N. Y.

Since the sequence of notes played is periodic, with period $2n-2$, and symmetrical within each period, k_i is either the $\{i+(2n-2)\lambda\}$th note or the $\{2n-i+(2n-2)\lambda\}$th note. But the jth note of the xth measure is the $\{j+m(x-1)\}$th note. Equating these expressions, we find that k_i will be the jth note of the xth measure if either

$$m(x-1) \equiv i-j \pmod{2n-2} \quad \text{or} \quad m(x-1) \equiv 2n-i-j \pmod{2n-2}.$$

Thus there is a solution only when the h.c.d. $(2n-2, m)$ divides $i-j$ or $2n-i-j$ or both; for example, there is no solution if $i-j$ is odd while m is even.

Also solved by the proposer, to whom the special case when $m=n$ and $i=j=1$ was suggested by a thirteen-year-old boy who had worked it out experimentally.

E 447 [1940, 708]. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find the locus of the center of a variable sphere which passes through a given point and touches two given planes.

Solution by R. K. Allen, Montpelier, Vt.

Restating the problem, we require the locus of a point which is equidistant from a given point and from two given planes. The locus of a point equidistant from the given point and either of the planes is a paraboloid of revolution having its focus at the given point and its axis perpendicular to the plane. Hence the desired locus is the ellipse in which this paraboloid meets the plane which bisects the angle between the two given planes.

Also solved by W. E. Buker, G. R. Kaelin, Nathan Newman, and the proposer.

E 448 [1940, 708]. *Proposed by Ruth Mason Ballard, Chicago.*

How many bridge hands are there with which it would be impossible to take a trick, no matter how the other cards are distributed or played, (a) if no trump were the bid, (b) if spades are trumps?

Solution by Nathan Newman, Bronx, N.Y.

In order that a hand may not take a trick, it must contain all the bottom cards, without a gap, in any suit that it has. Thus, if a hand contained five hearts, to be sure not to win a heart trick it must have the 2, 3, 4, 5, and 6 of hearts. Our problem thus resolves itself into finding the number of ways we can distribute 13 unnumbered cards among four suits in case (a), or among three suits in case (b). Since the hand in question may lead, we have to rule out the cases where it has a solid suit. Hence the required numbers are as follows:

$$\text{(a)} \qquad \binom{4+13-1}{13} - 4 = \binom{16}{3} - 4 = 556;$$

$$\text{(b)} \qquad \binom{3+13-1}{13} - 3 = \binom{15}{2} - 3 = 102.$$

Also solved by R. K. Allen, W. E. Buker, and (with a slip in calculation) by Jack Lorell.

E 449 [1940, 708]. *Proposed by Esther Szekeres, Shanghai, China.*

Given n positive numbers a_i, with

$$a_1 + a_2 + \cdots + a_n = 2n,\ a_i \geqq 1, \qquad (i = 1, 2, \cdots, n),$$

define $a_{n+j} = a_j$, $(j = 1, 2, \cdots)$ and

$$s_{i,\nu} = a_i + a_{i+1} + \cdots + a_{i+\nu}, \qquad (\nu = 0, 1, \cdots).$$

Prove that, for any $A \geqq 0$, there is an $s_{i,\nu}$ with $A < s_{i,\nu} \leqq A + 2$.

Solution by George Szekeres, Shanghai, China.

If $a_1 = \cdots = a_n = 2$, the statement is obvious. Suppose, then, that $a_i < 2$, and consider the greatest ν for which $s_{i,\nu-1} \leqq A$. If the theorem is not true, we

can assume that $s_{i,\nu}>A+2$, whence $a_{i+\nu}>2$. Thus, setting $i+\nu\equiv j \pmod n$ with $0<j\leqq n$, we have coördinated to every $a_i<2$ a unique $a_j>2$. Now consider a particular $a_j>2$, and let $a_{i_1}, \cdots, a_{i_k}$ be all the numbers a_i to which a_j is thus coördinated, arranged so that each of the k sums

$$a_{i_r}+a_{i_r+1}+\cdots+a_j, \qquad (r=1, 2, \cdots, k),$$

(save the last) includes the next. The last gives

$$a_{i_k}+a_{i_k+1}+\cdots+a_j>A+2,$$

and since $a_{i_k}<2$, we can assume that

$$a_{i_k+1}+\cdots+a_j>A+2$$

(since otherwise we should have $A<a_{i_k+1}+\cdots+a_j\leqq A+2$). But the first of the k sums gives

$$a_{i_1}+\cdots+a_{i_2}+\cdots+a_{i_k}+a_{i_k+1}+\cdots+a_{j-1}\leqq A.$$

Hence,

$$a_j>a_{i_1}+a_{i_2}+\cdots+a_{i_k}+2,$$

$$a_j+(a_{i_1}+a_{i_2}+\cdots+a_{i_k})>2(a_{i_1}+a_{i_2}+\cdots+a_{i_k}+1)\geqq 2(k+1).$$

For every $a_j\geqq 2$ we have such an inequality, possibly with $k=0$. Adding all these inequalities, we obtain $a_1+a_2+\cdots+a_n>2n$, contrary to the assumption.

E 450 [1940, 708]. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Evaluate

$$\lim_{n\to\infty}\left\{\frac{n^4}{e}\left(1+\frac{1}{n}\right)^n-n^4+\frac{n^3}{2}-\frac{11n^2}{24}+\frac{7n}{16}\right\}.$$

Solution by E. P. Starke, Rutgers University.

Consider the familiar expansion

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$+\frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)+\cdots,$$

which is valid for $n\geqq 1$. Collecting terms according to powers of $1/n$, we find

$$\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{\infty}\frac{1}{k!}-\frac{1}{n}\sum_{k=1}^{\infty}\frac{S(k,1)}{(k+1)!}+\frac{1}{n^2}\sum_{k=2}^{\infty}\frac{S(k,2)}{(k+1)!}$$
$$(1)\qquad -\frac{1}{n^3}\sum_{k=3}^{\infty}\frac{S(k,3)}{(k+1)!}+\cdots,$$

where $S(k, r)$ denotes the sum of all possible products formed from the first k positive integers taken r at a time. Using the evident relation

$$S(k, r) = S(k-1, r) + kS(k-1, r-1),$$

we easily verify the following formulas:

$$\begin{aligned} S(k, 1) &= \tfrac{1}{2}k(k+1), \\ S(k, 2) &= \tfrac{1}{8}(k-2)(k-1)k(k+1) + \tfrac{1}{3}(k-1)k(k+1), \\ S(k, 3) &= \tfrac{1}{48}(k-4)\cdots(k+1) + \tfrac{1}{6}(k-3)\cdots(k+1) \\ &\quad + \tfrac{1}{4}(k-2)\cdots(k+1), \\ S(k, 4) &= \tfrac{1}{384}(k-6)\cdots(k+1) + \tfrac{1}{24}(k-5)\cdots(k+1) \\ &\quad + \tfrac{13}{72}(k-4)\cdots(k+1) + \tfrac{1}{5}(k-3)\cdots(k+1). \end{aligned}$$

Substituting these values in (1), and replacing $\sum_{j=0}^{\infty} 1/j!$ by e wherever it occurs, we have

$$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - \frac{7e}{16n^3} + \frac{2447e}{5760n^4} - \cdots.$$

Thus the required limit is 2447/5760.

E 452 [1941, 65]. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find a multiple of 7 whose square has eight digits of the form $ababbbcc$.

Solution by E. P. Starke, Rutgers University.

From the doubled final digit, c must be 0 or 4. Since 49 is a divisor of $ababbbcc$, we may write

$$11c + 1011100b + 10100000a \equiv 0 \pmod{49},$$

$$60c - 15b + 120a \equiv 0 \pmod{49},$$

or

$$(1) \qquad b \equiv 4c + 8a \pmod{49}.$$

For $c=0$ (and $0 \leqq a, b \leqq 9$), this gives $a=1$, $b=8$, or else $a=b=7$; but no square can end in 800 or 700. For $c=4$, (1) has the single solution $a=5$, $b=7$, giving

$$57577744 = 7588^2.$$

Thus 7588 is the desired multiple of 7.

Also solved by D. H. Browne, W. E. Buker, L. R. Chase, M. L. Constable, William Douglas, Howard Eves, Evelyn Hesseltine, Helen T. Raudenbush, C. W. Trigg, Alan Wayne, and the proposer. Buker remarks that the restriction to a multiple of 7 is superfluous.

E 453 [1941, 65]. *Proposed by N. A. Court, University of Oklahoma.*

Given three skew lines a, b, c, for what positions of a point M will the har monic inverses of M with respect to the pairs b and c, c and a, a and b be coplanar

with M? (The harmonic inverse of M with respect to two skew lines is its harmonic conjugate with respect to the points in which the lines meet their transversal from M.)

Solution by Phyllis Barclay and Marian Wright, University of Oklahoma.

Let A, B, C be the points in which the required plane meets the lines a, b, c. From the conditions of the problem, M must be collinear with the pairs of points B and C, C and A, A and B. This is possible only when the points are collinear. M is any point on the regulus generated by lines meeting a, b, c.

Also solved by the proposer.

E 454 [1941, 65]. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A maze consists of stations A and B, n other stations, and just one direct passage between every pair of stations. Let N be the number of distinguishable routes from A to B, going through no station more than once; and let N' be the number of such routes which go through all the other stations on the way. What is the limiting value of the ratio N/N' as n increases without limit?

Solution by Howard Eves, Lindy's Lake, N. J.

The number of routes passing through k other stations between A and B is

$$n(n-1)(n-2)\cdots(n-k+1) = n!/(n-k)!;$$

for, on leaving A we have n choices for the first passage, then $n-1$ choices for the second, and so on for the k passages before finally passing to B. Accordingly, we have

$$N' = n! \quad \text{and} \quad N = \sum_{k=0}^{n} \frac{n!}{(n-k)!} = n!\sum_{r=0}^{n} \frac{1}{r!},$$

whence $N/N' = \sum_0^n 1/r!$. Thus we see that, as n increases, the ratio N/N' approaches e, the base for natural logarithms.

Also solved by W. E. Buker, L. R. Chase, and the proposer.

E 455 [1941, 65]. *Proposed by V. W. Graham, Dublin, Ireland.*

Given a fixed straight line l and a fixed point P outside it, consider two variable points Q and R on l, such that $\angle QPR$ is constant. Let S be the point in which l meets the bisector of this angle, and let C be the center of the circle PQR. Prove that CS passes through a fixed point.

Solution by D. H. Browne, Buffalo, N. Y.

Draw CM and PO perpendicular to l. Let CM meet the circle PQR in B, and let PO meet CS in D. Since PS bisects $\angle QPR$, while B bisects the arc QR, B lies on PS. Also $\angle QCB = \angle BCR = \angle QPR$, and since this is constant, so also is the ratio CM/MB. But, by similar triangles,

$$CM/MB = DO/OP.$$

Hence D is a fixed point.

Also solved by H. W. Bailey, W. E. Buker, W. B. Clarke, Howard Eves, L. M. Kelly, E. P. Starke, B. F. Yanney, G. A. Yanosik, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers, would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4006. *Proposed by P. D. Thomas, Norman, Okla.*

Find the equation of the family of curves, each curve having the property $LR=T^2$, where L is the distance of the tangent from the origin, R is the radius of curvature, and T is the length of the tangent from the point of contact to the y-axis.

4007. *Proposed by J. W. Clawson, Ursinus College.*

The straight lines l_i, $(i=1, 2, 3, 4)$, determine the complete quadrilateral Q. The four triangles determined by l_j, l_k, l_l have O_i, G_i for their circumcenters and centroids. The point P_i divides O_iG_i in the ratio $r:1$. Lines through these points parallel to l_i form the quadrilaterals Q_o, Q_g, Q_p. Prove that: (1) the four quadrilaterals are congruent; (2) the homothetic center of Q and Q_o is the orthic center of $O_1O_2O_3O_4$; (3) the homothetic center of Q and Q_g is the mean center of $G_1G_2G_3G_4$; (4) the homothetic center of Q and Q_p divides the line joining the preceding points in the ratio $r:1$; (5) these homothetic centers all lie on the common mid-diagonal line of the four quadrilaterals.

4008. *Proposed by V. Thébault, Le Mans, France.*

Given in a plane a polygon (P) of n sides having a center of symmetry S and two points M, M' symmetric with respect to S. Show that the nth pedal polygons of M and M' with respect to (P) are equal.

Note. Let Q be an arbitrary point in the plane of polygon $(A)\equiv A_1A_2\cdots A_n$. We say that the first pedal polygon of Q for the polygon (A) is the polygon $(B)\equiv B_1B_2\cdots B_n$ if the indicated consecutive vertices of (B) are the orthogonal projections of Q on the sides A_1A_2, A_2A_3, $\cdots$, A_nA_1 of (A); then the second pedal of Q for (A) is the first pedal of Q for (B), and so on.

SOLUTIONS

3950 [1940, 182]. *Proposed by P. Turán, Budapest, Hungary.*

Given

$$(1+z+z^2+\cdots+z^k)^n = c_0^{(k,n)} + c_1^{(k,n)}z + \cdots + c_{kn}^{(k,n)}z^{kn},$$

show that for kn odd

$$c_0^{(k,n)} \leqq c_1^{(k,n)} \leqq \cdots \leqq c_{(kn-1)/2}^{(k,n)} = c_{(kn+1)/2}^{(k,n)} \geqq c_{(kn+3)/2}^{(k,n)} \geqq \cdots \geqq c_{kn}^{(k,n)},$$

and for kn even

$$c_0^{(k,n)} \leqq c_1^{(k,n)} \leqq \cdots \leqq c_{kn/2}^{(k,n)} \geqq c_{(kn/2)+1}^{(k,n)} \geqq \cdots \geqq c_{kn}^{(k,n)}.$$

Solution by J. S. Frame, Brown University.

The inequalities may be proved by induction with respect to n. We let k be a fixed positive integer, and set

$$f_n(z) = (1+z+z^2+\cdots+z^k)^n = \sum_{p=0}^{kn} c_p^{(k,n)} z^p. \tag{1}$$

Then since we have

$$(1-z)f_{n+1}(z) = (1-z^{k+1})f_n(z), \tag{2}$$

we may equate coefficients of like powers of z and obtain

$$c_p^{(k,n+1)} - c_{p-1}^{(k,n+1)} = c_p^{(k,n)} - c_{p-k-1}^{(k,n)}, \tag{3}$$

provided we define $c_s^{(k,n)}=0$ for negative s. It is important to observe that since $f_n(1/z)=z^{-kn}f_n(z)$, the coefficients are symmetrical, and we have

$$c_p^{(k,n)} = c_{kn-p}^{(k,n)}. \tag{4}$$

We assume as our induction hypothesis that for some n we have

$$c_p^{(k,n)} - c_{p-1}^{(k,n)} \geqq 0 \quad \text{for} \quad p \leqq (kn+1)/2, \tag{5}$$

and note that this is true for $n=1$. By (4) and (5) we see that

$$c_p^{(k,n)} - c_{p-k-1}^{(k,n)} \geqq 0 \quad \begin{array}{l} \text{if} \quad p-k-1 \leqq p \leqq (kn+1)/2, \\ \text{or if} \quad p-k-1 \leqq kn-p \leqq (kn+1)/2. \end{array} \tag{6}$$

The second set of inequalities is equivalent to $(kn-1)/2 \leqq p \leqq [k(n+1)+1]/2$. Combining (3) and (6) we obtain the inequality (5) with n replaced by $n+1$, which completes the induction proof.

Solved also by the proposer.

Editorial Note. The solution by the proposer is similar to the above. He deduced the relation

$$c_p^{(k,n+1)} = \sum_{t=\max(0,p-k)}^{p} c_t^{(k,n)},$$

and used it for the case $0 \leqq p \leqq k$. For the remaining cases, he derived from this formula the formula (3) in the above solution.

It is easily found from the above formula or otherwise that $c_p^{(k,2)} = p+1$, $0 \leqq p \leqq k$; and then the induction proof shows that, for $n \geqq 2$,

$$c_p^{(k,n)} > c_{p-1}^{(k,n)}, \quad 1 \leqq p \leqq [kn/2]; \qquad c_{kn-p}^{(k,n)} = c_p^{(k,n)}.$$

3952 [1940, 244]. *Proposed by N. A. Court, University of Oklahoma.*

The spheres of a coaxal net which meet a given straight line in pairs of points in involution form a coaxal pencil.

Solution by the Proposer.

Let (N) be the given coaxal net of spheres and (P) the pencil conjugate to (N). If the sphere (V) of the net (N) meets the given line u in two points M, M' of an involution and O is the center of the involution, the sphere (O) having O for center and $r = OM \cdot OM'$ for the square of its radius, is orthogonal to (V). Now as (V) varies, r remains constant, hence the sphere (O) is fixed. Thus the variable sphere (V) is orthogonal to the spheres of the coaxal net determined by (O) and the coaxal pencil (P); consequently (V) describes a coaxal pencil conjugate to this net.

Editorial Note. If the center O of the involution of points on the given straight line u lies also on n, the radical axis of the coaxal net (N), then each sphere of (N) cuts u in a pair of points belonging to the involution. Hence the problem means that O does not lie on n. In this case each pair of spheres (V) of the net (N) which cut u in pairs of points of the involution have a fixed radical plane determined by n and O; hence the spheres (V) belong to a coaxal pencil of spheres.

3953 [1940, 244]. *Proposed by N. A. Court, University of Oklahoma.*

If L is a point on the radical axis r of a coaxal net of spheres, and q is the polar line of r with respect to a variable sphere (S) of the net, the locus of the circle of intersection of (S) with the plane Lq is a sphere.

I. *Solution by C. E. Springer, University of Oklahoma.*

Let the coaxal net of spheres be given by

$$x^2 + y^2 + z^2 + 2lx + 2my + d = 0, \tag{1}$$

where l and m are parameters. The line q is represented by $z = lx + my + d = 0$. The plane through q and the point $L(0, 0, 1/K)$ on r has the equation $lx + my - Kdz + d = 0$. On eliminating l and m between this and equation (1) we have the sphere $x^2 + y^2 + z^2 + 2Kdz - d = 0$.

II. *Solution by P. D. Thomas, Norman, Okla.*

The polar line q of r lies in the plane of centers of the coaxal net and it is perpendicular at R' to RS, where R is the intersection of r with the plane of centers and R' is the inverse of R with respect to (S). The polar plane of L is the plane qM through q perpendicular to LS and cutting r in M; the points L, M are conjugate with respect to (S) and also to each sphere of the net. Thus M is a fixed point for the net. The power T_m^2 of M with respect to (S) is the same for all spheres of the net; and hence the sphere (M), with center M and a radius whose square is T_m^2, and (S) have the radical plane Lq. The circle of intersection of Lq and (S) lies therefore on the fixed sphere (M); the center O of this circle is the intersection of SM with LR'.

The points L, M, and the midpoint of LM are centers of spheres orthogonal to (S). Hence these three spheres pass through the radical circle of the coaxal net and they belong to the conjugate coaxal pencil of the coaxal net.

Solved also by the proposer synthetically.

3954 [1940, 245]. *Proposed by Oystein Ore, Yale University.*

From three elements a, b, c in given order one can form two products, namely, $(ab)c$ and $a(bc)$ when the associative law is not assumed. Similarly four elements a, b, c, d give $N_4=5$ products; $[(ab)c]d$, $[a(bc)]d$, $a[(bc)d]$, $a[b(cd)]$, $(ab)(cd)$. Find the general expression for the number N_i of products with i factors.

I. *Solution by H. E. Vaughan, University of Illinois.*

If N_i is the total number of products which can be formed from i elements $a_1, a_2, \cdots, a_i$ in a given order, then the total number of products which can be formed is $i!N_i$. On removing from such a product the element a_i together with the pair of parentheses indicating the particular multiplication in which it was a factor, we obtain one of the $(i-1)!N_{i-1}$ possible products of the first $i-1$ elements. Hence, if we find the total number of distinct products obtained from the latter by inserting a new factor a_i and a pair of parentheses, we shall have an expression for $i!N_i$.

Each of the $(i-1)!N_{i-1}$ products contains $2i-2$ parentheses (we make the convention of not placing parentheses about the product as a whole). Hence the ith element can be introduced in $3i-4$ ways, after which the associated pair of parentheses may be introduced in two ways, making $2(3i-4)(i-1)!N_{i-1}$ products which, as we shall see, are not all different. However, two of these products can be alike only if (1) they arise from the same product of the first $i-1$ elements and (2) the element a_i is inserted just before an already present parenthesis in one case and just after it in the other. Since there are $2i-2$ parentheses present, there are $2(i-1)(i-1)!N_{i-1}$ such pairs of cases possible. We shall show that of the four products which can be derived from such a pair, only three are different.

There are two situations to be considered: 1. The parenthesis mentioned

above is the first of a pair. If a_i is inserted immediately after it, the result is of the form (a_iPQ), where P and Q are the factors of the original product which are bound together by the pair. On inserting parentheses we obtain either $(a_i(PQ))$ or $((a_iP)Q)$. If a_i is inserted immediately before the same parenthesis, so that a figure of the form $a_i(PQ)$ results, then insertion of the new pair of parentheses gives rise to $(a_i(PQ))$ or to one of the figures $(Ra_i)(PQ)$ or $a_i((PQ)R)$. 2. The parenthesis is the second of a pair. This case is transformed into the preceding by reading from right to left instead of from left to right.

Hence

$$\begin{aligned} i!N_i &= 2(3i-4)(i-1)!N_{i-1} - 2(i-1)(i-1)!N_{i-1} \\ &= 2(2i-3)(i-1)!N_{i-1}, \end{aligned}$$

or

$$N_i = \frac{2(2i-3)}{i} N_{i-1}.$$

Applying this successively we obtain, since $N_2 = 2(2\cdot 2-3)/2 = 2\cdot 1/2$,

$$\begin{aligned} N_i &= \frac{2(2i-3)2(2i-5)\cdots 2\cdot 3\cdot 2\cdot 1}{i(i-1)\cdots 3\cdot 2} \\ &= \frac{2^{i-1}(2i-3)(2i-5)\cdots 3\cdot 1}{i!} \\ &= \frac{2^{i-1}(2i-2)!}{2^{i-1}(i-1)!i!} = \frac{(2i-2)!}{(i-1)!i!} = \frac{1}{i-1}\binom{2i-2}{i-2}. \end{aligned}$$

II. *Solution by G. Szekeres, Shanghai, China.*

Since only products of two elements are defined, every product of the i elements is of the form $(a_1 \cdots a_k)(a_{k+1}\cdots a_i)$; and hence

$$N_i = N_1N_{i-1} + N_2N_{i-2} + \cdots + N_{i-1}N_1, \qquad N_1 = 1.$$

If we set formally $f(x) = N_1x + N_2x^2 + N_3x^3 + \cdots$, we have $[f(x)]^2 = f(x) - x$. This functional equation is satisfied by $\phi(x) = (1-\sqrt{1-4x})/2$, regular for $|x| < 1/4$, $\phi(0)=0$, $\phi'(0)=1$; whence $\phi(x) = f(x)$ for $|x| < 1/4$. Now $f'(x) = g(x) = (1-4x)^{-1/2}$, $(1-4x)g'(x) = 2g(x)$, $g^{(n+1)}(0) - 4ng^{(n)}(0) = 2g^{(n)}(0)$, $g(0) = 1$,

$$g^{(n+1)}(0) = \frac{(2n+2)(2n+1)}{n+1} g^{(n)}(0), \qquad g^{(n)}(0) = \frac{(2n)!}{n!}\cdot$$

Hence

$$N_i = \frac{(2i-2)!}{(i-1)!i!} = \binom{2i-2}{i-1} - \binom{2i-2}{i-2}.$$

Solved also by J. S. Frame, S. H. Lachenbruch, H. G. Landau, E. L. Post, E. P. Starke, and J. H. M. Wedderburn.

Editorial Note. The remaining solvers, with the exception of Starke, used the generating function as in solution II. Starke used the term couple for the product $(a_t a_{t+1})$; at least one couple occurs in each of the N_i products. The number of products in N_i each containing precisely g couples is denoted by M_{ig}, and he then derived the equations

$$\binom{i-g}{g}N_{i-g} = \sum_{u=g}^{q}\binom{u}{g}M_{iu}, \qquad 0 \leqq g \leqq q = [i/2], \qquad M_{i0} = 0.$$

Adding these equations with alternate change of sign, there results

$$\sum_{t=0}^{q}(-1)^t\binom{i-t}{t}N_{i-t} = 0.$$

Then by similar combinations of the latter and induction he derived the result

$$\sum_{t=0}^{q}(-1)^t\binom{i-h-t+1}{t}N_{i-t} = \binom{i+h-4}{h-2} - \binom{i+h-4}{h-4}.$$

Setting $h=i$ we obtain the desired result. These results are quite interesting, but their derivation is long and difficult.

Wedderburn remarked that this problem was proposed by P. Quarra, *Torino Atti*, vol. 53, 1918, pp. 634–637; and that an equivalent problem was proposed in this MONTHLY by P. Franklin and solved by C. F. Gummer, 2681 [1919, 127]. He also stated that when multiplication is commutative the problem is much harder and is probably not algebraic. See Wedderburn, *The functional equation* $g(x^2)=2ax+[g(x)]^2$, Annals of Math., vol. 24, 1922, pp. 121–140. See also, 2959 [1922, 129].

E. S. Pondiczery quoted the solution, using the generating function, in the above reference to Wedderburn. He stated that extensions of this problem are discussed in the article *On non-associative combinations*, by I. M. H. Etherington, Proc. Royal Soc. Edinburgh, vol. 59, 1939, pp. 153–162, and references to earlier solutions of this and equivalent problems, some more than a hundred years old, are given by this author and Wedderburn. After the appearance of the problem in print, the proposer referred to the above solution by Wedderburn. At the time of the preparation of this note, H. E. Vaughan discovered that 3674 [1935, 518] is related to this problem.

The second half of solution I may be replaced by the following. Each of the $(i-1)!\,N_{i-1}$ products contains $2(i-2)$ parentheses, that is, marks (and) , where we make the convention of placing parentheses about each product. The element a_i is introduced in two ways according as the introduced mark (is placed immediately before it or the introduced mark) is placed immediately after it. The pair $(a_i$ may be placed immediately before any one of the $i-1$ ele-

ments a_j; for this means that a_j is replaced by the new element (a_ia_j), where the enclosing pair of marks are those introduced. The pair $(a_i$ may be placed immediately before any one of the $i-2$ marks (already present, since the result is (a_iP), where P is a product. There are $i-2$ marks (and $i-1$ elements in each of the products considered, that is, $2i-3$ things. Hence, we obtain from each $2i-3$ distinct results. Similarly, there are $2i-3$ distinct ways of placing the pair $a_i)$. We thus obtain $2(2i-3)(i-1)!N_{i-1}$ distinct products; and, equating this to $i!N_i$, we have $N_i=2(2i-3)N_{i-1}/i=(2i-2)(2i-3)N_{i-1}/i(i-1)$. Hence, there results finally

$$N_i = \frac{(2i-2)!}{i!(i-1)!}\cdot$$

This leads to another way of formulating the problem. A given product of i elements in a given order may be indicated completely by the use of $i-1$ marks of the same kind, say) ; and we now consider the distribution of the $i-1$ marks) so that they determine a product in which each of the i elements enter as a factor in the original order. At the left the sequence of elements and marks must begin with at least two elements, and it must end with at least one mark. Denote such a sequence by $i_1j_1, i_2j_2, \cdots, i_kj_k$, where i_tj_t are a pair of positive integers denoting a sequence of i_t elements and j_t marks). We have a desired product of the i elements if

$$\begin{aligned} \sum_{t=1}^{u} (i_t - j_t) &\geqq 1, \qquad 1 \leqq u < k, \\ &= 1, \qquad u = k, \qquad \sum_{t=1}^{k} i_t = i. \end{aligned}$$

For, the j_1 marks replace j_1+1 of the i_1 elements preceding them by a single new element, and we must then have $i_1-(j_1+1)\geqq 0$. When we come to the second block there are $(i_1-j_1)+i_2$ elements, and we must again have $(i_1-j_1)+i_2-(j_2+1)\geqq 0$; and so on until the end as above. This property may be stated in a generalized form. We have n letters A and m letters B, $n-m\geqq 1$, which are arranged in a line so that, in passing along it from left to right, at each passage of a letter we have passed over more A's than B's. The inequality relations above for $n=m+1$ require a few easily supplied modifications for the generalization. We then have the problem of the determination of $N_{n,m}$, the number of ways of making such arrangements of the $n+m$ letters. Thus the present problem is equivalent to the one in 2681 proposed in 1918 by Franklin and solved by Gummer. This solution by Gummer required the solution of the generalization. We may regard this generalization as a generalization of the present problem where there are n elements in a given order and m operations of multiplication upon them, $n>m$, so that in the resulting products the elements have their original order, and there may be elements not used as factors. We shall give a determination of $N_{n,m}$ quite different from the one by Gummer.

Suppose that the last of the m letters B is after the jth A counting from the left (and before the $(j+1)$th A, if $j<n$), where $j\geqq m+1$. The number of different positions of the remaining $m-1$ letters B is then $N_{j,m-1}$; and hence

$$N_{n,m}=\sum_{j=m+1}^{n}N_{j,m-1}. \tag{1}$$

Thus $N_{n,1}=n-1$, as is easily seen independently, or with the use of (1) and setting $N_{j,0}=1$; and then $N_{n,2}=2+3+\cdots+(n-1)=(n+1)(n-2)/2=(n+1)n/2-(n+1)$. These results and further computations indicate that

$$N_{n,m}=\binom{n+m-1}{m}-\binom{n+m-1}{m-1}. \tag{2}$$

Suppose that this has been proved for $N_{j,m-1}$, $j\geqq m$. Then we must have

$$\begin{aligned}N_{n,m}&=\sum_{j=m+1}^{n}\left\{\binom{m+j-2}{m-1}-\binom{m+j-2}{m-2}\right\},\\&=\binom{n+m-1}{m}-\binom{2m-1}{m}-\left[\binom{n+m-1}{m-1}-\binom{2m-1}{m-1}\right],\\&=\binom{n+m-1}{m}-\binom{n+m-1}{m-1},\end{aligned}\tag{3}$$

where the summations easily follow from

$$\Delta_u\binom{u}{m}=\binom{u}{m-1},$$

and the proof is complete. We may also write

$$N_{n,m}=(n-m)\frac{(n+m-1)!}{n!m!}. \tag{4}$$

We have then two simple evaluations of $N_{n,n-1}$ and there is no need for the generating function.

It is interesting to consider the evaluation of $N_{n,m}$ by a method which is the reverse of the above and similar to Gummer's method. We may easily put (1) in the form

$$N_{n,m}=N_{n-1,m}+N_{n,m-1}, \tag{5}$$

and it is just as easy to derive (5) independently. This may be written formally

$$\begin{aligned}N_{n,m}&=(\overset{-1}{U_n}+\overset{-1}{U_m})N_{n,m}=(\overset{-1}{U_n}+\overset{-1}{U_m})^r N_{n,m}\\&=\sum_{j=0}^{r}\binom{r}{j}N_{n-j,m+j-r}.\end{aligned}\tag{6}$$

We now extend the meaning of $N_{n,m}$ so that $N_{m,n}=-N_{n,m}$ for n, m both positive integers or one of them zero. We define $N_{n,m}=0$ if a subscript is negative, and so on. Then set $r=n+m-1$, and all terms on the right of (6) drop out except for $j=n-1$, n; and we obtain

$$N_{n,m} = +\binom{n+m-1}{n-1} - \binom{n+m-1}{n}. \tag{7}$$

This is merely formal. In order to prove that we get the required formula we must show that each extension of the meaning of $N_{n,m}$ so that (5) is satisfied is necessarily the one chosen, first when a newly defined $N_{n,m}$ first enters the equation (5), and so on. In other words, we must show that there is one and only one way of extending the meaning so that (5) is always satisfied. For this we may use (6) with induction, but this appears to be a very tedious task. This method is not as easy as it may appear at first.

This last method is based upon the separation of the sequences which have the desired property, which we call P, into two classes according as the end letter is an A or a B. This suggests another related method which we now describe. The number of arrangements of the $n+m$ letters which have the property $-P$, that is, do not possess the property P, is easily seen to be ${}_{n+m}C_n-N_{n,m}$; and these fall into two classes (A), (B) according as the first letter at the left is an A or a B. It is obvious that any arrangement which begins with B must have the property $-P$, and the number of such is ${}_{n+m-1}C_n$. From the previous evaluation of $N_{n,m}$, we find by an easy computation that $\{A\}=\{B\}$, where $\{A\}$ denotes the number of elements in class (A). Hence, if we can discover a way of setting up a one-to-one correspondence between the elements of (A) and (B), we shall have an additional evaluation of $N_{n,m}$. It appears to be difficult to prove the existence of such a one-to-one correspondence. See Uspensky's *Introduction to Mathematical Probability*, pp. 151–153.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

An important new mathematical publication has been received, coming from the Universidad Nacional de Tucumán, República Argentina. It is entitled *Revista de Matemáticas y Física Teórica.* Volume 1, numbers 1 and 2, December, 1940, contains twenty-six articles in six languages. Four of the authors are living in North America. The editors are Dr. A. Terracini and Dr. F. Cernuschi.

THE NEW EDITOR-IN-CHIEF

As announced last March, Professor L. R. Ford was elected for a five-year term as Editor-in-Chief of the MONTHLY, beginning his services with the January, 1942, issue. Papers hereafter submitted for publication in the MONTHLY should be sent directly to him at the Illinois Institute of Technology, Chicago, Illinois.

DISTRIBUTION OF THE REPORT ON MATHEMATICAL EDUCATION FOR DEFENSE

W. L. HART, University of Minnesota

The report of the Subcommittee on Education for Service of the War Preparedness Committee, which was published recently in this MONTHLY (vol. 48, 1941, pp. 353–362), also has appeared in the *Mathematics Teacher*, and will appear soon in *School, Science and Mathematics*. Reprints of the report have been sent to the superintendents of schools in all cities of over 2500 population in the United States; the expense of this distribution was borne jointly by the American Mathematical Society, the Mathematical Association of America, and the National Council of Teachers of Mathematics. The publicity already provided for the report has placed it in the hands of the specified superintendents and all teachers of mathematics who belong to national organizations. In addition, it would be desirable for the report to reach as many as possible of the extremely large percentage of teachers of high school mathematics who do not belong to the National Council or to the Central Association of Science and Mathematics Teachers. Hence, reprints of the report are being offered for sale while the supply lasts, *only in units of 25 copies each*, at $1.25 per 25 copies, delivered to the purchaser. Orders for copies should be sent, with attached money order or check, to Professor J. R. Kline, American Mathematical Society, University of Pennsylvania, Philadelphia, Pa.

Members of the Association are urged to bring the possibility of purchase of reprints of the report to the attention of organized groups of teachers of secondary mathematics and superintendents of schools.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., May 3; Washington, Pa., October 25.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS, Manhattan, April 4–5.
KENTUCKY, Richmond, April 26; Lexington, October 25.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 10; Washington, D.C., December 6.
METROPOLITAN NEW YORK, Flushing, April 19.
MICHIGAN, Ann Arbor, March 15; Detroit, November 15.
MINNESOTA, St. Joseph, May 10.
MISSOURI, Columbia, April 18.
NEBRASKA, LINCOLN, May 3.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3.
OKLAHOMA, Tulsa, February 7.
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, Colorado Springs, April 18–19.
SOUTHEASTERN, Chapel Hill, N. C., March 28–29.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, April 4–5.
UPPER NEW YORK STATE, Ithaca, May 3.
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The Chauvenet Prize

The ASSOCIATION in 1925, with the aid of its president, Professor J. L. Coolidge, established a prize of one hundred dollars for a noteworthy expository paper published in English during successive periods of five years by a member of the ASSOCIATION. In 1928 Professor W. B. Ford, then president of the ASSOCIATION, gave five hundred dollars, the income from which was to supplement the original fund so that the prize could be awarded every three years. In 1936 an anonymous gift of one hundred dollars was added to this fund.

The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books. It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE should tend to stimulate such production.

Five awards have been made as follows:

1920-1924. G. A. Bliss, "Algebraic Functions and their Divisors," *Annals of Mathematics.*

1925-1928. T. H. Hildebrandt, "The Borel Theorem and its Generalizations," *Bulletin* of the American Mathematical Society.

1929-1931. G. H. Hardy, "An Introduction to the Theory of Numbers," *Bulletin* of the American Mathematical Society.

1932-1934. Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," *Annals of Mathematics;* "The Convergence of Fourier Series," *American Mathematical Monthly.*

1935-1937. G. T. Whyburn, "On the Structure of Continua," *Bulletin* of the American Mathematical Society.

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ZARISKI, Prof. OSCAR, Ph.D. (Rome) Johns Hopkins Univ., Baltimore, Md. *Dept. of Math.*
ZASLAVSKY, SAM, E.E. (C.C.N.Y.) Principal engineering draftsman, U.S. Navy Yard, Brooklyn, N.Y. *32 A Hampton Pl.*
ZELDIN, Asst. Prof. S. D., Ph.D. (Clark) Massachusetts Inst. of Tech., Cambridge, Mass. *Dept. of Math.*
ZIMMERMAN, B. C., A.M. (St. Louis Univ.) Punta Gorda, British Honduras

ZOCH, R. T., A.M. (George Washington) Asso. Meteorologist, U.S. Weather Bureau, Airport Station, Newark, N.J.
ZORN, Asso. Prof. M. A., Dr.res.mat. (Hamburg) Univ. of California at Los Angeles, Los Angeles, Calif.

INDIVIDUAL MEMBERS RECEIVED OCTOBER 15–31, 1941
(Not indexed geographically)

BRIGHT, H. F., A.B. (Lake Forest) Instr., San Angelo Coll., San Angelo, Tex. *623 Koberlin St.*
CHERRY, Prof. L. B., A.M. (Texas) Edinburg Jr. Coll., Edinburg, Tex. *418 N. Eighth St.*
COLEMAN, RUTH T., A.M. (Columbia) Instr., Bergen Jr. Coll., 1000 River Road, Teaneck, N.J.
COMER, LOUISE M., M.S. (Brown) Asst. Barnard Coll., Broadway and 119 St., New York, N.Y.
CONGER, L. H., Jr., A.M. (Harvard) Instr., Muskegon Jr. Coll., Muskegon, Mich.
CRANE, R. E., B.S. (Harvard) Engineer, Amer. Tel. and Tel. Co., 195 Broadway, New York, N.Y. *120 Early St., Morristown, N.J.*
FAY, E. A., A.M. (Harvard) Teaching asst., Univ. of Rochester, Rochester, N.Y. *Morey Hall, River Campus*
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GABRIELLE MARIE, Sister, M.S. (Catholic Univ.) Instr., Dunbarton Coll., 2435 Upton St. N.W., Washington, D. C.
GARRETT, E. T., A.B. (Cornell Coll.) War Dept., Air Corps Tech. School, Scott Field, Ill. *808 Union Ave., Belleville, Ill.*
GOOD, J. M., A.B. (Park Coll.) Grad. student, Brown Univ., Providence, R.I. *Dept. of Math.*
GOODE, C. J., A.B. (Fordham) Teacher, Brooklyn Prep. School, Brooklyn, N.Y. *119–34 193 St., St. Albans, N.Y.*
HUNTER, Asst. Prof. J. L., Ph.D. (Catholic Univ.) John Carroll Univ., Cleveland, Ohio
INNIS, MARY E., A.B. (Smith) Teaching fellow, Smith Coll., Northampton, Mass. *30 Belmont St.*
LADUE, MARY E., A.M. (Columbia) Lecturer, Barnard Coll., Broadway and 119 St., New York, N.Y.
MOELLER, Asso. Prof. E. L., B.S. (St. Ambrose) St. Ambrose Coll., Davenport, Iowa. *2131 Davie St.*
OLPIN, J. L., M.S. (Colorado) Instr., Chm. Sci. Comte., Gila Jr. Coll., Thatcher, Ariz.
PETERS, I. D., M.S. (West Virginia) Teaching fellow, West Virginia Univ., Morgantown, W.Va. *P.O. Box 554*
SCHMIDT, O. H., Cand.mag. (Copenhagen) Instr., Brown Univ., Providence, R.I.
SCHWARTZ, A. A., M.S. in Educ. (C.C.N.Y.) Senior Fiscal Acctg. Clerk, Railroad Retirement Board, New York, N.Y. *55 Parade Pl., Brooklyn, N.Y.*
SPICER, Asso. Prof. LUCY E., A.M. (Columbia) Registrar, Western State Coll. of Colorado, Gunnison, Colo. *Box 786*
THOMAS, Asso. Prof. P. D., A.M. (Oklahoma) Southeastern State Coll., Durant, Okla.
TOLLE, L. F., A.M. (St. John's Univ.) Chm. of Dept., Teachers Coll., St. John's Univ., Brooklyn, N.Y. *240–23 141 Ave., Rosedale, N.Y.*

RECAPITULATION OF MEMBERSHIP

Individual members October 14, 1941	2,253
Institutional members October 14, 1941	109
Total membership October 14, 1941	2,362
Total membership October 31,1941	2,385

GEOGRAPHICAL DISTRIBUTION OF INDIVIDUAL MEMBERS

UNITED STATES AND CANADA

ALABAMA. (26)

AUBURN. Doner, Fuller, Garrison, Martin, Sams.
BIRMINGHAM. Hess, Moore, Rayl.
DEMOPOLIS. Adams.
FLORENCE. Culmer.
MARION. Barnes.
MONTEVALLO. Jackson.
MONTGOMERY. Cox.
ST. BERNARD. Capesius.
UNIVERSITY. Clark, Cowan, Gaskell, Hummel, Kennedy, Lewis, Mancill, Ott, Scott, Seebeck, Wade, Warnock.

ARIZONA. (13)

FLAGSTAFF. Lampland, Mehlenbacher.
MIAMI. Harmon.
PHOENIX. Hannelly, Johnson, Riley.
TEMPE. Wexler.
TUCSON. Boldyreff, Graesser, Hohn, Leonard, Mewborn, Purcell.

ARKANSAS. (9)

ARKADELPHIA. Dorroh.
CONWAY. Lane.
FAYETTEVILLE. Adkisson, Comfort, Harding, Hosford, Kent.
MAGNOLIA. Wetzig.
MONTICELLO. Garrett.

CALIFORNIA. (107)

ALAMEDA. Sumner.
BAKERSFIELD. Cook.
BERKELEY. Bernstein, Evans, Lehmer, Levy, Noble, Putnam, Sperry, Williams, Wong, Yood.
BRAWLEY. Beckwith.
CLAREMONT. Hamilton, Jaeger, Sellew.
COLTON. Battin.
COMPTON. Gorman.
DAVIS, Baker, Roessler.
FRESNO. Morris.
FULLERTON. Ernsberger, Reynolds.
GLENDALE. Roberts.
HAYWARD. Rahn.
HOLLYWOOD. Campbell, Hand.
HUNTINGTON PARK. Niersbach.
LAGUNA BEACH. Moore.
LA JOLLA. McEwen.
LONG BEACH. Black.
LOS ANGELES. Ablow, Alexander, Ames, Bell, Buckman, Collier, Daus, Duncan, Gaver, Glazier, Hedrick, Hinds, Hoel, Hunt, James, Kaelin, McClellan, Mason, Orange, Perry, Puckett, Quilliam, Robb, Sherwood, Showman, Steed, Taylor, Thomas, Thompson, Trigg, Urner, White, Whyburn, Wicker, Worthington, Zorn.
MADERA. Fuller.
MARYSVILLE. Miller.
MENLO PARK. Phillips.
OAKLAND. Hesse.
PALO ALTO. Blichfeldt.
PASADENA. Bateman, Bell, Birchby, Glenn, Michal, Van Buskirk, Ward, Wear, White.
REDLANDS. Albert
RIVERSIDE. Wolfe.
SAN DIEGO. Klauber, Livingston, McConnell, C. E. Smith.
SAN FRANCISCO. Bass, Ivanoff, Koch, McCarty, Skarstedt, Sturges, Waider, Welch.
SANGER. Spearman.
SAN JOSE. Clarke, Heaslet, Myers.
SANTA ANA. Whiting.
SANTA MONICA. Adams.
STANFORD UNIVERSITY. Bacon, Szegö.
STOCKTON. Corbin.
TULARE. Maizlish.
WHITTIER. Mendenhall, Pyle.

CANADA. (39)

EDMONTON. Campbell, Cook, Sheldon.
HAMILTON. Findlay.
KINGSTON. Matheson, Miller.
LENNOXVILLE. Richardson.
LONDON. Cole, Kingston.
MONTREAL. Gauthier, Pall, Pelletier, Perry, Tate.
OTTAWA. Dubé.
QUEBEC. Tremblay, White.
REGINA. Oakland.
SACKVILLE. McEwen.
SASKATOON. Ferns, James, Pyke.
TORONTO. Beatty, Brauer, Coxeter, Dobson, Grant, Ling, Pounder, Robinson, Roseburgh, Synge.
VANCOUVER. Buchanan, Gage, Hull.
WINNIPEG. Warren, Wilson.
WOLFVILLE. Jeffery, Macphail.

CANAL ZONE

BALBOA HEIGHTS. Mc Nair.

COLORADO. (25)

BOULDER. Britton, DeLong, Hutchinson, Kempner, Kendall, McMaster, Nelson.
COLORADO SPRINGS. Lovitt, Sisam.
DENVER. Carmichael, Gorrell, Lewis, Recht, Sabin.
FORT COLLINS. Clark, Gunder, Macdonald.
GOLDEN. Everett, Fitterer, Hebel, Pawley, Wilson.
GREELEY. Mallory.

LORETTO. Cook.
PUEBLO. Hemenover.

CONNECTICUT. (42)

BLOOMFIELD. Rosenbaum.
BRANFORD. Gillette.
BRISTOL. Sabel.
HARTFORD. Bronstein, Dadourian, Donchian, Elston, Keffer, Kelly, Mitchell, Wyckoff.
LAKEVILLE. Northrop.
MIDDLETOWN. Arnold, Camp, Howland.
NEW BRITAIN. Ferry, Fuller, Weeber.
NEW HAVEN. Dunford, Engstrom, Hille, Kovarik, Leech, Longley, Miles, Ore, Tracey, Uhler, Whittemore, Wilson.
NEW LONDON. Bower, Dimick, Wilson.
PUTNAM. Alfred.
SIMSBURY. Howard.
STAMFORD. Kinsman.
STORRS. Cheney.
STRATFORD. Hall, Loring, Wood.
WEST HAVEN. Anderton.
WEST HARTFORD. Pease.

CUBA.

HAVANA. Corral y Alemán, Gonzàlez.

DELAWARE. (8)

DAGSBORO. Short.
LAUREL. Stauffer.
NEWARK. Harter, R. W. Jones, McDougle, Makarov, Rees, Wilson.

DISTRICT OF COLUMBIA. (62)

WASHINGTON. Adams, Avers, Barten, Bearman, Berry, Blake, Bertsch, Burington, Clark, Cromwell, Duerksen, Edmonston, Federico, Finan, Flaherty, Fleming, Goldberg, Greville, Hamilton, Johnston, Kaplan, Kennedy, Kullback, Lambert, Landry, Lloyd, McCamman, Maloney, Mears, Mertie, Miles, Moulton, O'Toole, Patterson, Ramler, Rasor, Reynolds, Rice, Sasuly, Schach, Schell, Schirrmacher, Schult, Shenton, Siller, Skelly, J. P. Smith, Sohon, Spencer, Stearn, Steinbach, Taylor, Tunell, Van Orstrand, Varnhorn, Watts, Weida, Wilson, Winston, Woolard, Wrench, Zoch.

FLORIDA. (20)

BROOKSVILLE. Blake.
GAINESVILLE. Davis, Dostal, Kokomoor, McInnis, Mead, Phipps, Simpson, Wilson.
JACKSONVILLE. Reiber.
LAKELAND. Fike, Reinsch.
ST. PETERSBURG. Gager, Story.
TALLAHASSEE. Fleming, Larson, E. R. Smith.
TAMPA. Rhodes.
WINTER HAVEN. Hart.
WINTER PARK. Weinberg.

GEORGIA. (39)

AMERICUS. Shuler.
ATHENS. Barrow, Beckwith, Callaway, Cumming, Hill, Stephens.
ATLANTA. Ballou, Carroll, Field, Fumerl, Green, Hanson, Hefner, Hook, Howe, Sewell, D. M. Smith, Starrett, Steen, Webb.
COLLEGEBORO. Moye.
DAHLONEGA. Barnes, Rogers.
DECATUR. Gaylord, Robinson.
DEMOREST. W. B. Smith.
EMORY UNIVERSITY. Lang, Messick.
FORT VALLEY. Pitts.
MACON. Bancroft, Bruce.
MILLEDGEVILLE. Nelson.
OXFORD. Moore.
ROME. Hightower, Thompson.
SAVANNAH. Murray, Williams.
YOUNG HARRIS. Miller.

HAWAII

KEALAKEKUA. Pyuen.

IDAHO. (4)

BOISE. Bridger, McFarland.
CALDWELL. Rankin.
MOSCOW. Dimsdale.

ILLINOIS. (149)

AURORA. Furman.
BATAVIA. Gutekunst.
BLOOMINGTON. Hunt.
BLUFFS. Carter.
CARBONDALE. Mayor, Wright.
CARTHAGE. Boatman.
CHARLESTON. Hendrix, Taylor.
CHICAGO. Aschenbrenner, Ballard, Bay, Campbell, Christman, Corliss, J. E. Davis, Ettinger, Feltges, Georges, Gere, Gerst, Godfrey, Gore, Haggard, Holland, Kinney, Kurzin, Lange, Mahoney, Mansfield, Mary Esther, Miller, Modesitt, Moran, Mullen, Nicolet, Petty, Rasmusen, Sachs, Schweitzer, Templin.
University of Chicago. Albert, Barnard, Bartky, Bliss, Brauer, Dickson, Everett, Graves, Hartung, Hestenes, Householder, Lane, Leavens, Logsdon, Lunn, Reichelderfer, Reid, Sanger, Z. L. Smith, White, Young.
Illinois Institute. Bibb, W. M. Davis, DeCicco, Ford, Krathwohl, Loch, Oldenburger, Perlin, Sadowsky, Wilcox.
DECATUR. Denton, Kiefer, Ploenges.
DE KALB. Hellmich, Stelford, Storm.
EUREKA. Newson.
EVANSTON. Buell, Curtiss, Hellinger, Holgate, Moulton, Newell, Paydon, Scott, Simmons, Wall, Wescott, Wood.
FREEPORT. Baumgartner, Mensenkamp.
GALESBURG. Heren, Smyth, Stephens.
HIGHLAND PARK. MacMartin.
JACKSONVILLE. Miller.
JOLIET. F. C. Smith.
KENILWORTH. Hart.
LAKE FOREST. Curtis.
LEBANON. Stowell.
LINCOLN. Balof.
MACOMB. Ayre, Ginnings, Schreiber.
MAYWOOD. Hildebrandt, Olson.

MONMOUTH. Beveridge.
MURPHYSBORO. Bock.
NORMAL. Atkin, Flagg, Larsen, Mills.
PEORIA. Comstock, Gault, Johanson.
RIVER FOREST. Dobbin.
ROCK ISLAND. Cederberg, Olmsted.
SANDWICH. Rumney.
SPRINGFIELD. Harman, Wells.
TAYLORVILLE. Dappert.
URBANA. Armstrong, Bailey, Bower, Carmichael, Coble, Crathorne, Hartley, Hattan, Hazlett, Ketchum, Levy, Miles, Miller, Moore, Niven, Pepper, Peters, Schwartz, Vaughan.
WHEATON. Boyce, Brandt, Taylor.
WINNETKA. Gadske, Humphrey, Jewell.

INDIANA. (63)

ANGOLA. Moore.
BLOOMINGTON. Artin, Hennel, Rothrock, Wells, Weyl, Williams, Wolfe.
COLLEGEVILLE. Zanolar.
CRAWFORDSVILLE. Carscallen, Polley.
EARLHAM. Long.
EAST CHICAGO. Burns.
GARY. Copp, Oursler.
GOSHEN. Hartzler, Lehman.
GREENCASTLE. Arnold, Edington, Greenleaf.
HANOVER. Meyer.
HOLY CROSS. Callahan.
INDIANAPOLIS. Beal, Getchell, McColgin, Suter, Welchons.
MARION. Porter.
MUNCIE. Edwards, Shively.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparó, Kelley, Maurus, Menger, Pepper.
OAKLAND CITY. Messick.
REYNOLDS. Erwin.
TERRE HAUTE. Kennedy, Shriner, Sousley.
UPLAND. Draper.
WEST BADEN SPRINGS. Muehlman.
WEST LAFAYETTE. Ayres, Bailey, Burr, Crain, Doan, Downing, Graves, Hadley, Hardman, Hazard, Hodge, Hughes, Keller, Lanczos, Little, Miller, Robbins, Shreve, Stone, Webster.

IOWA. (49)

AMES. Brandner, Daniells, Fleming, Gouwens, Herr, Lonseth, J. V. McKelvey M. M. McKelvey, McMaster, Robertson, E. R. Smith, Snedecor.
CEDAR FALLS. Kearney, Trimble, Van Engen, Wester.
CEDAR RAPIDS. Coffin, Yothers.
DECORAH. Ellingson.
DES MOINES. Neff, Westemeier.
DUBUQUE. Beach, Mary Resignata, Theobald.
FAYETTE. Deming.
FORT DODGE. Shannon.
GRINNELL, McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. Chittenden, Conkwright, Craig, Knowler, Lane, Oberg, Reilly, Rietz, Ward, Woods, Wylie.
IOWA FALLS. Kreider.
KEOKUK. West.
MOUNT VERNON. McGaw, Moots.
ORANGE CITY. Hattan.
SIOUX CITY. Graber, Rochford.
WAVERLY. Chellevold.

KANSAS. (51)

ATCHISON, Pretz, Sullivan.
BALDWIN. Garrett.
EL DORADO. Wrestler.
EMPORIA. Peterson, Tucker.
HAYS. Colyer, Grabbe.
HESSTON. Driver.
HIGHLAND. Culbertson.
INDEPENDENCE. Bell.
KANSAS CITY. Dougherty, Thornton.
LAWRENCE. Babcock, Black, Jordan, Mitchell, Price, G. W. Smith, Stouffer, Ulmer, Wheeler.
LEAVENWORTH. Ann Elizabeth.
LINDSBORG. Marm.
MANHATTAN. Babcock, Daugherty, Hyde, Janes, Lewis, Mossman, Remick, Sigley, Stratton, White.
NORTH NEWTON. Richert.
OTTAWA. Bemmels.
PITTSBURG. Shirk, R. G. Smith.
ST. MARY. Yenni.
SALINA. Arnoldy, Jensen.
STERLING. Bell.
TOPEKA. Harshbarger.
WICHITA. Beito, Deal, Greer, Hoare, Longenecker, Read. Reagan, Wedel.

KENTUCKY. (36)

ASHLAND. Taylor, Williams.
BEREA. Hutcherson, Pugsley.
BOWLING GREEN. Howard, Strayhorn, Yarbrough.
COVINGTON. Thuener.
GEORGETOWN. Hatfield.
HOPKINSVILLE. Nowlan.
LEXINGTON. Boyd, Brown, Cohen, Downing, John, Latimer, LeStourgeon, Pence, South, Wright.
LOUSVILLE. Bloom, Bullitt, Ford, Moore, Morrison, Simester, Stevenson, Straw.
MAPLE MOUNT. Sheeran.
MOREHEAD. Black, Fair.
MURRAY. Carman.
RICHMOND. Jenkins, Park.
WILLIAMSBURG. Baumgart.
WINCHESTER. Allison.

LOUISIANA. (46)

BAKER. Johnson.
BATON ROUGE. Sanders.
HAMMOND. Cordrey, Tucker.
LAFAYETTE. Buchanan, Loflin, Nolan, Sanders, Jr.
MONROE. Currie.
NATCHITOCHES. Blair, Killen, Maddox.
NEW ORLEANS. Buchanan, Cramer, Duren, Fleddermann, Frankenbush, Higgins,

Humphreys, Many, Menuet, Monasterio, Petersen, Spencer, Stevens, Thomson, Weiss.
PINEVILLE. Temple.
RUSTON. Gatewood, Gentry, Kaltenborn, Schroeder, P. K. Smith.
SHREVEPORT. Banks.
UNIVERSITY. Christensen, Freas, Karnes, Nichols, O'Quinn, Parker, Rickey, Rutt, Scott, H. L. Smith, White, Yates.

MAINE. (11)

BRUNSWICK. Hammond, Holmes, Korgen, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
LISBON FALLS. Schultz.
ORONO. Bryan, Kimball.
WATERVILLE. Ashcraft.

MARYLAND. (51)

ABERDEEN PROVING GROUND. Dederick, Hart, Stearn.
ANNAPOLIS. Ayres, Ball, Bingley, Bleick, Bramble, Buchanan, Church, Clements, Currier, Dillingham, Echols, Kells, Lamb, Leiper, Littauer, Lyle, Moore, Rawlins, Root, Scarborough, Sears, Wilson.
BALTIMORE. Bacon, Bankier, Celauro, Cohen, Hearn, Lewis, Mary Cordia, Morrill, Murnaghan, Reed, Roman, H. R. Smith, Torrey, Williamson, Zariski.
COLLEGE PARK. Gilbert, Hutchinson, Lancaster, Richeson, Vedova.
EMMITSBURG. Burke.
FREDERICK. Brown.
FROSTBURG. Hallett.
PORT DEPOSIT. Haviland.
SILVER SPRING. Majella.
WOODSTOCK. Phillips.

MASSACHUSETTS. (86)

AMESBURY. Dame.
AMHERST. Boutelle, Esty, Miller, Moore.
BELMONT. Jackson.
BOSTON. Brown, Bruce, Combellack, Fisher, Gould, Hemenway, Hubbard, Laurentine, Mode, Skofield, Spear, Stone, Weaver, Wilson.
BROOKLINE. McCarthy, Miller.
CAMBRIDGE. Beatley, Birkhoff, Cameron, Clifford, Coolidge, Douglass, Emmons, Franklin, Harvey, Huntington, Lennahan, MacLane, Moon, Phelps, Rule, Rulon, Stone, Walsh, Widder, Woods, Zeldin.
CHESTNUT HILL. Marcou, O'Donnell.
CHICOPEE. Madden.
DORCHESTER. Davis.
GROTON. Holt, Nash.
MARBLEHEAD. Oergel.
NORTHAMPTON. Benedict, McCoy, Montgomery, Munroe, Rambo.
NORTON. Garabedian, Watt.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY. Baker, Doak, Litzinger.
SWAMPSCOTT. Evans.
TUFTS COLLEGE. Mergendahl, Ransom.
TYNGSBORO. Richmond.
WELLESLEY. Copeland, Merrill, Russell, C. E. Smith, Stark, Young.
WESTON. Burke.
WILLIAMSTOWN. Agard, Hardy, Wells, Wray.
WOLLASTON. Dennison.
WORCESTER. Brown, Gay, Melville, Morley, O'Callahan, Rice, Wheeler.

MICHIGAN. (85)

ALBION. Ingalls, Sleight.
ANN ARBOR. Anning, Beckenbach, Begle, Bookstein, Bradshaw, Churchill, Coe, Copeland, Craig, Dwyer, Field, Fischer, Ford, Gaskell, Goldstine, Hildebrandt, Hinds, Hopkins, Kaplan, Karpinski, Love, Nyswander, Rainich, Rainville, Rosenthal, Rouse, Rufus, Running, Schorling, Stabler, Wilder.
BERRIEN SPRINGS. Woods.
DEARBORN. P. S. Jones.
DETROIT. Baldwin, Borgman, Butler, Coral, Fischer, Folley, Goldman, Johnston, McCarthy, Mary Paula, Morrow, Nelson, Pixley, Shires.
EAST LANSING. Barbour, Baten, Blanche, Grove, Heyda, Hill, Plant, Powell, Speeker, Stewart, Van Schaack, Welmers.
FLINT. Swanson.
GRAND RAPIDS. Warren, Wilson.
HART. Burdick.
HIGHLAND PARK. Peterson.
HILLSDALE. Beeler.
HOLLAND. Lampen.
HOWELL. Olson.
IRONWOOD. Eittreim, Field, McLaughlin.
KALAMAZOO. Ackley, Bartoo, Blair, Butler, Everett, Walton.
MARQUETTE. Spooner.
MILLFORD. McNeal.
MOUNT PLEASANT. Richtmeyer.
NEGAUNEE. Johnson.
OLIVET. Young.
YPSILANTI. Erikson, Lindquist.

MINNESOTA. (70)

ALBERT LEA. Blakeman.
ANNANDALE. Tucker.
COLERAINE. J. B. Davis, Tangjerd.
COLLEGEVILLE. Danzl, Winkelmann.
DULUTH. Cothran, Herr, Mercedes, Strane.
ELY. Cramer.
EVELETH. Pollard.
GILBERT. Schey.
HIBBING. Erickson.
MINNEAPOLIS. Amundson, Anderson, Bearman, Brink, Brooke, Bussey, Campaigne, Carlson, Dalaker, Eggers, Fattu, Gibbens, Hart, Hartig, Jackson, Johnson, Kirchner, Koehler, McEwen, Munro, Ness, Novak, Olmsted, Opatowski, Priester, Quaid, Saunders, Scammon, Scherberg, Shuman, Shumway, Stigler, Swanson, Thorp, Turrittin, Underhill.
MOORHEAD. Andersen, Mundhjeld.

New Ulm. McCutcheon.
Northfield, Carlson, Gingrich, Shover.
Rochester. Hickman.
St. Joseph. Claudette.
St. Paul. Bush, Camp, Ostrom, Polansky, Taylor, Thielman, Wegner, Wolf.
St. Peter. Rundstrom.
Virginia. Hancock.
Waseca. Hawkes.
Winona. De La Salle.

Mississippi. (18)

Cleveland. Ward.
Clinton. Hitt.
Goodman. Wilson.
Hattiesburg. Dearman.
Jackson. Babbitt, McCoy, Mitchell.
Meridian. Coker.
Raymond. MacDonald.
Smithville. Tubb.
State College. Cox, Ollivier, Pettis, C. D. Smith.
University. Bickerstaff, Hume, Quarles.
Wesson. Felder.

Missouri. (42)

Canton. Ingold.
Cape Girardeau. Michel.
Clayton. Haertter, Rosskopf.
Columbia. Blumenthal, Callaway, Cosby, Ewing, Ferguson, Haynes, Kelly, Shanks, Wahlin, Wehausen, Westfall.
Eldon. Cole.
Fayette. Fleet.
Fulton. Butchart, Sweazey.
Jefferson City. Jason, Talbot.
Kansas City. Cutting, Pierson.
Kirksville. Jamison.
Parkville. Crull.
Rolla. Hinsch.
St. Charles. Karr.
St. Louis. Callaghan, Case, Dunkel, Gove, Kruer, Middlemiss, Osborn, Rider, Roever, Siroky.
Springfield. Finkel, Graves, H'Doubler.
Warrensburg. Urban.
Webster Groves. Clarke.

Montana. (6)

Bozeman. Hurst.
Butte. Poole,
Garrison. Canning.
Helena. Topel.
Missoula. Carey, Merrill.

Nebraska. (27)

Chadron. Berry.
Gilead. Erwin.
Hastings. Hadlock, McDill.
Kearney. Hanthorn.
Lincoln. Basoco, Brenke, Camp, Candy, Congdon, Cox, Dribin, Gaba, Harper, Howie, Ogden, Pierce, Runge.
Omaha. Bettinger, Earl, Fitzpatrick, Marrin.
Peru. Hill.
Wayne. Boyce, Hove.
York. Christensen, Feemster.

Nevada

Reno. Vance, Wood.

New Hampshire. (19)

Concord. Conwell.
Dover. Hodgdon.
Durham. Lewis, Slobin.
Exeter. Adkins, Funkhouser, Pennell.
Hanover. Brown, Forsyth, Mathewson, Morgan, Nordstrom, Perkins, Robinson, Silverman, Wilder.
Keene. Goodrich.
Manchester. O'Leary
Plymouth. G. M. Smith.

New Jersey. (58)

Belleplain. Durell.
Convent Station. Kenna.
East Orange. Nordgaard.
Elizabeth. Nagle.
Hightstown. Harrison, Litterick.
Hoboken. Hazeltine, Murray.
Jersey City. McGrath, Schnefel.
Lawrenceville. Kimball, Mikesh.
Leonia. Karapetoff.
Montclair. Davis, Mallory, Turner.
Newark. Conklin, MacDonald, Strock.
New Brunswick. Bunyan, Galbraith, Grant, Meder, Morris, Nelson, Starke, Walter.
Patterson. McGlade.
Princeton. Adams, Alexander, Arnold, Bohnenblust, Eisenhart, Flood, Fubini, Gillespie, Kolchin, Lefschetz, Macphail, Morse, Mosteller, Olmsted, Scheffé, Tucker, Tukey, Veblen, von Neumann, Wedderburn, Wilks.
South Orange. Rauch, Stanwick.
Trenton. Shuster.
Upper Montclair. Campbell, Clifford, Fehr, Hildebrandt.
Westmont. High.
West Orange. Edison.

New Mexico. (17)

Albuquerque. Barnhart, Bauer, Buck, Haskins, Larsen, Newsom.
Las Vegas. Roberts, Rodgers.
Portales. Fleck, MacKay, McLaughlin.
Roswell. Harp.
Silver City. Mickelson.
Socoro. Reece.
State College. Branson, Heinzman, Swingle.

New York. (271)

Albany. Beaver, Birchenough, DoBell, Frankel, Lester, Snader, Stokes, Thornton.
Alfred. Lowenstein, Polan, Seidlin, Titsworth, Whitford.
Aurora. Hollcroft, Rusk.
Baldwin. Bowden.
Brookfield. Whitford.
Brooklyn. Berry, Borofsky, Boyer, Charosh, Cowles, Feld, Fleisher, Forman, H. M. Griffin, J. I. Griffin, Harkin, Hertzler, R. A. Johnson, Kaplan, Karnow, Kennison, Koch, Landers, Lavoie, Levy, Lieber, Locke, MacNeish, McCarthy, McMahon,

Maria, Milkman, Miller, Moore, Richardson, Rush, Simpson, Singer, F. E. Smith, Thompson, Whitford, Wolfe, Woodbridge, Zaslavsky.
BUFFALO. Browne, Gehman, Harrington, Kendall, Montague, Ott, Podmele, Pound.
CAZENOVIA. Price.
CLINTON. Brown, Carruth, Ferry, Patterson.
CROWN POINT. Henderson.
ELMIRA. Suffa, Wright.
FLUSHING. Archibald, A. B. Brown, Cairns, Cope, Raudenbush, Sard.
GENEVA. Durfee, Hubbs.
HAMILTON. Aude, Munshower, Wardwell.
HEMPSTEAD. Ollmann.
HOUGHTON. Davison.
ITHACA. Agnew, Carver, Curtiss, Ficken, Flexner, Hurwitz, B. W. Jones, Kac, Karapetoff, Randolph, Rhodes, Snyder, G. L. Walker, R. J. Walker.
KENMORE. Brockett.
LOUDONVILLE. Nickol.
NEW LEBANON. Lewis.
NEW YORK. Alfieri, Allison, Berger, Bergstresser, Berkeley, Bernard, Berry, Boehm, Bowden, Burgess, Campbell, Darraugh, D'Atri, Eichert, A. M. Ginsburg, J. Ginsburg, Greenberg, Grossman, Hlavaty, Jablonower, Joffe, Katsh, Keeler, Kirby, Kraitchik, Kubis, Landin, Lantz, Lazar, Longfellow, McKenna, Martin, Moore, Nehrbas, Newman, Oehler, Penney, Quilty, Ripandelli, Roll, Ruderman, Russ, Sheridan, Skelding, Stuckey, Wayne, Weaver.
Bell Tel. Labs. Foster, Fry, Gray, P. C. Jones, MacColl, Mead, Molina, Schelkunoff, Shewhart.
City College. Allen, Fagerstrom, Gill, Grove, Hubert, Hurwitz, Linehan, MacEwen, Newman, Post, Robinson, Turner, Whitford, Wirth, Wright.
Columbia University. Alman, Bakst, Fiske, Fite, Gentzler, Hawkes, Hoy, Kasner, Mehr, Mirick, Mullins, Précourt, Rayher, Reeve, Ritt, Siceloff, D. E. Smith, Swenson, Upton, Walker.
Cooper Union. Larkin, Lehmann, Miller, Reddick, Tanzola.
Hunter College. Anderson, Aroian, Bradley, Mrs. J. H. Bushey, J. H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Lawton, MacDuffee, Rees, Simons, Weisner, Whelan.
New York University. Bernstein, Cooley, Courant, Doermann, Flanders, Graham, John, Kline, Payne, Peters, Putnam, Roth, Schlauch, Tilley, Wahlert, Yanosik.
NIAGARA FALLS. O'Connor.
ONEONTA. Newton, Sanford.
PELHAM. Milos.
PELHAM MANOR. Harter.
POTSDAM. Waltz.
POUGHKEEPSIE. Fiske, Hopper, Wells.
ROCHESTER. Atkins, Betz, Chesna, Eastham, Gale, Green, Harding, Long, Seidel, Watkeys.
ST. BONAVENTURE. Roth, Scheier.
SCHENECTADY. Burkett, Fox, Morse, Poritsky, Snyder.
SYRACUSE. Campbell, Carroll, Decker, Harwood, Simpson, Taylor
TROY. Allen, H. K. Brown, Campbell, Merrill, Nash.
WEST POINT. Jones.
WILLISTON PARK. Buell.
WYOMING. Hartnell.

NORTH CAROLINA. (43)

BOONE. Wright.
CHAPEL HILL. Browne, Cameron, Garner, Gillis, Henderson, Hickerson, Lasley, Linker, Mackie, Reynolds.
CHARLOTTE. Douglass, O. M. Jones, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM. Boas, Dressel, Elliott, Gergen, Greenwood, Hickson, Patterson, Rankin, Thomas.
ELON COLLEGE. Westhafer.
GREENSBORO. Barton, Pegram, Strong.
GREENVILLE. Graham, ReBarker.
MARS HILL. Howell, Robinson.
RALEIGH. Bullock, Cell, Downing, Eason, Levine, Strobel.
RED SPRINGS. Prince.
SALISBURY. Dearborn, Patten.
WILSON. R. E. Smith.

NORTH DAKOTA. (6)

DICKINSON. Muggli.
FARGO. Householder, I. W. Smith.
GRAND FORKS. Mason, Staley.
JAMESTOWN. Jackson.

OHIO. (130)

ADA. Whitted.
AKRON. Bender, Selby.
ALLIANCE. Hildner.
ATHENS. Denbow, Marquis, Miller, Reed, Starcher.
BEREA. Dustheimer.
BLUFFTON. Linscheid.
BOWLING GREEN. Mathias, Overman.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Hancock, Justice, Lubin, Merriman, Moore, Reilly, E. S. Smith, Szász, Taylor, Yowell.
CLEVELAND. Boyce, O. E. Brown, Burington, Burwell, Erkiletian, Focke, Johnson, Jonah, Justin, Kelly, Morris, Musselman, Nassau, Patterson, Rinehart, Risley, Sauté, Simon, Thomas, Tolar, Topp, Torrance.
CLEVELAND HEIGHTS. Joliat.
COLUMBUS. Albert, Bamforth, Bareis, Beatty, Blumberg, Caris, M. E. Jones, Kuhn, LaPaz, Manson, Maple, Morris, Radó, Rasor, Reade, Rickard, Risley, Singer, Toops, Weaver, Wildermuth, Wylie.
DAYTON. Eagle, Schraut.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
FINDLAY. Roots.

GAMBIER. Bumer, MacNeille.
GRANVILLE. Ladner, Wiley.
HIRAM. Clarke.
KENT. Brooks, Harshbarger, Manchester, Olson, Rogers, Stelson.
LAKESIDE. Wolfe.
MARIETTA. Bennett, Sandt.
MOUNT ST. JOSEPH. Corona.
NAPOLEON. Yeager.
NEW LEXINGTON. Hoops.
NEW RICHMOND. Wishard.
NORTH CANTON. Schug.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Wagner, Yeaton.
OXFORD. Anderson, Pollard, Spenceley, Tappan, Wolfe, Young.
PAINESVILLE. Peters.
PATTERSON FIELD. Westbrook.
SEVEN MILE. Baird.
SOUTH EUCLID. Garvin.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Blackall, Brandeberry, Dancer, Koley, Mercedes, Welker.
WESTERVILLE. Glover.
WILMINGTON. Spinks.
WOOSTER. Fobes, Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan.
YOUNGSTOWN. Foard.

OKLAHOMA. (27)

ADA. Heimann, Winn.
ALVA. Hall.
CLAREMORE. Paine.
LANGSTON. Tinner.
NORMAN. Brixey, Court, Duval, Hassler, LaFon, McFarland, Randels, Reaves, Springer, Whitney.
SHAWNEE. Doerfler, Short.
STILLWATER. Allen, Barnett, Diamond, Flanders, Garretson, Hamilton, H. W. Smith, Zant.
TULSA. Ellis, Veatch.

OREGON. (19)

ALBANY. Porter.
ASTORIA. Ely.
CORVALLIS. Beaty, Kirkham, Milne, Sobczyk, Williams.
EUGENE. Aitchison. DeCou, Moursund, Peterson.
FOREST GROVE. Price.
GRANT'S PASS. Feinler.
MCMINNVILLE. Ramsey.
PORTLAND. Griffin, Hadley, Johnson, Merriss.
SALEM. Luther.

PANAMA.

PANAMA CITY. Linares.

PENNSYLVANIA. (156)

ALLENTOWN. Deck, Koehler, Kunkel.
ANNVILLE. Black.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Cutler, Fort, Latshaw, Pitcher, Rau, Raynor, Reynolds, Shook, Smail, D. M. Smiley, M. F. Smiley, Van Arnam.
BRYN ATHYN. Allen.
BRYN MAWR. Atkinson, Lehr, Wheeler, Williams.
CARLISLE. Ayres, Landis.
CHESTER. Williams.
COLLEGEVILLE. Clawson, Dennis, Manning.
DUBOIS. Cunningham.
EASTON. Benner, Cawley, Hatch, W. M. Smith.
ERIE. Benedicta, Kraus, Sullivan, Wells.
GEORGE SCHOOL. Bates.
GETTYSBURG. Clutz.
GREENSBURG. McNeil.
GROVE CITY. Carpenter, Renwick.
HARRISBURG. Whited.
HAVERFORD. Allendoerfer, Oakley, Wilson.
HAZELTON. Herpel.
HUNTINGDON. Stayer.
INDIANA. Stright.
JENKINTOWN. Durand.
KUTZTOWN. Knedler.
LANCASTER. Charles, Ikenberry, Long, Marburger, Murray, Worthington.
LATROBE. Seubert.
LEBANON. Heilman.
LEWISBURG. Gold, Miller, Richardson.
LOCKHAVEN. S. J. Smith.
MEADVILLE. Beisel.
MERCERSBURG. Johnson.
MILLERSVILLE. Boyer.
NEW KENSINGTON. Sturm.
NEW WILMINGTON. Black.
PAXTANG. McKee.
PHILADELPHIA. R. D. Brown, Campbell, Caris, Constable, Cross, Davis, Eggert, Evans, Fudge, Helms, Hill, Kline, Latshaw, Leifer, McDonough, Mitchell, Moore, Robertson, Safford, Schoenberg, Shohat, Wallace, Whitman.
PITTSBURGH. Baird, Blumberg, Briant, Brown, Bryson, Buker, Calkins, Cowley, Dines, Donaldson, Foraker, Hicks, Hoover, Johnson, Karpov, Moskovitz, Neelley, Olds, Petrie, Riggs, Rosenbach, Saibel, S. R. Smith, Starr, Taylor, Wagner, Whitman.
POTTSTOWN. Huff.
POTTSVILLE, Kleinschmidt.
READING. Speicher.
SCRANTON. Bertrand, Mary Daniel.
SHARON. Manning.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, F. W. Owens, H. B. Owens, Rupp, Sheffer, West.
SWARTHMORE. Brinkmann, Dresden, Marriott.
SWISSVALE. Kaplan.
UNIONTOWN. Mosesson.
VILLANOVA. Crawford.
WARREN. Lafferty.

WASHINGTON. Atchison, Bert, Brady, Dorwart, Shaub, Thomas.
WAYNESBURG. Moston.
YORK. Baker.

RHODE ISLAND. (15)

NEWPORT. Chase.
PROVIDENCE. Adams, Archibald, Bennett, Carlen, Frame, Gilman, Hall, Manning, McKenney, Précourt, Richardson, Smiley, Tamarkin, Watt.

SOUTH CAROLINA. (15)

CHARLESTON. Doyle, Dye, Hair, Hutchison, Reves, Saunders.
CLINTON. Spencer.
COLUMBIA. Coker, Coleman, Jackson, Williams.
GREENVILLE. Blackwell.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Stokes.

SOUTH DAKOTA. (9)

BROOKINGS. MacDougal, Walder, Wente.
MITCHELL. Knox.
RAPID CITY. Davis, Swanson.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILLION. Ekman.

TENNESSEE. (30)

CHATTANOOGA. Massey, Mays.
CLEVELAND. Hutto.
COOKEVILLE. Hutchinson, Moorman.
FOUNTAIN CITY. Keller.
HARROGATE. Bowling.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson.
KNOXVILLE. Blincoe, Cooley, Eaves, Gillis, Hughes, Lee, Pepper, Purviance.
LEBANON. Donnell.
MARYVILLE. Sisk.
MEMPHIS. Locke.
NASHVILLE. Blair, Falvey, Hyden, McPherson, N. P. Miser, W. L. Miser, Morrel, Van Horn, Wicht, Wren.

TEXAS. (77)

ABILENE. Burnam, Mullings, Tate.
ALPINE. Gilley.
AMARILLO. Layton, Whetstone.
ARLINGTON. Howard.
AUSTIN. Batchelder, Craig, Decherd, Dodd, Ettlinger, Greenwood, Lubben, Moore, Vandiver.
BROWNSVILLE. de la Garza.
BROWNWOOD. Freese.
CAMP BARKELEY. Goodpasture.
CANYON. Murray, York.
COLLEGE STATION. Basye, Blumberg, Chaney, Coleman, Edmonson, Klipple, Luther, McCulley, Moore, Porter.
DALLAS. Huff, Mouzon, Rees, Starr.
DENTON. Brown, Cooke, Hanson, White.
EL PASO. Leech, Schwid.
FORT WORTH. Sherer.
GEORGETOWN. Wapple.
HOUSTON. Blau, Bray, Dean, Lovett, Rees, Slotnick, Ulrich, Underwood.
HUNTSVILLE. Querry.
KINGSVILLE. Kennedy.
LUBBOCK. Hazlewood, Heineman, May, Michie, Sparks, Thompson, Underwood, Wakerling, Wallis.
MAYPEARL. Thomas.
PRAIRIE VIEW. Randall, Stephens.
SAN ANTONIO. Hurry, McNelly, Mary of Mercy, Schnepp, Tulloch.
STEPHENVILLE. McSweeny, Redden.
TEAGUE. Notley.
WACO. Baker.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams, Searcy.

UTAH. (7)

LOGAN. Bird.
ST. GEORGE. Everett.
SALT LAKE CITY. Gibson, Hayes, Horsfall, Pehrson, Stafford.

VERMONT. (10)

BURLINGTON. Bullard, Butterfield, Millington, Swift.
MIDDLEBURY. Bowker, Hazeltine, Perkins, Wiley.
NORTHFIELD. Dix.
RANDOLPH. Alliot.

VIRGINIA. (45)

ASHLAND. Simpson.
BLACKSBURG. Hatcher, O'Shaughnessy, Williams.
CHARLOTTESVILLE. Aylor, Hancock, Hedlund, McShane.
EMORY. Miller.
FARMVILLE. Taliaferro.
FREDERICKSBURG. Frick, Whitney.
FORTRESS MONROE. Pettis.
HAMPTON. W. B. Brown, Perkins.
HOLLINS. Allen.
LANGLEY FIELD. Pinkerton, Street.
LEXINGTON. Byrne, Knox, Paxton, Purdie, L. W. Smith.
LYNCHBURG. Larew, Wiggin.
MIDDLEBURG. Keppler.
MILLER SCHOOL. Watson.
NEWPORT NEWS. Raine.
NORFOLK. A. L. Smith.
RICHMOND. Drew, Gaines, Harris, Wheeler.
SALEM. Carpenter.
SOUTH BOSTON. Patten.
STAUNTON. Taylor.
SWEET BRIAR. Cole, Morenus.
UNIVERSITY. Linfield, Oglesby, Whyburn.
WILLIAMSBURG. Calkins, Gregory, Phalen, Stetson.

WASHINGTON. (22)

CENTRALIA. Van Arkel.
CHENEY. Bell.
LACEY. Cebula.
PULLMAN. Bauserman, Butler, Hacker, Knebelman.
SEATTLE. Ballantine, Beegle, Cramlet, Hal-

ler, Jerbert, McFarlan, Mullemeister, Winger.
SPOKANE. Carlson, Summers.
STEILACOOM. Chelius.
TACOMA. Martin.
WALLA WALLA. Bratton, Stewart.
YAKIMA. Whitney.

WEST VIRGINIA. (15)

HUNTINGTON. Hackney.
KINGSTON. Neely.
MONTGOMERY. Hall, Hines, Reckzeh, W. F. Smith.
MORGANTOWN. Davis, Eiesland, Reynolds, Turner, Vehse, Vest.
PAX. Milo.
WEST LIBERTY. Kiplinger.
WHEELING. Bagby.

WISCONSIN. (52)

BEAVER DAM. Newlin.
BELOIT. Bigelow, Conwell, Huffer.
CUDAHY. Sedlack.
LACROSSE. Adkins.
MADISON. Allen, Bernstein, Evans, Finch, Ingraham, Langer, March, Sokolnikoff, Tompkins, Trump, Ullsvik, Van Vleck.
MILWAUKEE. Bardell, Beckwith, Boehmer, Ericson, Fischer, Fitzpatrick, Jautz, Kenney, Knight, Luteyn, Marden, Mary Felice, Mary Gertrude, Nordhaus, Norris, Parkinson, Pettit, Roth, Vass, Wilczewski, Wolf.
OSHKOSH. Beenken, Price.
PLATTEVILLE. Harrell.
RIVER FALLS. Eide, Kirchen.
SHEBOYGAN. Battig.
SUPERIOR. Daoust, Flogstad, C. W. Smith.
WAUKESHA. Dancey, Hopkins.
WEST DE PERE. De Cleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (5)

LARAMIE. Barr, Bellamy, Neubauer, Rechard, Varineau.

FOREIGN MEMBERS

ARGENTINA.

BUENOS AIRES. Baidaff, Barral-Souto.

BELGIUM

UCCLE. Errera.

BRITISH HONDURAS.

PUNTA GORDA. Zimmerman.

CEYLON

VADDUKODDAI. Lockwood.

CHILE

SANTIAGO. Moreno, Salas-Edwards.

CHINA

CANTON. MacDonald.

EIRE

DUBLIN. Rowe.

FINLAND

HELSINGFORS. Ahlfors.

GREAT BRITAIN

BELFAST. McCrea.
CAMBRIDGE. Hardy.
CHIPPING NORTON. O'Hara.
LONDON. Dalal.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville, Todd.

HUNGARY

BUDAPEST. Arany.

INDIA

BANGALORE. Iyengar.
DACCA. Vijayaraghavan.
MADRAS. Durairajan.
POONA. Banerji.
SURAT. Shah.

ITALY

BOLOGNA. Bortolotti.
NAPLES. Crudeli.
ROME. Enriques, Laboccetta.

JAPAN

TOKYO. Kobayashi.

NEW ZEALAND

DUNDEDIN. Martyn.

PERU

LIMA. de Losada y Puga.

PORTUGAL

LISBON. Caraça, Cunha.

ROUMANIA

BUCHAREST. Claudian.
TIMISOARA. Sergescu.

SOUTH AUSTRALIA

ADELAIDE. Wilton.

SPAIN

SAN SEBASTIAN. Thébault.

STRAITS SETTLEMENTS

SINGAPORE. Oppenheim.

SWITZERLAND

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHÂTEL. DuPasquier.

SYRIA

BEIRUT. Jurdak.

UKRAINE

KIEFF. Kryloff.

UNION OF SOUTH AFRICA

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.

URUGUAY

MONTEVIDEO. Calcagno.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

PROVISIONAL FORM

(The By-Laws are here presented as amended in December 1939 and September 1940 and edited, without any change in substance, by the Secretary. This provisional form is here given to the members of the Association and, after a clarification of Art. III, Sec. 8(g), will be presented to the Association for its formal approval.)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by this membership in constituencies (hereinafter called "Regions") established by the Board.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Region shall elect biennially a Governor for a term of two years. Nominations shall be made by the Section or Sections of the Association existing within the Region, or, in the absence of such Sections, by a committee appointed for that purpose by the Governor representing the Region.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the Board. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Board shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook

County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to the American Annuitants' Table (Male) based upon three and one-half ($3\frac{1}{2}$) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the American Annuitants' Table (Male), three and one-half ($3\frac{1}{2}$) per cent; and the reserve thus computed shall be held as a liability.

7. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Honorary President for Life

H. E. Slaught, December 1933–May 1937

Presidents

E. R. Hedrick	1916	Dunham Jackson	1926
Florian Cajori	1917	W. B. Ford	1927–1928
E. V. Huntington	1918	J. W. Young	1929–1930
H. E. Slaught	1919	E. T. Bell	1931–1932
D. E. Smith	1920	Arnold Dresden	1933–1934
G. A. Miller	1921	D. R. Curtiss	1935–1936
R. C. Archibald	1922	A. J. Kempner	1937–1938
R. D. Carmichael	1923	W. B. Carver	1939–1940
H. L. Rietz	1924	R. W. Brink	1941–
J. L. Coolidge	1925		

Vice-Presidents

E. V. Huntington	1916	F. D. Murnaghan	1928, 1939
G. A. Miller	1916	E. T. Bell	1929, 1930
D. N. Lehmer	1917, 1918	W. C. Graustein	1929, 1930, 1940
Oswald Veblen	1917	Arnold Dresden	1931
J. W. Young	1918, 1926	C. N. Moore	1931
R. G. D. Richardson	1919	W. H. Bussey	1932
H. L. Rietz	1919	G. C. Evans	1932
Helen A. Merrill	1920	E. B. Stouffer	1933
E. J. Wilczynski	1920	E. P. Lane	1934
R. C. Archibald	1921	L. L. Dines	1935
R. D. Carmichael	1921, 1922	N. A. Court	1936
B. F. Finkel	1922	T. C. Fry	1936
A. B. Chace	1923	T. H. Hildebrandt	1937
L. P. Eisenhart	1923	E. J. Moulton	1937, 1938
J. L. Coolidge	1924	H. E. Buchanan	1938
Dunham Jackson	1924, 1925	W. L. Hart	1939
A. A. Bennett	1925, 1933, 1934	R. W. Brink	1940
W. B. Ford	1926	B. H. Brown	1941-
A. J. Kempner	1927, 1928, 1935	R. E. Langer	1941
Clara E. Smith	1927		

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(Appointed by the Board after 1918)

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(Appointed by the Board. Discontinued after 1939)

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W. A. HURWITZ.................1919–1921
A. A. BENNETT................1922
H. P. MANNING................1921–1922
W. B. FORD...................1923–1925
J. L. COOLIDGE................1923
A. J. KEMPNER................1924–1939
W. B. CARVER.......1932–1936, 1937–1939
E. J. MOULTON................1937–1939

EDITORS-IN-CHIEF AFTER 1939

E. J. MOULTON................1940–1941
L. R. FORD...................1942–

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E. P. LANE....................1928–1933
W. B. FORD...................1929–1934
E. R. SMITH...................1929
W. L. HART...................1930–1935
LAO G. SIMONS................1930–1931
L. L. DINES...................1931–1933
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J. L. WALSH..................1934–1936
ARNOLD DRESDEN..............1935–1940
J. O. HASSLER................1935–1936
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CONTENTS

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THE TWENTY-FOURTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-fourth summer meeting of the Mathematical Association of America was held at the University of Chicago, Chicago, Illinois, on Monday to Thursday, September 1–4, 1941, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meetings of the Institute of Mathematical Statistics and the Econometric Society. Six hundred forty-nine were in attendance at the meetings, including the following three hundred thirteen members of the Association:

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C. H. YEATON, Oberlin College

OSCAR ZARISKI, Johns Hopkins University

The meetings of the week were notable in that they formed a worthy part of the celebration of the fiftieth anniversary of the founding of the University of Chicago, the theme of this celebration being "New Frontiers in Education and Research." With this in mind, the university collaborated with the American Mathematical Society in arranging a special program of three lectures on "Trends in Research" and conferences on algebra and on the theory of integration, and for one lecture under the auspices of the Mathematical Association. The three lectures and the two conferences will be reported in full form in the *Bulletin* of the Society.

The colloquium of the Society consisted of three lectures on "Mathematical relations and structures" by Professor Oystein Ore of Yale University on Wednesday, Thursday, and Friday mornings. The sixteenth Josiah Willard Gibbs Lecture was delivered by Professor Sewall Wright Wednesday evening on the subject "Statistical genetics and evolution." The Society held sessions for the reading of short papers in morning and afternoon sessions on Tuesday and Friday; the large number of papers necessitated three or four sections at each of these times.

The Institute of Mathematical Statistics held sessions on Tuesday morning and afternoon and on Wednesday afternoon. The Econometric Society held joint sessions with a section of the Society Tuesday morning and afternoon, and separate sessions on Wednesday and Thursday mornings. These proceedings will be reported in the official journals of these two societies.

Most of those attending the meetings lived in Judson and Burton Courts and had their meals in the Burton Court dining room. Lounges in these courts and in Eckhart Hall furnished convenient meeting places for social hours. The informal receptions on Monday and Thursday evenings and the teas given on Tuesday and Thursday afternoons by the ladies of the department of mathematics of the University of Chicago were well attended and greatly enjoyed.

Three hundred forty-one attended the joint dinner of the four organizations on Thursday evening in Hutchinson Commons. After a sumptuous dinner President Marston Morse acted as toastmaster in introducing first Professor Dresden, who recounted the early history of the department of mathematics of the University of Chicago and the work of Moore, Bolza, Maschke, Dickson, Slaught, Bliss and others. Professor Wilks described the activities of the Institute of Mathematical Statistics and of the Econometric Society, and the desire of each to utilize to the utmost the contributions which mathematics can bring. Professor D. R. Curtiss gave his reminiscences of the days when he studied in France at a period when French mathematics was at its brightest. Professor Bliss read an account, sent by Professor Eisenhart, of the National Science Fund established by the National Academy of Science; this is being given publicity in the scientific press. A resolution was presented by Professor Griffin extending our felicitations to the University of Chicago upon its semicentennial and expressing our hearty thanks to the university for the fine hospitality and coöperation in making the meetings a success, to the department of mathematics and the committee on arrangements, and particularly to Professor R. G. Sanger, for providing effectively for the comfort of members in attendance, and our thanks to the ladies of the department for their gracious hospitality at the afternoon teas and the evening receptions and for their activity in arranging recreational opportunities for members and guests.

The Mathematical Association held sessions Monday morning and afternoon and a joint session with the Society Wednesday afternoon. The program committee, consisting of Professors L. R. Ford, M. R. Hestenes, and T. H. Hildebrandt, chairman, merit the thanks of the Association for the formation of an

excellent program. This follows, together with abstracts of some of the papers numbered in accordance with their place on the program.

First Session of the Association

1. "A finite geometry based on the Desargues configuration" by Professor H. F. MAC NEISH, Brooklyn College.
2. "Universal functions of polygonal numbers" by Professor LOIS W. GRIFFITHS, Northwestern University.
3. "Vortices in fluid flow" by Dr. HILLEL PORITSKY, General Electric Company.

It is expected that these papers will appear in early issues of the MONTHLY.

Second Session of the Association

1. Brief business meeting.
2. "Undergraduate mathematical research" by Professor F. L. GRIFFIN, Reed College.
3. "Matrix methods in the solution of algebraic equations" by Professor RUFUS OLDENBURGER, Illinois Institute of Technology.
4. "Frequency distributions of the product and quotient of two statistical variables" by Professor C. C. CRAIG, University of Michigan.

1. At the business meeting an amendment with regard to the exemption from dues of teachers who have retired from active service was adopted in the form sent to members in advance of the meeting; and, on the recommendation of the Board of Governors, the Association elected Professor L. E. Dickson to honorary life membership.

2. Professor Griffin's paper will appear in an early issue of the MONTHLY.

3. Professor Oldenburger said that with an algebraic equation $f(x)=0$ of degree n, possessing real or complex coefficients, is associated a set of matrices $A_0, \cdots, A_n$ constructed in a simple manner from the coefficients of $f(x)$, where these matrices have the property that the rank m of one of them determines the ranks of all of the others. This rank m enables one to obtain a strong bound on the multiplicities of the roots of $f(x)=0$. For each i such that $i \geqq m$, the solutions $\{\xi\}$ of $A_i\xi=0$ correspond in a one-to-one manner to certain polynomials $\{F_i(x)\}$ of degree i. These polynomials form a linear family L_i of polynomials. The equation $f(x)=0$ has a root α repeated r or more times if and only if the family L_{n-r+1} contains a polynomial $F_{n-r+1}=(x-\alpha)^{n-r+1}$. Thus the study of roots of $f(x)=0$ is reduced to the study of algebraic equations of type $F_i(x)=0$ with coincident roots. From intimate relations which exist between the families L_i and L_j for $i \neq j$ and properties of equations with coincident roots, the problem of determining the equations of type $F_i(x)=0$, $(i=m, \cdots, n)$, with coincident roots is much simplified.

4. Professor Craig's paper will probably appear in the January, 1942, issue of the MONTHLY

Joint Session of the Association with the American Mathematical Society

1. "The polygonal regions into which a plane is divided by n straight lines" by Professor W. B. Carver, retiring president of the Association.

2. "The calculus of variations," Lecture on Trends in Research, by Professor G. A. Bliss, University of Chicago.

1. The retiring address of President Carver will appear in the December, 1941, issue of the Monthly.

2. Professor Bliss' lecture was one of the four arranged in collaboration with the University of Chicago. The paper was concerned with the theory of the calculus of variations for multiple integrals which is much less advanced than the corresponding theory for simple integrals. A general problem was formulated in such a way as to include all cases without side conditions. A new method of deducing necessary conditions from the first variation, due to Hestenes and Carson, was discussed. The necessary condition of Weierstrass was originally deduced in a simpler case by Levi, and for multiple integrals in general by McShane and Graves. This condition has an unexpectedly restricted form, not yet completely understood, for the case of multiple integrals containing more than one dependent variable. The most modern forms of the conditions of Jacobi were discussed. The theory of fields and sufficiency proofs is still in an incomplete state. Special fields have been discussed by Carathéodory and Weyl, and Smiley has deduced the most general form of the invariant integral of a field. Sufficiency proofs in rather unsatisfactory forms have been formulated, but a theory approximating the completeness of that for simple integrals is still lacking.

Meeting of the Board of Governors

Sixteen members of the Board were present at the meeting Monday evening, including seven regional governors.

The following fourteen persons were elected to membership on applications duly certified:

G. C. Bartoo, A.M.(Michigan) Prof., Western Michigan Coll., Kalamazoo, Mich.
W. D. Bemmels, A.M.(Syracuse) Prof., Ottawa Univ., Ottawa, Kans.
Eupha A. Buck, A.M.(New Mexico) Teaching asst., Univ. of New Mexico, Albuquerque, N. M.
C. H. Butler, Ph.D.(Missouri) Instr., Western Michigan Coll., Kalamazoo, Mich.
Paul Cramer, A.M.(Illinois) Instr., Ely Junior Coll., Ely, Minn.
M. O. González, D.P.M.S.(Havana) Prof. Auxiliar de Anal. Mat., Univ. of Havana, Havana, Cuba.
J. W. Green, Ph.D.(California) Asst. Prof., Univ. of Rochester, Rochester, N. Y.
C. E. Heilman, A.M.(Duke) Asso. Prof., Elizabethtown Coll., Lebanon, Pa.
C. S. Jewell, A.M.(Allegheny) Teacher, retired, Chicago High Schools. 459 Provident Ave., Winnetka, Ill.
Maurice Kraitchik, Dr. of Math.(Liege; Brussels) Prof., New School for Social Research, New York, N. Y.
E. D. Miller, A.B.(Stanford) Instr., Yuba Junior Coll., Marysville, Calif.
Philip Newman, Ph.D.(Columbia) Instr., Townsend Harris High School, Coll. of City of New York, N. Y.

PETRE SERGESCU, Lic.ès Sc.(Paris), Dr.ès Sc.(Bucarest) Prof., Fac. des Sciences, Univ. of Cluj, Timisoara, Roumania.
L. V. TORALBALLA, Ph.D.(Michigan) Asso. Actuary, Govt. Service Ins. System, Manila, P. I.

The Board voted New York City as the meeting place in December 1942, on recommendation of the joint committee of the Association and Society.

On recommendation of the Finance Committee the Board voted (1) to employ the Cleveland Trust Company as financial adviser for 1942; (2) to appoint a committee to study the Association library, with a view to its discontinuance since its usefulness is very limited; (3) to adopt a definite budget for 1942 as presented; (4) to appropriate funds to cover part of the expenses to meetings for the regional governors and for the Finance and Executive Committees, as contemplated in the Langer report of 1939; (5) to finance the printing of the first of the Slaught Memorial Papers, when it is to appear, from the interest of the Houck Fund, appropriate mention of the Houck bequest being made.

On recommendation of the Executive Committee, it was voted (1) to provide for the selection of the three regional governors whose election remains to be provided for; (2) to approve the establishment of the Metropolitan New York Section, the twenty-fourth Section to be formed; (3) to propose an amendment to Article III, Section 8 (f) of the By-Laws, inserting after the word "filled" the words "in the case of the Second Vice-President and the members of the Finance Committee," it appearing to be wiser to make a single nomination in the case of the editor, the secretary-treasurer, and the associate secretary, and to secure the consent of the nominee beforehand.

The Board discussed possible modifications and improvements in the details of the Putnam Competition, these matters to be taken up with the Putnam trustees for further consideration. It is the outstanding desire of the officers of the Association, as well as of the Putnam trustees, to discover strong traits of originality in the papers of the contestants and to develop the team spirit in the Competition.

The secretary-treasurer stated his wish and purpose to retire at the end of 1942, to which term he was appointed, in the conviction that a younger person should succeed to this position.

W. D. CAIRNS, *Secretary-Treasurer*

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The sixteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on Saturday, May 3, 1941. Professor J. S. Taylor, chairman of the Section, presided at both morning and afternoon sessions.

The attendance was sixty-one, including the following twenty-five members of the Association: C. S. Atchison, L. C. Bagby, J. O. Blumberg, Elizabeth F. Brown, W. E. Buker, L. L. Dines, L. T. Dunlap, F. A. Foraker, W. O. Gordon,

B. P. Hoover, R. P. Johnson, E. L. Kaplan, Sister Marie Gertrude McNeil, M. L. Manning, David Moskovitz, L. T. Moston, E. G. Olds, J. B. Rosenbach, E. A. Saibel, H. C. Shaub, R. G. Sturm, J. S. Taylor, W. J. Wagner, E. D. Wells, E. A. Whitman.

The invitation of Washington and Jefferson College, to hold the next meeting of the Section at Washington, Pennsylvania, was accepted. This meeting is scheduled for Saturday, October 25, 1941.

After the opening address by Professor L. L. Dines of Carnegie Institute of Technology, the following six papers were read:

1. "Method of finite differences applied to problems in vibrations and stability" by Professor E. A. Saibel, Carnegie Institute of Technology.

2. "The use of models and films in teaching triple integration" by Professor E. A. Whitman, Carnegie Institute of Technology.

3. "Arithmetic of life" by Dr. R. G. Sturm, Aluminum Research Laboratories, New Kensington.

4. "Dynamic stability of rail vehicles" by B. F. Langer, Westinghouse Research Laboratories, East Pittsburgh, introduced by the Secretary.

5. "An investigation of the polar conics of ternary cubics" by Margaret Taylor, University of Pittsburgh, introduced by Professor Foraker.

6. "A simple quaternary group, G_{60}, and some geometric interpretations" by J. D. Donaldson, University of Pittsburgh, introduced by Professor Foraker.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Saibel discussed the problem of finding the natural frequencies or the stability loads of an elastic system by a method of replacing the differential equation by a difference equation. This led to a set of linear homogeneous equations; the determinant of the coefficients being set equal to zero gave the characteristic equation, from which approximations to the characteristic values of the original problem were found. The physical interpretation of the method was discussed and applications of the method to chain, beam, membrane, and plate problems were shown.

2. Professor Whitman discussed the use of models as an aid in the teaching of triple integration and showed models designed for this purpose. Among them were the models shown in the January, 1941, number of this MONTHLY, pp. 45–48. He also discussed the use of animated pictures and showed his film "A Triple Integral." This film traces the process of triple integration from the element of volume, and shows the part each successive integration plays in finding the volume. The idea of the limit of a summation is emphasized throughout the film.

3. Dr. Sturm pointed out that our commonly accepted number system can be traced with reasonable certainty to the experiences of individuals with the digits on their hands. Our experience with things indicates that if we give away that which we have, we no longer have it. But experiences with thoughts reveal that if we give away ideas, we still have the ideas that we have given away. The

basic postulate of this experience is that one may give indefinitely without diminishing the giver. Experiences in the realm of the personality reveal that if we give of our personality, we find that our capacity to give further has increased. The more we give of our personality the more we have to give. Pursuing these basic arithmetics still further, we find that if we follow through the arithmetic of things, we reach our modern mathematics and sciences. If we follow the arithmetic of ideas, we find our modern educational system, and if we follow the arithmetic of the personality, we find our fraternal organizations and religions. Integrating all of these arithmetics we can find expression for the whole of life's experiences for individuals, home groups, and governments, which, in its broadest sense, is business.

4. Mr. Langer considered, in a study of the dynamic stability of rail vehicles, the simple case of a four-wheel vehicle on a rigid track. The differential equations for the lateral motion of such a vehicle can be set up, but cannot be solved completely because of the discontinuity which occurs when a flange strikes a rail. It is possible to obtain a step-by-step solution which is too laborious for practical use but which, nevertheless, discloses the advantage of keeping the polar moment of inertia of the vehicle about a vertical axis to a minimum. Another method, of great practical importance, is to assume the form of the lateral motion and calculate the net energy input involved. When this quantity is positive, the motion is unstable. The results are in good agreement with the observed behavior of models and full-size vehicles.

5. Miss Taylor remarked that the discriminant of the polar conics of a ternary cubic is a conic, the "dividing" conic; and she showed that on the dividing conic the polar conics of the ternary cubic are parabolas. If the dividing conic itself is a proper circle, ellipse, or hyperbola, and when rotated is in standard form, the polar conics of points inside the dividing conic are ellipses, the polar conics of points outside the dividing conic are hyperbolas. If the dividing conic is a proper circle, ellipse, or hyperbola, and when rotated is expressed in the negative of the standard form, the reverse is true.

6. Mr. Donaldson presented a study of a collineation group of 60 matrices, whose elements are the fifth roots of unity (w, w^2, w^3, w^4, and $w^5=1$), and combinations of these roots ($a=w+w^4$ and $b=w^2+w^3$). These matrices are used as the coefficients of linear transformations in tetrahedral coördinates. In the geometric interpretations, the transformations are first examined for invariant points and lines. The quadric surface $x_1x_4+x_2x_3=0$ is a general type and is invariant under all the transformations of the group. The cubic surface $x_1^2x_3+x_3^2x_4=x_1x_2^2+x_2x_4^2$ is invariant under the transformations, and the twenty-seven lines of the surface are found. The equation of the cubic surface is then put into the forms $x_4/x_1=(x_1x_3-x_2^2)/(x_2x_4-x_3^2)$ and $x_2/x_3=(x_1^2+x_3x_4)/(x_1x_2+x_4^2)$. These are set equal to two parameters and inspected for various values of the parameters when the resulting equations are subjected to the transformations of the group.

DAVID MOSKOVITZ, *Secretary*

ORGANIZATION MEETING OF THE METROPOLITAN NEW YORK SECTION

A meeting of teachers of college and secondary mathematics was held at Queens College, Flushing, New York, on Saturday, April 19, 1941, for the purpose of organizing a Metropolitan New York Section of the Mathematical Association of America. Professor F. H. Miller of Cooper Union presided at the morning session and Professor T. F. Cope of Queens College presided at the afternoon session.

The attendance was about one hundred eight, including the following forty-four members of the Association: R. G. Archibald, L. A. Aroian, C. A. Bergstresser, Samuel Borofsky, A. B. Brown, S. S. Cairns, H. R. Cooley, T. F. Cope, Richard Courant, J. E. Darraugh, D. R. Davis, W. B. Fite, Edward Fleisher, R. M. Foster, Marion C. Gray, Harriet M. Griffin, J. I. Griffin, N. A. Hall, D. C. Harkin, L. S. Kennison, Morris Kline, Nathan Lazar, C. H. Lehmann, C. C. MacDuffee, H. F. Mac Neish, Mrs. A. J. Maria, Sallie P. Mead, F. H. Miller, E. C. Molina, C. K. Payne, H. W. Raudenbush, Jr., H. W. Reddick, Mina S. Rees, Moses Richardson, Selby Robinson, Arthur Sard, L. W. Sheridan, L. P. Siceloff, James Singer, J. A. Swensen, H. E. Wahlert, Helen M. Walker, Mary Evelyn Wells, Jack Wolfe.

At the beginning of the afternoon session, Dr. Paul Klapper, president of Queens College, welcomed the group to Queens College. At the close of the afternoon session a business meeting was held at which proposed by-laws for the Section were approved and the following officers were elected for the coming year: Chairman, T. F. Cope, Queens College; Vice-Chairman, J. A. Swensen, Andrew Jackson High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union.

The following eight papers were presented:

1. "Finite geometry" by Professor H. F. Mac Neish, Brooklyn College.

2. "Some enumerative problems of electrical network theory" by John Riordan, Bell Telephone Laboratories, introduced by R. M. Foster.

3. "Problems of maxima and minima" by Professor Richard Courant, New York University.

4. "The nature of actuarial work" by E. B. Whittaker, The Prudential Insurance Company of America, introduced by Professor Cope.

5. "Trends in the high school curriculum in mathematics with respect to the whole country and to New York City" by Dr. J. A. Swensen, Andrew Jackson High School.

6. "Suggestions for correlating senior high school and college mathematics" by Professor D. R. Davis, New Jersey State Teachers College.

7. "The concept of area in plane geometry" by Professor C. C. MacDuffee, Hunter College.

8. "Important statistical concepts needed by the well-educated non-statistician" by Professor Helen M. Walker, Teachers College, Columbia University.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Mac Neish's paper will appear in an early issue of this MONTHLY.

2. Mr. Riordan discussed the number N of ways n abstract (electrical) elements may be connected in series-parallel arrangements, particularly for n large. He gave a derivation of a generating identity for N, published without proof by P. A. MacMahon in 1892. This was used to give recurrences, corresponding respectively to the Euler and Gupta schemes for the partition function from which N may be calculated. The second of these leads to an inequality showing that the values of N for n large are less than $A4^{n-1}n^{-3/2}$. An argument was given to show the plausibility of the better bound $A\lambda^{n-1}n^{-3/2}$ with A about 3/7, λ about 3.56. A table was given showing values of N for $1 \leqq n \leqq 30$.

4. Mr. Whittaker discussed the difficulties insurance companies are having in selecting new men for their actuarial departments who combine mathematical ability with the personal qualities necessary for success in actuarial work. After analyzing the present membership of the Actuarial Society, he discussed the nature of actuarial work by following a student in his course, showing that an actuary is part lawyer, part doctor, and part accountant as well as a mathematician. The successful actuary ends up as a business executive and social engineer rather than as a pure mathematician, and the companies in their initial selection of men are looking primarily for the type of student who will ultimately develop into executive material.

5. Dr. Swensen discussed the First International Congress in the Teaching of Mathematics, held in Rome in 1908, and its effect upon the formation of the Mathematical Association of America in 1915. The first president of the Association, E. R. Hedrick, appointed the National Committee on Mathematical Requirements. Its report appeared in 1923 and has been the basis for most of the curriculum revisions in secondary mathematics in this country since that time. Thus the report made in 1940 by the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics is mainly along the lines of the report of 1923. The same is true of the report of the Committee on the Function of Mathematics in General Education, also made in 1940. The Standing Committee in Mathematics made up of teachers from New York City public high schools issued syllabi in integrated mathematics for the ninth and tenth school years in 1935. These syllabi deviated from the Recommendation of the National Committee of 1923 and included finite differences and coördinate geometry.

6. Professor Davis discussed certain ways of correlating senior high school and college mathematics. From the view-point of high school mathematics, this correlation is chiefly for the information and guidance of the student. From the view-point of college mathematics correlation may be made through generalizations, logical organization of material, and a more comprehensive view of the various branches of mathematics and of their interdependence and applications. Finally, suggestions were made as to how better correlation could be secured.

7. Professor MacDuffee defined two polygons as equivalent if they can be cut into the same finite number of polygons which are congruent in pairs. Equivalence is determinative, reflexive, symmetric, and transitive, so it is an abstract relation of equality. Area is a pure abstraction devised to transform the relation of equivalence into a relation of equality. That is, area is that which equivalent polygons have in common.

8. Professor Walker stated that every major statistical idea has two natures. There is a sensory world of mass phenomena in which variation from individual to individual, from group to group, appears at first to be the one inescapable fact. Underneath its apparent chaos there are discernible patterns, relationships which dominate its swarming inconsistencies, limits to its uncertainties. There is another world of definitions, postulates and proofs, the abstract structures of which match the stochastic world at certain points in much the same way that a road map matches the countryside. The competent research worker must know *both* worlds.

The remarkable discoveries in the physical sciences in recent years have had a supporting base in the widespread popular interest in scientific topics. This may be contrasted with the wide popular misunderstanding of the nature of statistical inference and the purposes of statistical inquiry. Even as great scientists have concerned themselves with the scientific education of the non-scientist, so it is time for the leading statisticians to put serious thought upon the statistical education of the non-statistician.

H. E. WAHLERT, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The eighteenth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska, Lincoln, on May 3, 1941. The chairman of the Section, Professor A. L. Candy, presided.

The attendance was twenty-six, including the following sixteen members of the Association: M. A. Basoco, E. M. Berry, A. K. Bettinger, C. C. Camp, A. L. Candy, H. M. Cox, D. M. Dribin, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, F. S. Harper, E. Marie Hove, J. M. Howie, F. E. Marrin, O. J. Peterson, Lulu L. Runge.

The following officers were elected for the coming year: Chairman, J. M. Earl, University of Omaha; Secretary-Treasurer, Lulu L. Runge, University of Nebraska; Member of Executive Committee, F. S. Harper, University of Nebraska. The Section accepted the invitation to meet at the University of Omaha, May 9, 1942.

The following papers were read:

1. "Sub-correlation in instructional research" by Professor H. M. Cox, University of Nebraska.

2. "On the conformal mapping of the region exterior to two non-intersecting circles" by Professor M. A. Basoco, University of Nebraska.

3. "A rare mathematical book" by Professor C. C. Camp, University of Nebraska.

4. "Globular map projections" by Professor E. M. Berry, State Teachers College, Chadron.

5. "A partition formula" by D. L. Christensen, University of Nebraska, introduced by the Secretary.

6. "Mathematics of finance and actuarial science in accounting courses" by Professor W. A. Dwyer, Creighton University, introduced by the Secretary.

7. "The theory of algebraic functions as an algebraic theory" by Dr. D. M. Dribin, University of Nebraska.

8. "On Euler's critical load for a slender column" by Professor M. A. Basoco, by title.

9. "A theorem on sets of ideals in solvable extensions of algebraic number fields" by Dr. D. M. Dribin, by title.

Abstracts of the papers follow in the order numbered above:

1. Professor Cox applied the principles and processes of the analysis of variance and covariance to an investigation of marks in a freshman course at the University of Nebraska. Charts illustrating the effects of sub-correlation were displayed.

2. Professor Basoco determined an analytic function $\tau = f(z)$ so that the region outside two non-intersecting circles in the z-plane was mapped conformally onto the entire τ-plane, the two circles mapping into two parallel straight line segments. The mapping function was a simple combination of the Weierstrass P and ζ functions.

3. Professor Camp exhibited Peter Gray's *Tables for the formation of Logarithms and Antilogarithms to 24 or any less number of places*. He compared the method employed in this brief table with that of the *Logarithmetica Britannica*, the latest type of table being published now, and showed the advantages and disadvantages of both.

4. Professor Berry compared the two map projections which are called globular, and showed there was a maximum difference of about 2.3 per cent. The general equations for perspective projections were set up and specialized for the globular, stereographic, and other projections.

5. Mr. Christensen developed a formula which expressed the number of partitions of a number with parts of a generalized type as a function of two other partition functions; one, the number partitions of a number with parts as multiples of the generalized type, and the other the number of partitions of a number with a given maximum number of repetitions of parts of the generalized type.

6. Professor Dwyer stated the results of his study of performance in the actuarial science portion of an accounting course at Creighton University of students who have, and those who have not, had mathematics of finance.

7. Dr. Dribin gave an exposition of the algebraic foundation of the theory of algebraic functions, exhibiting the fundamental algebraic nature of this branch of analysis as shown by F. K. Schmidt and others.

8. Professor Basoco's paper has appeared in this MONTHLY, vol. 48, 1941, pp. 303–309.

9. Dr. Dribin proved the following theorem which is important in the theory of solvable class fields: Let π be a set of prime ideals in an algebraic number field k, such that all ideals in π are equivalent (in the broad sense). If K is a normal extension of k with solvable group, and if π^* represents the set of all prime ideals in K which are divisors of ideals in π, then the elements of π^* are all equivalent (in the broad sense).

LULU L. RUNGE, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Benedict, St. Joseph, Minnesota, on Saturday, May 10, 1941. A morning session, held at 10:30 o'clock, was followed by luncheon and an afternoon session at 2:15 o'clock. Professor A. J. Strane of the Duluth Junior College presided at the morning session, and Professor R. W. Brink of the University of Minnesota, National President of this Association, presided at the afternoon session.

Seventy-five persons attended the meeting, including the following thirty-five members of the Association: R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, H. H. Campaigne, Sister M. Claudette, Arthur Danzl, Brother Louis De La Salle, Margaret C. Eide, Gladys Gibbens, Clara L. Hancock, W. L. Hart, H. E. Hartig, W. N. Herr, Dunham Jackson, C. J. Kirchen, W. H. Kirchner, W. R. McEwen, Margaret P. Martin, W. D. Munro, J. M. H. Olmsted, Isaac Opatowski, F. J. Polansky, G. C. Priester, R. B. Saunders, M. G. Scherberg, C. Grace Shover, A. J. Strane, L. W. Swanson, F. J. Taylor, H. P. Thielman, Ella Thorp, H. L. Turrittin, A. L. Underhill, K. W. Wegner; and Sister Thomas à Kempis, institutional member representative.

At the business session officers were elected for the coming year as follows: Chairman, E. J. Camp, Macalester College; Secretary, A. L. Underhill, University of Minnesota; Executive Committee, Arthur Danzl, St. John's University, C. Grace Shover, Carleton College, and Clara L. Hancock, Virginia Junior College.

The following ten papers were presented:

1. "Diffusion processes near a mercury electrode" by Dr. Rolf Landshoff, College of St. Thomas, introduced by Professor Bush.

2. "Some fields of characteristic two" by Dr. J. M. H. Olmsted, University of Minnesota.

3. "On uniform convergence" by Dr. M. G. Scherberg, University of Minnesota.

4. "A descriptive definition of the system of integers" by Professor L. E. Bush, College of St. Thomas.

5. "Note on the convergence of series of Jacobi polynomials" by Professor Dunham Jackson, University of Minnesota.

6. "A generalization of the incomplete Gamma function and the symmetric functions of the theory of equations" by Dr. Isaac Opatowski, University of Minnesota.

7. "Mathematics in national defense" by Professor W. L. Hart, University of Minnesota.

8. "Note on the oscillations of a projectile" by W. D. Munro and L. W. Swanson, University of Minnesota.

9. "An example of the structure of a group" by Dr. H. H. Campaigne, University of Minnesota.

10. "On earthquake waves" by Professor H. P. Thielman, College of St. Thomas.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Simple metal ions in solution are deposited on a fresh mercury surface at a rate which depends on the voltage between mercury and solution. The current voltage relation is determined by the following processes. The ions are carried to the mercury surface by the process of diffusion. At the surface they are discharged and the atoms which are thus formed diffuse into the interior of the mercury. At the surface actually two processes are taking place; the formation of atoms from ions which is proportional to the concentration of ions, and the reverse process which is proportional to the concentration of the atoms. The difference of the two gives the resultant rate. The theoretical treatment leads to the following mathematical problem. Dr. Landshoff gave a description of the equation of diffusion with different diffusion coefficients for the solution and the mercury. At the beginning, the concentration must be uniform throughout the solution and zero in the mercury. At the interface we have the boundary condition

$$D_1 \frac{\partial c_1}{\partial x} = D_2 \frac{\partial c_2}{\partial x} = k_1 \overset{0}{c_1} - k_2 \overset{0}{c_2}.$$

The function $c(x, t)$ is determined by a method described and applied to other boundary conditions in Riemann-Weber, vol. 2, 1927, p. 234. It can be written down explicitly in terms of the error integral.

2. Dr. Olmsted gave examples of finite and infinite fields of characteristic two, showing relations to the solutions of quadratic equations and the game of Nim.

3. Attention was called by Dr. Scherberg to the concept of uniform convergence at a point, which was explicitly formulated by W. H. Young in 1902. It was noted that text-books generally ignore this concept and some even deny that such a concept is possible. $\text{Lim}_{x\to 0}\,(x \sin 1/x)$ was presented as an example showing how uniform convergence at a point differed from the ordinary concept of uniform convergence.

4. Professor Bush proposed the following definition: A *system of integers* is a set $\mathfrak{J}$ satisfying the following postulates:

A. There is a bi-unique correspondence between $\mathfrak{J}$ and itself. If x corresponds to y in this correspondence, we call y the *successor* of x and denote it by x^+, and we call x the *predecessor* of y and denote it by y^-.

B. Set $\mathfrak{J}$ contains a proper non-empty sub-set $\mathfrak{N}$ such that if x is an element of $\mathfrak{N}$, then also x^+ is an element of $\mathfrak{N}$.

C. Set $\mathfrak{J}$ contains no proper non-empty sub-set $\mathfrak{M}$ such that if x is an element of $\mathfrak{M}$, then also x^+ and x^- are both elements of $\mathfrak{M}$.

Professor Bush showed that Postulate A may be broken up into four parts, namely: For each element x of $\mathfrak{J}$, (A1) there exists an element x^+ in $\mathfrak{J}$, (A2) there exists an element x^- in $\mathfrak{J}$, (A3) x^+ is unique, and (A4) x^- is unique. The six postulates, A1, A2, A3, A4, B, and C are shown to be independent. In terms of these postulates we can define addition, multiplication, *etc.*, and from them obtain all the properties of the ordinary rational integers. The existence of a natural number system is assumed.

5. A simple treatment of the convergence of series of orthogonal polynomials is possible if the polynomial of the nth degree in the orthonormal set is bounded as n becomes infinite (see, *e.g.*, this MONTHLY, vol. 34, pp. 527–545). It is known that the Jacobi polynomials possess the requisite property of boundedness in the interior of the interval. A general proof of this fact is not entirely elementary. Professor Jackson pointed out in the present note, however, that the conclusion is readily obtained from first principles if each of the exponents in the weight function, assumed to be algebraically greater than -1, is an integer or half an integer.

6. In a problem of mathematical biology the following integral has occurred:

$$y_n(x) = \int_0^x e^{-k_n x}dx \int_0^x e^{(k_n - k_{n-1})x}dx \int_0^x e^{(k_{n-1}-k_{n-2})x}dx \cdots \int_0^x e^{(k_2-k_1)x}dx,$$

where k_i are n constants. Dr. Opatowski showed that the following expansion, convergent for all finite values of x, holds:

$$y_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(n+i)!} h_i(k_1, k_2, \cdots, k_n)x^{n+i},$$

where $h_0 = 1$ and h_i is, for $i > 0$, the complete homogeneous symmetric function of $k_1, k_2, \cdots, k_n$ of degree i (the aleph function of Wronski; see *Enc. d. math. Wiss.*, vol. I part B, 3b, p. 465). If $k_1 = k_2 = \cdots = k_n = 1$, then

$$y_n(x) = \int_0^x e^{-x}x^{n-1}dx \Big/ \int_0^\infty e^{-x}x^{n-1}dx,$$

which is the incomplete Gamma function of Karl Pearson. (See K. Pearson, *Tables of the Incomplete Gamma Function*, London, 1922.)

7. The content of Professor Hart's paper was similar to that included in *Mathematics in the defense program* by Marston Morse and W. L. Hart, this MONTHLY, vol. 48, 1941, pp. 293–302.

8. In F. R. Moulton's analysis of the motion of a rotating projectile it is important to consider the function $F(w)=(dw/dt)^2$, where $w=\cos\theta$ and θ is the angle of yaw. It is necessary to show that $d^2w/dt^2=\frac{1}{2}[dF(w)/dw]$. This is done as follows:

$$\frac{d^2w}{dt^2}=\frac{d}{dw}(\sqrt{F(w)})\frac{dw}{dt}=\frac{1}{2}\frac{1}{\sqrt{F(w)}}\frac{dF(w)}{dt}\frac{dw}{dt}=\frac{1}{2}\frac{dF(w)}{dw},$$

which breaks down when $F(w)=0$. By a consideration of the equations of motion, Mr. Swanson and Mr. Munro were able to show that the relation can be derived without this restriction.

9. In teaching group theory one sometimes has difficulty in conveying the idea of "structure" of a group. Dr. Campaigne gave the following simple example which illustrates the idea. Consider a function $f_p(n)$ which is defined to have one of the values ± 1 for n an integer, and which is periodic with period p. The set of all such functions with p fixed form a group, G_p, under multiplication. The union of all the groups G_p is an infinite group G. The set of all G_p form a lattice, isomorphic with the lattice of positive integers in which the partial ordering is defined in terms of divisibility. The classes into which the elements of G fall according to various properties can be readily visualized, and in a sense all questions concerning G can be answered. Thus its "structure" can be exhibited. These functions were pointed out to Dr. Campaigne by Mr. Vincent C. Harris of Wells, Lamont, Smith, Corp., who has explicit formulas for some of them.

10. Under certain assumptions Professor Thielman derived the equation of that path of an earthquake wave which would make the time of transmission through the interior of the earth a minimum. The Bateman-Herglotz integral equation was derived, and a method of solving it by means of fractional integration and differentiation was indicated. (See H. P. Thielman, *The application of functional operations to a class of integral equations occurring in physics*, Philosophical Magazine, vol. 11, Suppl. Feb. 1931, pp. 523–535.) Professor Thielman then expressed in polar coördinates the path of the wave in the form of an exponential equation. This equation was shown to satisfy the imposed conditions of the path, and an explicit formula for the velocity of earthquake waves was given in terms of the distance from the center of the earth.

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THE INVERSIVE PLANE

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1. Introduction. The study of inversive geometry in the plane is controlled analytically by a group of transformations which includes all *homographies*,

$$azw + bz + cw + d = 0, \tag{1.1}$$

and all *antigraphies*,

$$az\bar{w} + bz + c\bar{w} + d = 0, \tag{1.2}$$

where, in each, the variables and coefficients are complex quantities and $\bar{w}$, as is customary, indicates the conjugate complex quantity of w.

A homography is involutory if $b=c$ and is then called a *polarity*. An involutory antigraphy is called an *inversion*, and is written

$$a_{11}z\bar{w} + a_{10}z + a_{01}\bar{w} + a_{00} = 0, \tag{1.3}$$

where $a_{ik}=\bar{a}_{ki}$, and a_{ii} is a real quantity. It may be shown that any homography or antigraphy is the resultant, respectively, of an even or odd number of inversions. Thus, homographies alone constitute a sub-group of the inversive group; but antigraphies alone do not form a group [**1**].

The so-called Argand representation of real points (x, y) and (u, v), on the same cartesian coördinate system, by complex numbers $z=x+iy$ and $w=u+iv$, $i=\sqrt{-1}$, enables one to illustrate graphically many analytical results in inversive geometry. Such a representation, however, insofar as it tends to conceal the rôle played by imaginary elements, fails its purpose. Therefore, in view of the importance of the concept of imaginaries in any geometry, the question as to how that concept fits into inversive theory is an interesting one.

It is the purpose of this exposition to answer that question by showing how the concept of imaginaries, in inversive geometry, gives way to the more fundamental concept of a correspondence in the Argand plane.

2. The complex plane. We shall understand the *complex plane* to consist of the totality of points (x_1, x_2, x_3), where the x_i are complex homogeneous coördinates and are not all zero. The complex plane is derived from the real ordinary plane, in which points are in a (1, 1) correspondence with real number-pairs (x, y) of a cartesian coördinate system, by extending the latter to include, first, the imaginary elements, and second, the ideal elements.

If this extension is made in such a way that, when $x_3 \neq 0$,

$$x = x_1/x_3, \qquad y = x_2/x_3,$$

and

$$x_1 : x_2 : x_3 = x : y : 1,$$

the complex plane as now conceived consists of all real and imaginary, ordinary and ideal points (x_1, x_2, x_3), excluding $(0, 0, 0)$. If a point has coördinates x_i pro-

portional to three real numbers, it is a *real ordinary* point if $x_3 \neq 0$, or a *real ideal* point if $x_3=0$; if, on the other hand, its coördinates are not proportional to three real numbers, it is an *imaginary ordinary* point if $x_3 \neq 0$, or an *imaginary ideal* point if $x_3=0$. The conjugate complex point $\overline{P}$ of a point $P(x_1, x_2, x_3)$ has coördinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$; and if a point coincides with its conjugate it is a real point.

The locus of points whose coördinates satisfy a homogeneous linear equation with complex coefficients,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

is a straight line; and the line is *real* or *imaginary* according as the coefficients of its equation are or are not proportional to three real numbers. The locus of all ideal points is a real line, *viz.*, $x_3=0$, called the *ideal line* of the plane; all other lines are *ordinary lines*.

A real line contains a single infinity of real points, but an imaginary line contains one and only one real point. The one real point on an imaginary line is also the one real point on the conjugate imaginary line,

$$\bar{a}_1x_1 + \bar{a}_2x_2 + \bar{a}_3x_3 = 0,$$

and has the coördinates

$$(a_2\bar{a}_3 - a_3\bar{a}_2,\ a_3\bar{a}_1 - a_1\bar{a}_3,\ a_1\bar{a}_2 - a_2\bar{a}_1).$$

An ordinary line (a_1 and a_2 are not both zero) contains one and only one ideal point. The ideal point is found as the intersection of the ordinary line with the ideal line; it has the coördinates $(a_2, -a_1, 0)$, and may be either real or imaginary.

3. The isotropic plane. On every ordinary point (p_1, p_2, p_3), $p_3 \neq 0$, in the complex plane there is a pencil of ordinary lines

$$mx_1 - x_2 - (mp_1 - p_2)x_3/p_3 = 0, \tag{3.1}$$

where the parameter m is, of course, the slope of the line if the line is real. If the line is imaginary, we define m to be its slope.

Consider, now, in this pencil the involutory correspondence $mm'+1=0$, which pairs every line m of the pencil with its perpendicular line m' in the pencil. This involution, therefore, associates with (3.1) the line

$$x_1 + mx_2 - (p_1 + mp_2)x_3/p_3 = 0;$$

and it also defines on the ideal line $x_3=0$ an involution of ideal points in which the point $(1, m, 0)$ corresponds doubly to the point $(m, -1, 0)$. We call this involution of ideal points the *orthogonal involution*, and it is to be noted that the orthogonal involution is independent of the ordinary point (p_1, p_2, p_3). The *fixed*, or *invariant*, elements of any correspondence are those elements which correspond to themselves in the correspondence. Hence, we have the following:

DEFINITION. *The circular points of the complex plane are the fixed, or invariant, points of the orthogonal involution.*

The parameters of the circular points are $m = \pm i$; and we label the points $\Omega(1, i, 0)$ and $\bar{\Omega}(1, -i, 0)$, for they are conjugate imaginary ideal points.

DEFINITION. *An isotropic line, or simply an isotropic, is an imaginary line through a circular point.*

In fact, every line through Ω except the ideal line, which is real, is an isotropic; and similarly for $\bar{\Omega}$. There are, consequently, two pencils of isotropics:

(1) The pencil of isotropics through Ω, called the λ-system, is defined by the equation

$$x_1 + ix_2 - \lambda x_3 = 0, \tag{3.2}$$

which assigns to every isotropic through Ω one and only one (complex) value of λ, and conversely to every λ one and only one isotropic through Ω.

(2) In a similar manner, the μ-system of isotropics on $\bar{\Omega}$ is defined by

$$x_1 - ix_2 - \mu x_3 = 0, \tag{3.3}$$

which sets up a (1, 1) correspondence between the imaginary lines on $\bar{\Omega}$ and the (complex) values of μ.

From this, it follows that there is a (1, 1) correspondence between the ordinary points of the complex plane and the ordered complex number-pairs (λ, μ), where λ and μ are respectively the values assigned to the isotropics, one from the λ-system and one from the μ-system, passing through the corresponding ordinary point. Thus, the complex number-pair (λ, μ) constitutes a set of nonhomogeneous *isotropic* coördinates [2] for the ordinary point (x_1, x_2, x_3); and, if $x_3 \neq 0$,

$$x_1 : x_2 : x_3 = (\lambda + \mu) : i(\mu - \lambda) : 2,$$

$$\lambda = \frac{x_1 + ix_2}{x_3}, \qquad \mu = \frac{x_1 - ix_2}{x_3}.$$

DEFINITION. *The ordinary isotropic plane consists of the totality of ordinary points* (λ, μ).

We observe that the ordinary isotropic plane is coextensive with the ordinary complex plane, and for a real point the isotropic coördinates are conjugate complex numbers.

We introduce now the *double* homogeneous number-pair $(\lambda_1, \lambda_2; \mu_1, \mu_2)$ for the nonhomogeneous coördinates (λ, μ), where

$$\lambda = \lambda_1/\lambda_2, \qquad \mu = \mu_1/\mu_2, \qquad \lambda_2\mu_2 \neq 0.$$

It is evident that $(\rho\lambda_1, \rho\lambda_2; \sigma\mu_1, \sigma\mu_2)$ represents the same point for all values of ρ and σ different from zero.

Furthermore, we replace the condition $\lambda_2\mu_2 \neq 0$ by the agreement that λ_1 and λ_2 shall not both vanish, and similarly for μ_1 and μ_2; and we adjoin to the λ-sys-

tem *one ideal isotropic*, with parameter $\lambda_2=0$, and to the μ-system *one ideal isotropic*, with parameter $\mu_2=0$. The plane is now extended to include the points of the two ideal isotropics; and, under this extension, the ideal isotropic of the λ-system intersects every ordinary isotropic of the μ-system at an *imaginary ideal* point whose coördinates take the form $(1, 0; \mu_1, \mu_2)$, with $\mu_2\neq 0$; and similarly, the ideal isotropic of the μ-system intersects every ordinary isotropic of the λ-system at an *imaginary ideal* point $(\lambda_1, \lambda_2; 1, 0)$, with $\lambda_2\neq 0$. There remains to be considered the intersection of the two ideal isotropics $\lambda_2=0$ and $\mu_2=0$; we define this intersection to be the *real ideal* point $(1, 0; 1, 0)$.

DEFINITION. *The isotropic plane consists of the totality of points* $(\lambda_1, \lambda_2; \mu_1, \mu_2)$, *where neither* λ_1 *and* λ_2 *nor* μ_1 *and* μ_2 *vanish simultaneously.*

We remark, at this point, that the complex and isotropic planes differ fundamentally with respect to their ideal regions. The ideal region of the complex plane consists of one real line and the points on it, whereas the ideal region of the isotropic plane consists of a pair of conjugate imaginary isotropics and the points on them. The real ideal region of the former is a line and the real ideal region of the latter is a point.

4. The inversive plane. Kasner, in 1900, made the first study of inversive geometry from an invariantive point of view [**3**, **4**], in accordance with a principle set forth by Klein in his *Erlanger Programme* of 1872. Proceeding from the known isomorphism of inversive geometry in the plane and projective geometry on a nondegenerate quadric with imaginary rulings, Kasner developed the theory of complex inversive geometry in terms of minimal or, as we call them, isotropic coördinates. In other words, *the plane of complex inversive geometry is the isotropic plane.* In the years following, Morley developed inversive theory with the aid of a coördinate system which has been called "complex," "conjugate," "circular," *etc.* For various reasons we prefer the last designation.

In the Argand plane an ordered pair of real points, (x, y) and (u, v), is represented by two complex numbers $z=x+iy$ and $w=u+iv$, in the order indicated. Thus, z and w, in that order, constitute the circular coördinates of the *ordered real point-pair* or, to be more concise, the *couple* $[z, w]$. The same two points in the opposite order shall be called the *conjugate couple* $[w, z]$ to $[z, w]$; and if $w=z$, the couple $[z, z]$ is *self-conjugate* and represents a single real point with the single circular coördinate z.

Circular coördinates are not isotropic coördinates but there is an intimate connection between them, and also a confusion sometimes as to their significance. Isotropic coördinates, to mention the most obvious difference, represent complex (imaginary and real) points and circular coördinates represent real (distinct and coincident) pairs of points. On the other hand, the complex inversive plane, *i.e.*, the isotropic plane, is connected with the essentially real plane of circular coördinates, *i.e.*, the Argand plane, by a $(1, 1)$ correspondence between the complex points of the former and real couples of the latter. We exhibit this $(1, 1)$ correspondence as follows:

In the isotropic plane the equation $\lambda=\lambda'+i\lambda''$, where λ' and λ'' are real constants, or $\alpha_1\lambda_1+\alpha_2\lambda_2=0$ in homogeneous form, is the equation of an isotropic of the λ-system. If $\alpha_1\neq 0$, the isotropic is ordinary and contains one and only one real ordinary point, whose nonhomogeneous coördinates (x, y) satisfy equation (3.2), *i.e.*,

$$x + iy = \lambda' + i\lambda''.$$

Let us represent this real ordinary point by the circular coördinate $z=\lambda'+i\lambda''=-\alpha_2/\alpha_1$. If $\alpha_1=0$, the isotropic is ideal and contains the real ideal point, which has no nonhomogeneous coördinates. Let us represent this point by the symbol $z=\infty$, read "z equals infinity"; and let us further agree to add this symbol to the complex number system according to the usual conventions for combining it with other complex numbers α, *viz.*, $\alpha\pm\infty=\infty$, $\alpha/0=\infty$, $\alpha/\infty=0$, *etc.* Conversely, corresponding to every point of the Argand plane, extended to include one ideal point $z=\infty$, there is one and only one isotropic of the λ-system.

In an entirely similar manner, every isotropic of the μ-system, $\mu=\mu'+i\mu''$ or $\beta_1\mu_1+\beta_2\mu_2=0$, contains one and only one real point. If $\beta_1\neq 0$, the real point is ordinary and its nonhomogeneous coördinates (u, v) satisfy equation (3.3),

$$u - iv = \mu' + i\mu''.$$

We represent this point (u, v) by the circular coördinate $w=\mu'-i\mu''=-\bar{\beta}_2/\bar{\beta}_1$. And if $\beta_1=0$, the isotropic is ideal and contains the real ideal point $w=\infty$. Conversely, to every point of the extended Argand plane there corresponds one and only one isotropic of the μ-system.

Thus we establish a (1, 1) correspondence between *complex points* $(\lambda_1, \lambda_2; \mu_1, \mu_2)$ of the isotropic plane and *real couples* $[z, w]$ of the Argand plane, where $z=\lambda_1/\lambda_2$ if $\lambda_2\neq 0$ and $z=\infty$ if $\lambda_2=0$, and similarly, $w=\bar{\mu}_1/\bar{\mu}_2$ if $\mu_2\neq 0$ and $w=\infty$ if $\mu_2=0$. Reciprocally,

(1) If z and w are two ordinary points, the couple $[z, w]$ represents the complex ordinary point $P(z, 1; \bar{w}, 1)$. If $w=z$, the point $P(z, 1; \bar{z}, 1)$ is real. If $w\neq z$, P is imaginary; and, moreover, the conjugate couple $[w, z]$ represents the conjugate point $\bar{P}(w, 1; \bar{z}, 1)$.

(2) If z is an ordinary point, the couple $[z, \infty]$ represents the imaginary ideal point $P(z, 1; 1, 0)$ on the ideal isotropic of the μ-system; and the conjugate couple $[\infty, z]$ represents the conjugate imaginary point $\bar{P}(1, 0; \bar{z}, 1)$ on the ideal isotropic of the λ-system.

(3) The self-conjugate couple $[\infty, \infty]$ represents the real ideal point $(1, 0; 1, 0)$.

5. (n, n) **correspondences.** A functional relation between two complex variables z and w may be *direct*, such as $F(z^m, w^n)=0$, or *indirect*, such as $F(z^m, \bar{w}^n)=0$. In either type, m and n represent the degrees of $F=0$ considered, respectively, as a polynomial in z and in w. Thus, in the Argand plane each type sets up an (m, n) correspondence between the points of the plane such that to every point z (or w) there correspond n (or m) points w (or z) of the same plane.

We restrict our discussion to the equiordinal correspondences for which $m=n$; and then $F(z, w)=0$ and $F(z, \bar{w})=0$ define (n, n) correspondences between points of the plane. In either type the coefficients are complex; and so, for example, we indicate by $\overline{F}(z, w)=0$ the equation whose coefficients are respectively the conjugate complex numbers of the coefficients of $F(z, w)=0$, and by $\overline{F}(\bar{z}, \bar{w})=0$ the *conjugate complex equation* of $F(z, w)=0$.

a. Direct type, $F(z, w)=0$. The (n, n) correspondence thus defined assigns to every point z (or w) n *image points* w (or z). A point z and an image point w constitute a couple $[z, w]$ which *satisfies* the equation $F(z, w)=0$. If w is an image of z *and also* z is an image of w, the couple $[z, w]$ and its conjugate $[w, z]$ both satisfy $F=0$. Hence, we have the following:

DEFINITION. *A point z and an image point w are doubly corresponding and $[z, w]$ is an invariant or fixed couple if and only if the conjugate couple $[w, z]$ also satisfies the equation of the correspondence.*

A point z of a fixed couple $[z, w]$ will satisfy an equation of order $2n^2$ in z, obtained by eliminating w from $F(z, w)=0$ and $F(w, z)=0$. For, unless it vanishes identically, the eliminant of two nth order polynomials in w is homogeneous of order $2n$ in the coefficients which, in turn, are each of order n in the variable z. Hence, there are $2n^2$ fixed couples some of which may be self-conjugate. But if the eliminant should vanish identically then, of course, every couple which satisfies $F(z, w)=0$ is a fixed couple.

DEFINITION. *An invariant or fixed point is a self-conjugate fixed couple.*

Thus, a fixed point is a point z which coincides with an image point. It satisfies $F(z, z)=0$, a polynomial of order $2n$. Consequently, if there are $2n^2$ fixed couples, $2n$ of them will be self-conjugate, *i.e.*, fixed points, and the remaining $2n(n-1)$ will be conjugate in pairs. Therefore, the correspondence $F(z, w)=0$ will have $n(n-1)$ pairs of doubly corresponding points, or every pair of corresponding points will be doubly corresponding. In the latter case, the correspondence is involutory in character.

DEFINITION. *A singular point is a point which has less than n distinct image points.*

The singular points z (or w) are found by requiring that the discriminant of $F(z, w)=0$, considered as a polynomial in w (or z) should vanish. There are, consequently, $2n(n-1)$ singular points z, and an equal number of singular points w.

DEFINITION. *A singular couple is a pair of singular points in a definite order.*

Example. $n=1$. The simplest functional relation of the direct type is the homography (1.1). Eliminating w from (1.1) and

$$awz + bw + cz + d = 0,$$

it is evident that

(1) the homography has two fixed points given by

$$az^2 + (b + c)z + d = 0;$$

(2) if $b=c$, the homography is a polarity, *i.e.*, every couple which satisfies it is a fixed couple.

b. Indirect type, $F(z, \bar{w})=0$. As above, this defines an (n, n) correspondence assigning to every point z (or w) n images w (or z). A pair of points, say $z=a'+ia''$ and $w=b'+ib''$, is a pair of corresponding points if and only if $F(a'+ia'', b'-ib'')$ vanishes; and we emphasize that w, and not $\bar{w}$, is the image of z. Furthermore, we define the terms fixed couple, fixed point, singular point, and singular couple for the indirect type in the same words as used for the direct type. It will be convenient, at times, to write the correspondence in the form

$$F(z, \bar{w}) \equiv \sum_{i,k=0}^{n} a_{ik}z^i\bar{w}^k = 0;$$

and, at other times, we use the so-called matrix form

$$F(z, \bar{w}) = \begin{array}{c|ccccc} & 1 & z & z^2 & \cdots & z^n \\ \hline 1 & a_{00} & a_{10} & a_{20} & \cdots & a_{n0} \\ \bar{w} & a_{01} & a_{11} & a_{21} & \cdots & a_{n1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{w}^n & a_{0n} & a_{1n} & a_{2n} & \cdots & a_{nn} \end{array} = 0,$$

where the determinant $|a_{ik}|$ is an invariant under inversive transformations [5].

The point z of a fixed couple $[z, w]$ satisfies the eliminant (as to $\bar{w}$) of $F(z, \bar{w})=0$ and $F(w, \bar{z})=0$ or, what is the same since every complex equation implies its conjugate, the eliminant of $F(z, \bar{w})=0$ and $\bar{F}(\bar{w}, z)=0$. This eliminant, if it does not vanish identically, is of order $2n^2$ in z. There are, therefore, $2n^2$ fixed couples and/or fixed points. If the eliminant vanishes identically, every couple which satisfies $F(z, \bar{w})=0$ is a fixed couple and the fixed points, if any, satisfy the equation $F(z, \bar{z})=0$.

If the equation of the correspondence is separated into its "real" and "imaginary" parts, *i.e.*,

$$F(z, \bar{w}) \equiv Q(x, y; u, v) + iR(x, y; u, v) = 0,$$

we see that $F=0$, in complex variables with complex coefficients, is equivalent to $Q=0$ and $R=0$, two equations in real variables with real coefficients. These equations, for a given point (x, y), determine the image points (u, v), and reciprocally.

We also observe that the loci

$$Q(x, y; x, y) = 0 \quad \text{and} \quad R(x, y; x, y) = 0,$$

if they intersect, will intersect at the fixed points of the correspondence; but if

they do not intersect, the correspondence has no fixed points. In the latter case, however, the imaginary points (conjugate in pairs) common to the two loci are represented by the fixed couples (conjugate in pairs) of the correspondence $F=0$.

Example. $n=1$. For $n=1$ we have the antigraphy (1.2). Upon eliminating $\bar{w}$ from (1.2) and

$$\bar{a}\bar{w}z + \bar{b}\bar{w} + \bar{c}z + \bar{d} = 0,$$

we have point z of the fixed couple $[z, w]$ given by the quadratic

$$(a\bar{c} - \bar{a}b)z^2 + (a\bar{d} - \bar{a}d + c\bar{c} - b\bar{b})z + (c\bar{d} - \bar{b}d) = 0.$$

There are several cases to be considered:

(1) If the quadratic has roots r_1 and r_2, and if $[r_1, r_2]$ satisfies (1.2), the antigraphy has $[r_1, r_2]$ and its conjugate $[r_2, r_1]$ as fixed couples.

(2) If the roots r_1 and r_2 are such that $[r_1, r_1]$ and $[r_2, r_2]$ both satisfy (1.2), the antigraphy has r_1 and r_2 as fixed points.

(3) If the coefficients of the quadratic all vanish or, what is equivalent, if a and d are reals and $c=\bar{b}$, every couple which satisfies (1.2) is a fixed couple. The antigraphy then is the inversion (1.3).

6. The self-conjugate equation. If an equation of the indirect type is separated into "real" and "imaginary" parts,

$$F(z, \bar{w}) \equiv Q(x, y; u, v) + iR(x, y; u, v) = 0,$$

the locus of fixed points $F(z, \bar{z})=0$ is equivalent to

(1) the real equation $Q(x, y; x, y)=0$, if R vanishes identically when $u=x$ and $v=y$; or,

(2) the real equation $R(x, y; x, y)=0$, if Q vanishes identically when $u=x$ and $v=y$.

DEFINITION. *The equation $F(z, \bar{z})=0$ is self-conjugate, i.e., real, if and only if $F(z, \bar{z}) \equiv \pm \bar{F}(\bar{z}, z)$.*

The *image equation* $F(z, \bar{w})=0$ is said to be *self-conjugate* if and only if $F(z, \bar{z})=0$ is self-conjugate.

From this it follows that if

$$F(z, \bar{z}) \equiv \sum_{i,k=0}^{n} a_{ik}z^i\bar{z}^k = 0$$

is self-conjugate, then $\bar{a}_{ik} = \pm a_{ki}$, which implies that a_{ii} is either real or pure imaginary (*i.e.*, a real multiplied by i). Moreover, $F(z, \bar{w})=0$ implies

$$\bar{F}(\bar{z}, w) \equiv \sum_{i,k=0}^{n} \bar{a}_{ik}\bar{z}^i w^k = \pm \sum_{i,k=0}^{n} a_{ki}w^k\bar{z}^i = \pm F(w, \bar{z}) = 0;$$

and, therefore, every couple $[z, w]$ which satisfies the image equation $F(z, \bar{w})=0$ is a fixed couple since $F(w, \bar{z})=0$ also.

In the Argand plane a self-conjugate equation in z and $\bar{z}$ is real and its locus is the locus (with or without a real trace) of fixed points of an indirect (n, n) correspondence. In this correspondence every pair of corresponding points z and w is a pair of doubly corresponding points, *i.e.*, $[z, w]$ and its conjugate $[w, z]$ are fixed couples of $F(z, \bar{w})=0$. In the complex inversive plane, these fixed couples represent a pair of conjugate complex points on the locus $F(z, \bar{w})=0$; and these points are imaginary and distinct if $w \neq z$, real and coincident if $w=z$.

The number n associated with the locus $F(z, \bar{w})=0$ is invariant under all inversive transformations of the plane. We define it, therefore, to be the *inversive degree* of the curve. A curve of inversive degree n is called a *bi-n-ic*.

To summarize, the *general* indirect (n, n) correspondence $F(z, \bar{w})=0$ is the complex equation of a complex locus in complex inversive geometry. Every couple $[z, w]$ which satisfies $F=0$ represents a complex point on the locus. The fixed couples of the correspondence represent pairs of conjugate complex points on the locus; and the fixed points, if any, are real points on the locus. However, if $F(z, \bar{w})=0$ is self-conjugate, it is the locus of pairs of conjugate complex points $[z, w]$ and $[w, z]$ and/or, when $w=z$, of real points. The locus of these real points, which are the fixed points of the correspondence, has a real equation but it may or may not have a real trace.

The study of curve theory in complex inversive geometry is not properly to be restricted to the study of the self-conjugate equation. However, because of limitations of space, we shall so restrict our discussion in what follows.

7. Circles and biquadratics. As illustrations of the preceding section we consider curves of inversive degrees one and two.

a. The circle. $n=1$. The simplest locus in the inversive curve theory is that defined by the self-conjugate equation (1.3) of the inversion

$$a_{11}z\bar{w} + a_{10}z + a_{01}\bar{w} + a_{00} = 0.$$

This equation is *bilinear* in z and $\bar{w}$ and is the image equation of

$$a_{11}z\bar{z} + a_{10}z + a_{01}\bar{z} + a_{00} = 0, \tag{7.1}$$

the equation of the locus of fixed points, if any, of the inversion. If $a_{11}\neq 0$, equation (7.1) may be put in the form

$$(z + a_{01}/a_{11})(\bar{z} + a_{10}/a_{11}) = (a_{10}a_{01} - a_{11}a_{00})/a_{11}^2. \tag{7.2}$$

A couple $[z, w]$ which satisfies (1.3) represents, in the complex inversive plane, a point on the bilinear curve (1.3). If $w=z$, this point is real. Therefore, when $a_{11}\neq 0$ the real points on the locus are, by (7.2), equidistant from the point $-a_{01}/a_{11}$, and the real trace (if any) of the bilinear is none other than the familiar circle of the real ordinary plane.

Consequently, we borrow the term "circle" from real ordinary geometry and herewith, by definition, make it synonymous with the term "bilinear" in inver-

sive geometry. The *real circle*, defined by the self-conjugate equation (1.3), as an inversive locus includes the curves for which $a_{11}=0$. If $a_{11}=0$, the circle contains the ideal point and always has a real trace. If $a_{11}\neq 0$, the circle has a real trace, has no real trace, or is *null* according as $(a_{10}a_{01}-a_{11}a_{00})$ is greater than, less than, or equal to zero.

If the couple $[z, w]$ satisfies the equation (1.3) of the circle, we know it is a fixed couple. Hence, w is the image of z and z is the image of w in the inversion (1.3). We define z and w to be ***inverse points with respect to the circle.***

b. The biquadratic. $n=2$. For $n=2$, the self-conjugate equation in z and $\bar{z}$ is the equation of the locus of fixed points of the (2, 2) correspondence defined by

$$\sum_{i,k=0}^{2} a_{ik}z^i\bar{w}^k = 0. \tag{7.3}$$

A couple $[z, w]$ which satisfies (7.3) represents, in the complex inversive plane, a complex point on the *real biquadratic* (7.3); in the Argand plane, it is simply a pair of image points with respect to the biquadratic.

A biquadratic and a circle have certain couples in common. These are found by the elimination of $\bar{w}$ from the equations of the two curves. The eliminant is a quartic in z, which gives four points z_i of the common couples $[z_i, w_i]$. In each common couple, w_i is the inverse of z_i with respect to the circle and an image of z_i with respect to the biquadratic. There are, therefore, four common couples and, since the equations of the curves are self-conjugate, the common couples may be conjugate in pairs and/or self-conjugate. The self-conjugate couples represent real points of intersection of the two curves.

In general, a circle determines with a bi-n-ic $2n$ common couples and/or points; and conversely, a curve which determines with a circle $2n$ couples and/or points is a bi-n-ic. A bi-n-ic and a bi-m-ic have $2mn$ common couples and/or points.

A biquadratic has four singular points z_i and four singular points w_j. Since its equation is self-conjugate, the z_i set and the w_j set consist of the same four points, say s_i. Combined, these give sixteen singular couples $[s_i, s_j]$, four of which, *viz.*, $[s_i, s_i]$, are self-conjugate and represent real points in the plane. Hence, we have the following:

DEFINITION. *A focus of a biquadratic is a point* $[s_i, s_i]$ *whose images coincide.*

We remark that the singular couples $[s_i, s_j]$, for $j\neq i$, and their conjugates represent, in the complex inversive plane, twelve imaginary points which are conjugate in pairs. In fact, any singular couple $[s_i, s_j]$ represents a point of intersection of two isotropics, with parameters $\lambda=s_i$ and $\mu=\bar{s}_j$, which are tangents to the curve. Hence, this point is a complex focus in accordance with Plücker's definition [6]. In particular, if $j=i$, then $\bar{\mu}=\lambda$, and the point of intersection of isotropic tangents (the Plücker focus) is real and is precisely the inversive focus as defined above.

There are other singularities of an inversive curve, but we shall merely mention them at this time:

(1) If two foci coincide at point N, then N is a *node* of the curve. A node may be a point where the curve cuts itself, or it may be isolated.

(2) If a focus coincides with its coincident images at point C, then C is a *cusp* and is a point on the curve. If $n=2$, a cusp is equivalent to three coincident foci; but if $n>2$, a focus which coincides with an image is a point on the curve but is not a cusp unless the image coincident with it is the pair of coincident images.

(3) If a focus S has a repeated image which is also a focus S', then S and S' are singularities of still higher order.

8. Conclusion. Historically, the failure to make a sharp distinction between projective and inversive terminology, if indeed any distinction was ever made, was a handicap in understanding the fundamentals of inversive geometry. For example, the P-circle, *i.e.*, the "projective circle," defined as a conic through the circular points, is not the same as the I-circle, *i.e.*, the "inversive circle"; for some of the latter enjoy properties analogous to those of P-lines. Similarly, the class of P-bicircular quartics, defined as quartics with a pair of imaginary nodes at the circular points, is not identical with the class of I-biquadratics; for this latter includes curves analogous to the P-circular cubics, P-conics, P-cartesians, *etc.*

The distinctive terminologies are not merely different ways of saying the same thing; but are, in fact, consequences of the different definitions of the associated geometries. The idea contained in the *Erlanger Programme*, mentioned earlier, *viz.*, that a geometry is defined by the controlling group of transformations, was a fruitful idea for the study of higher geometry over a long period of time; and an understanding of it is still necessary for a real appreciation of the meaning of a geometric terminology.

With this in mind, therefore, we observe in closing that a nodal biquadratic is analogous to a central conic, ellipse or hyperbola according as the node is isolated or not, or any inverse of a central conic; a cuspidal biquadratic is analogous to a parabola, or any inverse of a parabola; and a biquadratic with two nodes breaks down into a pair of circles, with the nodes appearing as the common couples and/or the common points of the two circles.

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2. W. C. Graustein, Introduction to Higher Geometry, New York, 1930.

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5. Morley and Patterson, On algebraic inversive invariants, American Journal of Mathematics, vol. 52, 1930, pp. 413–424.

6. B. C. Patterson, Projective Geometry, New York, 1937, Chap. XV.

A SELECTED LIST OF MATHEMATICS BOOKS FOR COLLEGES

There are frequent requests for lists of mathematics books suitable for college libraries. Such a list was published in this MONTHLY in 1917 by the Library Committee of the Mathematical Association of America.* A second list was published here in 1925 by members of the mathematics department of the University of California.† Beginning in the December, 1937, issue of the MONTHLY, the department of Mathematics Clubs published from time to time‡ short lists of books which were considered of special interest to Clubs, including a list of bibliographies (vol. 44, p. 656).

In 1937, Miss Margaret Shields of the Princeton University Library, in collaboration with Dr. J. F. Randolph and other mathematicians at Princeton, prepared a selected check list of mathematics books for college libraries. The bibliography presented below is based on this Princeton list, with additions suggested by mathematicians from many institutions; in particular, the list has been brought up to date by mathematicians at Northwestern University.

It is to be noted that this bibliography does *not* include standard textbooks in high school or junior college mathematics, textbooks in algebra, trigonometry, analytic geometry, differential and integral calculus, or actuarial science. Nor does it include books given in the 1925 list in this MONTHLY, except where those books have been extensively revised. And it is to be noted also that the bibliography is not complete, but merely a selected list which is deemed generally useful for intermediate work, including senior college and early graduate years. Starred titles are intended to designate the more advanced treatises.

For the convenience of individuals who wish to fit a list to a budget, presumptive prices have usually been included, but these prices are not guaranteed. For imported books, it is often more convenient to order through American dealers than to order directly from foreign publishers.

The abbreviated form in which publishers are indicated should cause no difficulty, since full names and addresses are given in the *United States Catalog* and the *Trade List Annual* which are available in bookshops and libraries.

GENERAL

Norman Alliston. *Mathematical Snack Bar.* Heffer, Cambridge, 1936. $3.00.

R. C. Archibald. *Outline of the History of Mathematics.* 5th edition. Math. Assoc., 1941. $0.75.

Felix Auerbach. *Lebendige Mathematik.* Hirt, 1929.

W. W. R. Ball. *Mathematical Recreations and Essays.* 11th edition. Revised by H. S. M. Coxeter, 1939. Macmillan, 1940. $2.60.

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* Vol. 24, pp. 368–376. Members of the Committee were: Florian Cajori, E. S. Crowley, Solomon Lefschetz, W. R. Longley, R. E. Root, and W. B. Ford. The list included 160 titles, classified as books (1) for freshmen, (2) for sophomores, (3) for juniors, and (4) for seniors and first-year graduate students. The report contained brief statements regarding many of the books.

† Vol. 32, pp. 462–468. The list was prepared by B. A. Bernstein, Florian Cajori, E. R. Hedrick, C. A. Noble, and T. M. Putnam. The list included 150 titles, some of which had been given in the earlier list (vol. 24, pp. 368–376). They were classified as (1) elementary and (2) advanced.

‡ Vol. 44, p. 656; vol. 45, pp. 44, 183, 245, 317, 385, 688. These lists were presented by Professor and Mrs. F. W. Owens, editors of the department.

E. T. Bell. *The Development of Mathematics.* McGraw, 1940. $4.50.
——— *The Search for Truth.* Reynal, 1934. $3.00.
——— *The Queen of the Sciences.* Stechert, 1938. $1.50.
——— *The Handmaiden of the Sciences.* Williams, 1937. $2.00.
——— *Men of Mathematics.* Simon, 1937. $5.00.
M. Black. *Nature of Mathematics.* Harcourt, 1934. $3.50.
H. R. Cooley, David Gans, Morris Kline, and H. E. Wahlert. *Introduction to Mathematics.* Houghton, 1937. $3.25.
Richard Courant and Herbert Robbins. *What is Mathematics?* Oxford, 1941. $3.75.
Tobias Dantzig. *Aspects of Science.* Macmillan, 1937. $3.00.
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B. Datta and A. V. Singh. *History of Hindu Mathematics—A Source Book.* Part 1. Stechert, 1935. $3.00.
E. J. Dijksterhuis. *Archimedes.* Noordhoff, 1938.
H. Dörrie. *Triumph der Mathematik: Hundert Berühmte Probleme aus Zwei Jahrtausend.* Hirt, 1933. $2.70.
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G. W. Dunnington. *Carl Fredrich Gauss—Inaugural Lecture on Astronomy and Papers on Foundations of Mathematics.* Trans. and edited by Dunnington. Louisiana Univ. Press, 1937. $1.00.
G. H. Hardy. *A Mathematician's Apology.* Cambridge Univ. Press, 1941. $1.00.
*David Hilbert and Paul Bernays. *Grundlagen der Mathematik.* Springer, 1934. (*Grundlehren der Mathematischen Wissenschaften*, vol. 40.) $11.30.
L. T. Hogben. *Mathematics for the Million.* Norton, 1937. $3.75.
Edward Kasner and James Newman. *Mathematics and the Imagination.* Simon, 1940. $2.75.
C. J. Keyser. *Portraits of Famous Philosophers Who Were Also Mathematicians—with Biographical Notes.* 12 folders. Scripta Mathematica, 1939. $3.00.
———, D. E. Smith, Edward Kasner, and Walter Rautenstrauch. *Scripta Mathematica Forum Lectures.* Yeshiva College, New York, 1937. $1.00.
Felix Klein. *Elementary Mathematics from an Advanced Standpoint.* Part I. *Arithmetic, Algebra, Analysis.* Macmillan, 1932. $3.50. Part II. *Geometry.* Macmillan, 1939. $3.50. Part III. *Präzisions- und Approximationsmathematik.* Springer, 1928.
G. W. H. Kowalewski. *Grosse Mathemaitker.* Lehmann, 1938.
*C. I. Lewis and C. H. Langford. *Symbolic Logic.* Appleton, 1934. $5.00.
Mayme I. Logsdon. *A Mathematician Explains.* Univ. of Chicago Press, 1936. $1.75.
L. T. More. *Isaac Newton, a Biography.* Scribner's, 1934. $4.50.
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G. Prasad. *Some Great Mathematicians of the 19th Century.* Benares Mathematical Society, 1933–34. 2 vols. $5.25.
W. V. O. Quine. *Mathematical Logic.* Norton, 1940. $4.00.
Hans Rademacher and Otto Toeplitz. *Von Zahlen und Figuren.* Springer, 1930.
*Bertrand Russell. *Principles of Mathematics.* Norton, 1938. $5.00.
Vera Sanford. *Short History of Mathematics.* Houghton, 1930. $3.25.
George Sarton. *Study of the History of Mathematics.* Harvard Univ. Press, 1936. $1.50.
Semicentennial Publications of the Amer. Math. Soc. 1938. 2 vols.
Vol. 1. R. C. Archibald. *A Semicentennial History of the Mathematical Society.* $3.00.
Vol. 2. *Semicentennial Addresses.* $4.50.
D. E. Smith. *Portraits of Eminent Mathematicians with Brief Biographical Sketches.* Portfolio 1, 12 portraits; portfolio 2, 13 portraits. Scripta Mathematica, 1936. $2.50, $3.00.
——— *Source Book in Mathematics.* McGraw, 1929. $5.00.
——— and Jekuthiel Ginsburg. *History of Mathematics in America before 1900.* Open Court, 1934. (*Carus Monograph*, no. 5.) $2.00.

David Eugene Smith Presentation Volume. *Osiris. Studies on the History and Philosophy of Science, and on the History of Learning and Culture.* Bruges, Belgium, 1936. $6.00.
J. W. U. Sullivan. *Isaac Newton.* Macmillan, 1938. $2.50.
Alfred Tarski. *Einführung in die Mathematische Logik.* Springer, 1937. $2.25.
Friedrich Waismann. *Einführung in das Mathematische Denken.* Gerold, 1936. $2.25.
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A. C. Aitken. *Determinants and Matrices.* Oliver, 1939.
A. A. Albert. *Introduction to Algebraic Theories.* Univ. of Chicago Press, 1940. $1.75.
*——— *Structures of Algebras.* 1939. (*A. M. S. Colloquium Publications*, no. 24.) $4.00.
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Garrett Birkhoff. *Lattice Theory.* 1940. (*A. M. S. Colloquium Publications*, no. 25.) $2.50.
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R. D. Carmichael. *Introduction to the Theory of Groups of Finite Order.* Ginn, 1937. $5.00.
E. J. Cartan. *Théorie des Groupes Finis et Continus et l'Analysis Situs.* Gauthier, 1930. (*Mémorial des Sciences Mathématiques*, vol. 42.) $0.60 unbound.
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E. J. M.

PYTHAGOREAN POINTS LYING IN A PLANE

I. A. BARNETT, University of Cincinnati, and
C. W. MENDEL, University of Illinois

Consider the system of Diophantine equations

$$x^2 + y^2 = z^2, \qquad Ax + By + Cz + D = 0, \tag{1}$$

where the integers A, B, C are relatively prime. Geometrically, we are seeking the lattice points common to the cone and the plane given by the equations (1). In Section 2 we dispose of the case $D=0$ and the results agree with those which would be arrived at geometrically. If the plane does not pass through the origin, a variety of possibilities may arise. If the section cut out is an ellipse, there are, of course, only a finite number of lattice points, possibly none. If the section is a parabola, then it is shown in Section 3 that there are always an infinitude of lattice points. If the section is a hyperbola with rational asymptotes,* there are only a finite number of solutions, possibly none (Section 5); but, if the asymptotes are not rational, then there is either no lattice point or an infinitude of lattice points (Section 6).

1. Analytical statement of the problem. Any three integers x, y, z satisfying the system (1) will be called a solution of the problem. If we were to restrict our attention to the case where x, y, z were all positive, then all the solutions of the equation $x^2+y^2=z^2$ would be given by

$$x = \frac{\rho}{2}(u^2 - v^2), \qquad y = \rho uv, \qquad z = \frac{\rho}{2}(u^2 + v^2), \tag{2}$$

where u and v are two arbitrary relatively prime positive integers with $u>v>0$ and ρ is an arbitrary positive integer subject to the restriction that ρ is even if the product uv is even.† If, however, we desire all solutions of $x^2+y^2=z^2$ admitting all possible integral values for the unknowns, we readily see that they are given by (2), subject to the restrictions

$$\begin{cases} \rho,\ u,\ v \text{ arbitrary integers with } u \text{ and } v \text{ relatively prime,} \\ \rho \text{ is even when the product } uv \text{ is even.} \end{cases} \tag{3}$$

Substituting the expressions (2) in the equation $Ax+By+Cz+D=0$ we obtain

$$\tfrac{1}{2}(C + A)\rho u^2 + B\rho uv + \tfrac{1}{2}(C - A)\rho v^2 + D = 0. \tag{4}$$

From the form of the original system (1), it is clear that we may expect a sharp distinction between the homogeneous case $D=0$ and the non-homogeneous case $D\neq 0$. For, if (ξ, η, ζ) is one solution of the system (1), then $(k\xi, k\eta, k\zeta)$ will also be a solution for every integer k in the homogeneous case and for no integer k,

* A straight line is said to be rational if it is the intersection of two planes whose equations have integral coefficients.

† L. E. Dickson, Introduction to the Theory of Numbers, Theorem 40, p. 41.

other than $k=1$, in the non-homogeneous case. For this reason we shall consider two solutions (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) of the system (1) as equivalent if each is an integral multiple of a third solution.

2. The given plane contains the origin. In this case equation (4) becomes

$$(C + A)u^2 + 2Buv + (C - A)v^2 = 0. \tag{5}$$

A necessary and sufficient condition that the quadratic equation (5) be solvable in integers is that the discriminant be a perfect square. Hence, we must have

$$A^2 + B^2 - C^2 = E^2, \tag{6}$$

where E is a positive integer or zero. The solutions u and v of (5) will then satisfy the relation

$$u(C + A) = v(- B \pm E),$$

and we may take $u=-B\pm E$, $v=C+A$. While it is true that these numbers u and v may not be relatively prime, nevertheless, in view of our definition of equivalent solutions, it suffices to take $u=-B\pm E$, $v=C+A$, $\rho=2$ in the formulas (2) and thus obtain two solutions. Every other solution will be equivalent to one or the other of these two. In case $E=0$, the two solutions coincide and all solutions are equivalent.

Summarizing the results in the case $D=0$, we see that *when $A^2+B^2-C^2$ is a perfect non-zero square, then there will always be two solutions to which all others are equivalent*; *if $A^2+B^2-C^2$ is zero,* there will be one such solution*; *and in all other cases, no solution other than the trivial one* (0, 0, 0).

3. The given plane does not contain the origin. If† $C+A\neq 0$, equation (4) is equivalent to the equation

$$[(C + A)u + Bv]^2 - \Delta v^2 = d(C + A), \tag{7}$$

where

$$\Delta = A^2 + B^2 - C^2, \qquad d = - 2D/\rho. \tag{8}$$

We see at once from equation (4) that ρ must be a divisor of $2D$. We shall find it convenient to subdivide the discussion of equation (7) into several sub-cases.

(a) $\Delta<0$. In this case there are at most a finite number of solutions, possibly none, since the form defined by the left member of (7) is definite. An example in which there are no solutions is given by the system

$$x^2 + y^2 = z^2, \qquad x + y + 5z = 1,$$

so that equation (7) becomes $(6u+v)^2+23v^2=12/\rho$, and it is clear that this last equation has no solution no matter what integral value ρ may assume.

* In this case the plane is tangent to the cone.

† If $C+A=0$, a parallel discussion may be made by multiplying equation (4) by $C-A$. If both $C+A$ and $C-A$ vanish, the second equation of the system (1) reduces to $y=$const. and the problem can be treated directly.

(*b*) $\Delta=0$. We shall prove that in this case *there is always an infinitude of solutions.* Since the triple of integers A, B, C satisfies the relation $A^2+B^2=C^2$, these integers may be represented by

$$(9) \qquad A = \tfrac{1}{2}\sigma(m^2 - n^2), \qquad B = \sigma mn, \qquad C = \tfrac{1}{2}\sigma(m^2 + n^2),$$

where m and n are relatively prime and $|\sigma| = 1$ or 2 according as the product mn is odd or even (since the g.c.d. of A, B, C must be unity).

In view of equations (9), equation (7) becomes

$$m^2\sigma^2(mu + nv)^2 = - 2D\sigma m^2/\rho.$$

If* $C+A\neq 0$ so that $m\neq 0$, we may write the last equation in the form

$$(10) \qquad (mu + nv)^2 = - 2D/\rho\sigma.$$

Let $-2D/\sigma=kl^2$, where k contains no square factor >1, and choose $\rho=kt^2$, where t is a divisor of l. Then equation (10) becomes

$$(11) \qquad mu + nv = r,$$

where $r=l/t$. Since m and n are relatively prime, there exist two relatively prime integers α and β for which we have the relation

$$(12) \qquad m\alpha + n\beta = 1.$$

The general solution of (11) is given by

$$(13) \qquad u = r\alpha + \lambda n, \qquad v = r\beta - \lambda m,$$

where λ is an arbitrary integer. If we solve equations (13) for r and λ we find, in view of (12),

$$(14) \qquad r = mu + nv, \qquad \lambda = \beta u - \alpha v.$$

From equations (13) and (14) we see that the g.c.d. of u and v is equal to the g.c.d. of r and λ. Since λ is arbitrary, we may choose it to be relatively prime to r. Thus, for each such choice of λ we obtain a relatively prime pair (u, v) which when substituted in (2) yields a solution of the original system (1), provided only that we do not have ρ odd and the product uv even simultaneously. A careful analysis of the parities of the various integers which enter into the problem shows that only in the case B even does this provision necessitate any restrictions on ρ and λ. In that case, if D also is even, ρ must be even; if D is odd, ρ is necessarily odd and we must employ only values of λ opposite in parity to the product $\alpha\beta$ defined by the relation (12).

As an illustration of the procedure, consider the system

$$x^2 + y^2 = z^2, \qquad 3x + 4y + 5z = 1.$$

Here $m=2$, $n=1$, $\sigma=2$, $k=l=1$, $r=t=\pm 1$, $\rho=1$. For the case $t=1$ equation (12) becomes $2\alpha+\beta=1$, of which a solution is $\alpha=0$, $\beta=1$ so that by (13) $u=\lambda$

* If $C+A=0$, equation (10) may be obtained directly from (1) and (2).

and $v=1-2\lambda$. Since in this case B is even and D odd, λ must be opposite in parity to the product $\alpha\beta$ and hence odd. The formulas (2) yield

$$x = -2s(1+3s), \qquad y = -(1+2s)(1+4s), \qquad z = 1+6s+10s^2,$$

where λ has been replaced by $2s+1$. These formulas for x, y, z, together with those obtained for $t=-1$, give us the complete solution of the system with which we started.

4. Some remarks on the case $\Delta>0$. The general equation of the type

$$w^2 - \Delta v^2 = L, \tag{15}$$

where $\Delta>0$ has been solved by Lagrange.* In Section 6 we shall have occasion to refer to his method, but for the present we observe that equation (7) which has the form

$$w^2 - \Delta v^2 = -2D(C+A)/\rho, \tag{16}$$

where $w=(C+A)u+Bv$, is for each integer ρ of the type (15). Of course, we must restrict ourselves to those integers ρ which are divisors of $2D$.

We see at once that necessary conditions for an integral solution (w, v) of equation (15) are that each of the numbers Δ and L be a quadratic residue† of the other. As applied to the equation (16) this says that we must use for ρ only such divisors (positive or negative) of $2D$ as will make

(*i*) Δ a quadratic residue of $-2D(C+A)/\rho$,

(*ii*) $-2D(C+A)/\rho$ a quadratic residue of Δ.

It is always possible to choose a ρ to satisfy the first condition, but not necessarily the second. Clearly, the choice $\rho=2D$ will suffice for (*i*). For, recalling that $\Delta=A^2+B^2-C^2$, we see that $\Delta\equiv B^2 \pmod{C+A}$, and hence Δ is a quadratic residue of $A+C$. More precisely, the condition imposed on ρ by (*i*) is that it must contain every divisor of $2D$ for which Δ is a quadratic non-residue. Employing the values of ρ which are thus found to be admissible, we get for the quantity $-2D(C+A)/\rho$ certain values $L_1, L_2, \cdots, L_k$ for each of which Δ will be a quadratic residue.

A practical procedure for the application of condition (*ii*) is to employ sequentially the various prime divisors of Δ and delete from the set $L_1, L_2, \cdots, L_k$ any integer which is a quadratic non-residue of one or more of these primes. If Δ is divisible by 4, one must also delete from the L's all integers of the form $4m+3$, since such integers are quadratic non-residues modulo 4. Finally, if Δ and L_i contain a common divisor p which occurs exactly e times in Δ and f times in L_i, with $f\leqq e$, then a solution of the congruence $x^2\equiv p^f L_i' \pmod{p^e}$, where $L_i=p^f L_i'$, can exist only if f is even. This condition may throw out others of

* Oswald's Klassiker der Exacten Wissenschaften No. 146.

† We shall say that r is a quadratic residue of s if there exists an integer x such that x^2-r is divisible by s. In all other cases, r will be called a non-residue of s. In the case where r and s are relatively prime, this agrees with the usual terminology.

the set $L_1, L_2, \cdots, L_k$. It may, of course, turn out that there will be no integer left, in which case there would not exist a solution of equation (16). An illustration of this is given by Example 1 below. The conditions (*i*) and (*ii*), however, are not sufficient to insure a solution of equation (16), as is seen from Example 2 below.

We must observe, however, that the existence of one or more solutions of equation (16) is still not sufficient to insure the existence of solutions of the original system (1). There still remain two restrictions on the u and v entering in the formulas (2). First, we have the congruential condition

(*iii*) $w-Bv\equiv 0 \pmod{C+A}$,

which follows from the relation $w=(C+A)u+Bv$. The second restriction is one of parity, *viz.*,

(*iv*) if ρ is odd, both u and v must be odd,

which is a restatement of the condition (3) of Section 1. The examples in the next section will illustrate how the restrictions (*iii*) and (*iv*) may again prevent us from reaching a solution of the system (1).

The following two examples show how (*i*) and (*ii*) may be applied.

Example 1. Consider the system of equations

$$x^2 + y^2 = z^2, \qquad 6x + 4y + z = 1.$$

Here equation (16) takes the form $w^2-51v^2=14/\rho$, where $w=7u+4v$. Since $D=-1, \rho=\pm 1, \pm 2$. We note that 51 is a quadratic residue of 2 and 7, so that no restriction on ρ arises from the condition (*i*). Hence, $L_1=7$, $L_2=-7$, $L_3=14$, $L_4=-14$. We may readily verify that no one of these four numbers is a quadratic residue of 17, so that for no value of ρ is the condition (*ii*) fulfilled, and hence, there exists no integral solution of the system with which we started.

Example 2. Let the system (1) be given by

$$x^2 + y^2 = z^2, \qquad 3x + 3y - z = 1.$$

Equation (16) now takes the form $w^2-17v^2=4/\rho$, where $w=2u+3v$, and since $D=-1, \rho=\pm 1, \pm 2$. It is readily seen that conditions (*i*) and (*ii*) are satisfied for these four values of ρ and, in fact, we have the solution (66, 16) for $\rho=1$. However, we shall show that there exist no solutions of the system of equations given above. For $\rho=\pm 1$, u and v must both be odd so that w and v must both be odd. Taking residues of $w^2-17v^2=4/\rho$ with $\rho=\pm 1$, we find that $w^2-v^2\equiv \pm 4 \pmod 8$, and this last congruence is impossible with w and v odd. Next, let $\rho=\pm 2$ and we have $w^2-v^2\equiv \pm 2 \pmod 4$ which is again impossible no matter what integers w and v are used. This example also shows that there exist equations of the type (15) possessing no solutions, *viz.*, $w^2-17v^2=\pm 2$.

5. The case Δ a square. This case, which is readily disposed of, is that of a hyperbolic section with rational asymptotes. Equation (16) becomes

$$[(C + A)u + (B + E)v][(C + A)u + (B - E)v] = - 2D(C + A)/\rho, \tag{17}$$

where we have set $\Delta = E^2$. Clearly, this equation can have only a finite number of solutions for u and v, possibly none. One has only to solve the various simultaneous linear systems obtained by setting the two factors in the left member of (17) equal to two complementary divisors of the right member, using only such values of ρ as are admissible by (*i*) and (*ii*). In the two examples that follow there exist solutions of (17), but none for the system (1), due to the restrictions (*iii*) and (*iv*).

Example 1. Consider the system

$$x^2 + y^2 = z^2, \qquad x + 46y + 34z - 89 = 0.$$

We find that $E = 31$ and equation (16) becomes $w^2 - 31^2v^2 = 2\cdot 89\cdot 35/\rho$. The values of ρ to be tested for the restrictions (*ii*) are $\rho = \pm 1, \pm 2, \pm 89, \pm 178$ (condition (*i*) is automatically satisfied). We find, after a short computation, that the values of ρ that need to be considered are $\rho = -1, -2, 89, 178$. A further study of the equation $w^2 - 31^2v^2 = 2\cdot 89\cdot 35/\rho$ shows that only for $\rho = -2$ is there a solution. In this case, $(w, v) = (27, 2)$, for example, satisfies the equation. Equation (17) becomes $(7u + 3v)(5u + 11v) = 2\cdot 89/\rho$, and it may readily be shown that even in the case $\rho = -2$ the last equation has no solution, so that there are no solutions of the system above.

Example 2. For the system

$$x^2 + y^2 = z^2, \qquad 57x + y - 55z = 1,$$

equation (17) becomes $(2u - 14v)(2u + 16v) = 4/\rho$ and it is clear that $\rho = \pm 1$. It is readily observed that the only solution of this last equation has $v = 0$, so that the product uv is even and since ρ is odd, the condition (*iv*) is violated and there is no solution of the system.

6. The case Δ positive and not a square. Before proceeding with the discussion of equation (16) for this case, we shall find it convenient to give one result from Lagrange's work, which will be found pertinent to our problem. Suppose that p and q are two given, relatively prime integers which satisfy the equation (15), so that

$$p^2 - \Delta q^2 = L. \tag{18}$$

Also, let (s, t) be a solution of the ordinary Pell equation

$$\xi^2 - \Delta \eta^2 = 1. \tag{19}$$

Then, it may be readily verified that

$$w = ps + \Delta qt, \qquad v = pt + qs, \tag{20}$$

is also a solution of (15). But equation (19) is known* to possess an infinitude of solutions given by

$$s_n + t_n\sqrt{\Delta} = (S + T\sqrt{\Delta})^n, \qquad (n = 0, 1, 2, \cdots), \tag{21}$$

* Dickson, Introduction to the Theory of Numbers, p. 113.

where the pair (S, T) is supposed the least positive solution of equation (19). It follows from this that equation (15) has either no solution or an infinitude of solutions. It must not be supposed, however, that all solutions of equation (15) are given by formulas* (20). We shall also find it convenient to have the recursion formulas

$$(22) \qquad s_{n+2} = (S^2 + \Delta T^2)s_n + 2ST\Delta t_n, \quad t_{n+2} = (S^2 + \Delta T^2)t_n + 2STs_n,$$

which are derived from the formulas (21).

We are now in a position to prove the following result. *If Δ is positive and not a square, then the system of Diophantine equations* (1) *has either no solution or an infinitude of solutions.* The result will be established if we show that, starting with a solution of equation (16) satisfying conditions (*iii*) and (*iv*), we obtain an infinitude of solutions of (16) also satisfying these same conditions. Identifying the w, v, and $-2D(C+A)/\rho$ of (16) with the p, q, and L of (18), we are given that $p - Bq \equiv 0 \pmod{C+A}$. Hence, since $\Delta = A^2 + B^2 - C^2$, we have by (20),

$$(23) \qquad \begin{aligned} w - Bv &= ps + \Delta qt - B(pt + qs) \\ &= (p - Bq)(s - Bt) - (C^2 - A^2)qt \equiv 0 \pmod{C + A}, \end{aligned}$$

so that all solutions (w, v) of (16) given by (21) satisfy the property (*iii*).

Finally, we are given that corresponding to the solution $(w, v) = (p, q)$ of equation (16), the integers $u = (p - Bq)/(C+A)$ and $v = q$ satisfy the parity condition (*iv*). This of course is no restriction unless ρ is odd, in which case the product uv must be odd. We must now show that the product of the two integers $(w - Bv)/(C+A)$ and v corresponding to an infinitude of solutions of (16) given by (20), must also be odd in case ρ is odd. We shall show in fact that all solutions given by (20), with (s, t) given by (21) and with *even* n, will fulfill this last requirement. If we observe that $S^2 + \Delta T^2 \equiv S^2 - \Delta T^2 = 1 \pmod 2$, we see from equation (22) that

$$s_{n+2} \equiv s_n, \qquad t_{n+2} \equiv t_n \pmod 2,$$

and, hence, that

$$(24) \qquad s_{2n} \equiv 1, \qquad t_{2n} \equiv 0 \pmod 2,$$

since $s_0 = 1$, $t_0 = 0$ by (21). Considering now only solutions (w, v) given by formulas (20) and (21) with n even, we have by (23) and (24),

$$\frac{w - Bv}{C + A} \equiv \frac{p - Bq}{C + A}, \qquad v \equiv q \pmod 2.$$

Hence, if the product of the two integers on the right side of these congruences is odd, so also is the product of the two on the left, and thus the parity condition (*iv*) is satisfied for this infinitude of solutions.

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* Oswald's Klassiker, p. 74.

COMPUTATION OF FLAT TRAJECTORIES WITH HIGH ANGLES OF DEPARTURE

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1. Introduction. The purpose of this note is to set forth a method of computing the normal trajectory of a projectile. The method seems particularly applicable to flat trajectories, whether or not they are level.

Like the methods of Siacci and Popoff,* the present method uses successive approximations and exhibits the time, position, and velocity of the projectile as functions of a "pseudo-velocity" w. However, for rising trajectories our w is a much closer estimate of the velocity v than is Siacci's, or Popoff's in his first method (*loc. cit.*, pp. 11–19). In Popoff's generalized method (*loc. cit.*, pp. 19–29) the estimate of w can be as good as ours; in fact, ours was suggested by his. But for this case we obtain simpler formulas of solution than Popoff did. An important feature of the present method is that we do not use the crude estimate ρ_0 for the density ρ; we first obtain, by a single quadrature, a closer approximation which is then used in computing the time, position, and velocity.

L. S. Dederick, of the Ballistic Laboratory at the Aberdeen Proving Ground, was so kind as to read the first draft of the manuscript, and made several valuable remarks. In particular, he pointed out the improvement by Hitchcock and Kent of Siacci's method, which since then has been published.† I believe that the present method is capable of giving more accurate approximations than that of Hitchcock and Kent, although I have not established this belief by computation. But, as Dederick remarked, the method of Hitchcock and Kent is superior to mine in convenience, since at least a part of the work in their method can be based on tables dependent only on the drag function and not on the trajectory, while the present method calls for special quadratures for each trajectory. I publish this note in the hope that some reader may be able to devise a method retaining the best features of both types of solution.

The notation will be that used by Dederick in the article cited.

2. The equations of motion. If we choose a coördinate system with origin at the muzzle, the X-axis horizontal in the plane of the initial velocity of the projectile, and the Y-axis vertically upward, the equations of motion are‡

$$\begin{aligned} X'' &= -EX', \\ Y'' &= -EY' - g, \end{aligned} \tag{1}$$

where primes denote time derivatives, g is the gravitational acceleration, and

$$E = G(v)H(Y)/C; \tag{2}$$

* K. Popoff, Les méthodes d'intégration de Poincaré et le problème général de la balistique extérieur, Mémorial de l'Artillerie Française, tome V (1^{er} fascicule de 1926), pp. 3–71.

† L. S. Dederick, The mathematics of exterior ballistic computations, this MONTHLY, vol. 47, 1940, pp. 628–634.

‡ Dederick, *loc. cit.*

here $G(v)$ is an experimentally determined function of the velocity v, H is the ratio of air density at altitude Y to air density at sea level, and C is a constant, the *ballistic coefficient.*

It is convenient, though by no means essential, to change to another system of coördinates. We choose the x-axis in the direction of the initial velocity, and the y-axis at an angle ϕ with the x-axis and below it. If the x- and y-axes make the respective angles α, β with the horizontal, so that $\alpha-\beta=\phi$, we readily compute

$$
\begin{aligned}
X &= x \cos \alpha + y \cos \beta, \\
Y &= x \sin \alpha + y \sin \beta,
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
x &= (- X \sin \beta + Y \cos \beta) \csc \phi, \\
y &= (X \sin \alpha - Y \cos \alpha) \csc \phi.
\end{aligned} \tag{4}
$$

From equations (1) we deduce

$$
\begin{aligned}
x'' &= - Ex' - g \cos \beta \csc \phi, \\
y'' &= - Ey' + g \cos \alpha \csc \phi,
\end{aligned} \tag{5}
$$

or

$$
\begin{aligned}
dv_x/dt &= - Ev_x - g \cos \beta \csc \phi, \\
dv_y/dt &= - Ev_y + g \cos \alpha \csc \phi.
\end{aligned} \tag{6}
$$

3. A linear approximation for the velocity. Let k be a positive number. If $\mathbf{v}$ is a vector joining the origin to a point of the line $x+y=k$ with x and y non-negative, we readily find that the length v of $\mathbf{v}$ is at most k and at least $k \cos (\phi/2)$. Thus if we restrict our attention to an arc of trajectory along which the tangent turns through not more than ϕ, we will have

$$
(v_x + v_y) \cos (\phi/2) \leqq v \leqq v_x + v_y. \tag{7}
$$

The estimate v_x+v_y therefore approximates closely to v if ϕ is not large; on the arcs mentioned, the proportionate error is at most $1-\cos (\phi/2)$, and since the error is zero when the motion is parallel to the y-axis we can extend the arc somewhat beyond this without much error.

4. Rough approximation to $H(Y)$. The approximation to the velocity in §3 is quite good if ϕ is small; for instance, if ϕ is 8°, along an arc with change of direction 11° the error in the approximation is less than one per cent. If the trajectory is not level, a much greater error is introduced by the approximation $H(Y)\equiv 1$ (as in the original Siacci method and the Popoff method). Hence we shall devise a better approximation.

Suppose that we are given a crude approximation to $H(Y)$ as a function of v_x+v_y; for instance, $H(Y)\equiv 1$. We denote this crude approximation by $h_0(v_x+v_y)$, and introduce the abbreviations

$$w = v_x + v_y, \tag{8}$$

$$a = Cg \csc \phi(\cos \beta - \cos \alpha). \tag{9}$$

By adding the equations (6) member by member and recalling (2), we obtain

$$dw/dt = - C^{-1}\{G(v)H(Y)w + a\}. \tag{10}$$

From this we wish to deduce an approximation to the time-function $t(w)$, which for each value of w expresses the time after firing at which v_x+v_y assumes the value w. We replace (10) by an approximate form, by substituting w for v and $h_0(w)$ for $H(Y)$. The solution of the approximation to (10) is not $t(w)$, but is an approximation $t_0(w)$, and we see readily that

$$\frac{dt_0}{dw} = \frac{-C}{a + wG(w)h_0(w)}. \tag{11}$$

Recall that the initial values of the variables are

$$t = 0, \quad x = 0, \quad y = 0, \quad v_x = v_0, \quad v_y = 0, \quad w = v_0, \tag{12}$$

the direction of the initial velocity being along the x-axis. The sum $x+y$, as a function of w, satisfies the equation

$$x + y = \int_0^t (x' + y')dt = \int_{v_0}^w [v_x(w) + v_y(w)]\frac{dt}{dw}dw = \int_{v_0}^w w(dt/dw)dw. \tag{13}$$

If we replace dt/dw by the approximation (11), we obtain an approximation $x_0(w)+y_0(w)$ for $x+y$, namely,

$$x_0(w) + y_0(w) = \int_{v_0}^w \frac{-Cw\,dw}{a + wG(w)h_0(w)}. \tag{14}$$

We suppose this computed by a mechanical quadrature.

If x and y were known, formula (3) would give us Y. But we know only an approximation for $x+y$. So we replace formula (3) for Y by an approximate form, substituting $\sin \alpha$ for $\sin \beta$. Since we have only changed the coefficient of y by the amount $\sin \beta - \sin \alpha$, and the initial values of y and v_y are zero, this is a good approximation for a few seconds. We assume the standard structure of the atmosphere,

$$H(Y) = \exp(-\lambda Y), \tag{15}$$

where λ is a known constant.

This yields our new approximation for H,

$$H \approx h_1(w) = \exp\left[\lambda C \sin \alpha \int_{v_0}^w \frac{w\,dw}{a + wG(w)h_0(w)}\right]. \tag{16}$$

5. Remarks on the approximation to $H(Y)$. As remarked in §4, for $h_0(w)$ we can choose 1. This, however, is rather crude. An improvement could be made with little effort. For instance, if a trajectory has already been computed for some projectile with ballistic constant not vastly different from C, and with initial velocity and quadrant angle of departure somewhere near v_0 and α, respectively, we can use the relative density H on this trajectory as our choice of $h_0(w)$, not bothering to keep more than say two significant figures. Or, choosing first $h_0(w)\equiv 1$, we can obtain $h_1(w)$ by (16), and then repeat the computation with $h_1(w)$ in place of $h_0(w)$. This first integration in (16) should not be carried beyond two significant figures.

However, even if we use (16) to obtain $h_1(w)$, then with h_1 in place of h_0 use (16) to obtain a new estimate $h_2(w)$, and so on, it is still not true that the sequence of functions $h_n(w)$ will converge to the true value. There is a small error introduced by our having replaced $G(v)$ by $G(w)$, and a larger one introduced by our replacing $\sin\beta$ by $\sin\alpha$ in the formula (3) for Y. It is therefore of some interest to observe that from rather crude heuristic considerations we may expect that at any given time, the vertical distance of the projectile below its original line of flight does not change much with change of quadrant angle of departure. Dederick has been so kind as to furnish me with the data of certain trajectories for the 37 mm. anti-aircraft gun. For this I find that the vertical drop below the initial line of flight is given approximately by the formula

$$\Delta(t) = \tfrac{1}{2}gt^2/(1 + .035t/C), \tag{17}$$

where C is the ordinary ballistic coefficient (in inch units). This gives the drop with a maximum error of 3 meters at time five seconds, 19 meters at time ten seconds, and 44 meters at time fifteen seconds. For certain other trajectories, given in Report No. 114 of the Ballistic Laboratory, Aberdeen Proving Ground, formula (17) is fairly accurate.

In terms of rectangular coördinates the drop is, recalling (4),

$$\Delta = X\tan\alpha - Y = (X\sin\alpha - Y\cos\alpha)\sec\alpha = y\sin\phi\sec\alpha. \tag{18}$$

Therefore

$$\begin{aligned} x\sin\alpha + y\sin\beta &= (x+y)\sin\alpha - y(\sin\alpha - \sin\beta) \\ &= (x+y)\sin\alpha - \Delta\csc\phi\cos\alpha(\sin\alpha-\sin\beta) \\ &= (x+y)\sin\alpha - \Delta\cos\alpha\cos\tfrac{1}{2}(\alpha+\beta)/\cos\tfrac{1}{2}\phi. \end{aligned} \tag{19}$$

Suppose then that we have a first approximation $h_0(w)$ to $H(Y)$. Presumably it will be worth while to have a better approximation than the crude one $h_0\equiv 1$; we could, for instance, first apply the method of §4 with $h_0=1$, finding $h_1(w)$ to at most two or three significant figures, and then start afresh with this new function as our $h_0(w)$. From (11) we obtain by integration the function $t_0(w)$, to say two significant figures. With this and (17) we find $\Delta(t)$. By (14) we compute $x_0(w)+y_0(w)$, say to three significant figures. Now by (19) we can replace the approximation (16) by the better one

$$H \approx h_1(w) = \exp\left[\lambda\left\{C \sin\alpha \int_{v_0}^{w} \frac{w\,dw}{a + wG(w)h_0(w)} + \frac{\Delta(t(w))\cos\alpha\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}\phi}\right\}\right]. \tag{20}$$

A slight variant on this procedure would be to form a rough estimate of Δ as a function of w by using a previously computed trajectory for some projectile with ballistic coefficient and initial velocity not too widely different from C and v_0, respectively.

6. Approximation to the trajectory. Let us first write equations (6) in the form

$$\begin{aligned} \frac{dv_x}{dw} &= (-Ev_x - g\cos\beta\csc\phi)\frac{dt}{dw}, \\ \frac{dv_y}{dw} &= (-Ev_y + g\cos\alpha\csc\phi)\frac{dt}{dw}, \end{aligned} \tag{21}$$

in order that w shall be visibly exhibited as independent variable. If we use the approximation w for v and the approximation $h_1(w)$ for $H(Y)$, the solutions of these new equations (with (8)) will be functions $t_1(w)$, $v_{x1}(w)$, $v_{y1}(w)$ which should be good approximations for the solutions $t(w)$, $v_x(w)$, $v_y(w)$ of equations (21). The new equations are

$$\begin{aligned} \frac{dv_{x1}}{dw} &= (-E_1v_{x1} - g\cos\beta\csc\phi)\frac{dt_1}{dw}, \\ \frac{dv_{y1}}{dw} &= (-E_1v_{y1} + g\cos\alpha\csc\phi)\frac{dt_1}{dw}, \\ w &= v_{x1} + v_{y1}, \end{aligned} \tag{22}$$

where

$$E_1(w) = G(w)h_1(w)/C. \tag{23}$$

By adding the first two of these equations member by member, and using the third of them together with (9) and (23), we obtain

$$\frac{dt_1}{dw} = \frac{-C}{wG(w)h_1(w) + a}, \tag{24}$$

so that

$$t_1(w) = \int_{v_0}^{w} \frac{-C\,dw}{wGh_1 + a}. \tag{25}$$

From (22), (23), and (24) we obtain

$$\frac{dv_{x1}}{dw} = \frac{Gh_1}{wGh_1 + a} v_{x1} + \frac{Cg \cos \beta}{\sin \phi(wGh_1 + a)}, \tag{26}$$

$$\frac{dv_{y1}}{dw} = \frac{Gh_1}{wGh_1 + a} v_{y1} - \frac{Cg \cos \alpha}{\sin \phi(wGh_1 + a)}. \tag{27}$$

In order to solve these equations, we define a new function,

$$\Lambda(w) = \exp \int_{v_0}^{w} \frac{Gh_1 dw}{wGh_1 + a}, \tag{28}$$

whose derivative is easily seen to satisfy the equation

$$\Lambda'(w) = \Lambda(w) \cdot \frac{Gh_1}{wGh_1 + a}. \tag{29}$$

With the help of (29), we readily verify the identity

$$\frac{d}{dw}\left(\frac{w}{a\Lambda}\right) = \frac{1}{(wGh_1 + a)\Lambda}. \tag{30}$$

The last two equations enable us to show, by substitution, that the solutions of (26) and (27) are

$$v_{x1} = \frac{Cg \cos \beta}{a \sin \phi} w + c_1\Lambda(w), \tag{31}$$

$$v_{y1} = \frac{-Cg \cos \alpha}{a \sin \phi} w + c_2\Lambda(w), \tag{32}$$

where c_1 and c_2 are constants. These constants we evaluate by means of the initial conditions (12). We find that

$$v_{x1} = \frac{Cg \cos \beta}{a \sin \phi} w + v_0\left(1 - \frac{Cg \cos \beta}{a \sin \phi}\right)\Lambda(w), \tag{33}$$

$$v_{y1} = -\frac{Cg \cos \alpha}{a \sin \phi} [w - v_0\Lambda(w)]. \tag{34}$$

Having these solutions for v_x and v_y as functions of w, we can find x and y by means of a quadrature in the following way. Equation (25) determines w as a function $w_1(t)$ of the time. Accordingly,

$$v_{x1} = \frac{dx_1}{dt} = \frac{dx_1}{dw}\cdot\frac{dw_1}{dt} = \frac{dx_1}{dw} \Big/ \frac{dt_1}{dw},$$

whence

$$x_1(w) = \int_{v_0}^{w} \frac{dx_1}{dw} dw = \int_{v_0}^{w} v_{x1}(w) \frac{dt_1}{dw} dw. \tag{35}$$

This, with (24) and (33), determines $x_1(w)$ by a quadrature. In a like manner,

$$y_1(w) = \int_{v_0}^{w} v_{y1}(w) \frac{dt_1}{dw} dw. \tag{36}$$

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

TWO ISOPERIMETRIC PROBLEMS

M. S. KNEBELMAN, State College of Washington

In a paper entitled *Ein isoperimetrisches Problem*,* G. Bol proved that a curve of given perimeter and prescribed "corners" enclosing the greatest area is obtained from a circle by replacing an arc by a pair of tangents forming each corner, his method of proof being based on Minkowski's inequality for mixed areas. In this note we show that this result is half of a pair of dual theorems, and our proof involves only freshman mathematics.

1. Polygons with prescribed sides. In this section we shall prove the companion to Bol's theorem. Suppose we have given n (straight) rods of lengths $a_1, a_2, \cdots, a_n$; let $\sum_1^n a_i = 2p$ and impose the condition that $a_i < p$, that is, that each rod is shorter then the sum of the remaining ones.

THEOREM 1. *The polygon of greatest area with prescribed sides is cyclic. The area and the radius of the circumscribing circle are independent of the order of the sides.*

Proof. If $n=3$ there is no question of maximum, and since any triangle is cyclic the theorem is obviously true. For $n=4$ let $0<\omega<2\pi$ be the sum of either pair of opposite angles; the area S of the quadrilateral is given by†

$$S^2 = (p - a_1)(p - a_2)(p - a_3)(p - a_4) - a_1a_2a_3a_4 \cos^2 \omega/2.$$

Obviously for maximum S, $\cos \omega/2 = 0$ and $\omega = \pi$, from which it follows that the quadrilateral is inscribable in a circle.

The theorem for any value of n may now be proved by induction; assume it to be true for polygons of n sides and consider a polygon of $n+1$ given sides. Let $P_1, P_2, \cdots, P_{n+1}$ be its vertices; if this polygon is not cyclic its area may be

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* Niew Arch. Wiskde., vol. 20, 1940, pp. 171–175.

† *Cf.* any old-fashioned trigonometry book.

increased by keeping, say, $P_1P_2P_3$ fixed and making the polygon of n sides $P_1P_3 \cdots P_{n+1}$ cyclic by our assumption. If P_2 is not on this circle we keep, say, $P_1P_{n+1}P_n$ fixed and make the polygon cyclic, thus increasing the area further. This may be kept up as long as the polygon of $n+1$ sides is not cyclic, and consequently our proposition is true for $n+1$ sides if it is true for n. Being true for $n=4$, it is true for $n=5$, *etc.*, and so it is true in general, which proves the first part of Theorem 1.

To prove the second part of the theorem we need only observe that the interchange of any two sides may be accomplished by successive interchanges of consecutive sides, and since the area and circumcircle of a triangle are unaltered when two sides are interchanged, the polygon and its circumcircle are also unaltered. This completes the proof of our theorem.

This result has an interesting immediate generalization; given a number of rods and a number of strings, the greatest area is enclosed by them when the strings are arcs of a circle and the rods are chords of the circle. The order in which the rods and strings are joined is immaterial. It may also be observed that the rods need not be straight; the same result holds if the rods are simple arcs of plane curves.

2. Polygons with prescribed angles. We suppose now that we are to construct a polygon of n vertices with angles $\alpha_1, \alpha_2, \cdots, \alpha_n$, where $\sum_1^n \alpha_i = (n-2)\pi$, $0 < \alpha_i < \pi$, and with prescribed perimeter $2p$ enclosing the greatest area.

THEOREM 2. *The angles and perimeter of a convex polygon being given, the greatest area is enclosed by it when the polygon circumscribes a circle. The order in which the angles are placed is immaterial.*

Proof. If $n=3$ there is no question of maximum, since a circle may be inscribed in any triangle. For $n=4$ we first consider the case when the two pairs of opposite sides are parallel. If α is one of the angles, a and c are a pair of adjacent sides, and S is its area, we have $2S = ac \sin \alpha$, $a+c=p$, so that

$$2S = a(p-a)\sin\alpha = -a^2 \sin\alpha + ap\sin\alpha.$$

This quadratic function has a maximum for $a = p/2 = c$, and the parallelogram is a rhombus and therefore circumscribes a circle. Hence, we consider the case when opposite sides are not parallel for at least one pair (Fig. 1). By the law of sines we have

$$(1)\qquad 2S\sin\theta = a^2 \sin\alpha \sin\beta - b^2 \sin\gamma \sin\delta,$$

$$2p = a + b + a\,\frac{\sin\alpha + \sin\beta}{\sin\theta} - b\,\frac{\sin\gamma + \sin\delta}{\sin\theta},$$

while

$$\theta = \pi - (\alpha+\beta) = (\gamma+\delta) - \pi.$$

A little trigonometric manipulation gives

$$p \sin \frac{\theta}{2} = a \cos \frac{\alpha}{2} \cos \frac{\beta}{2} - b \cos \frac{\gamma}{2} \cos \frac{\delta}{2} \cdot \tag{2}$$

Substituting the value of b from (2) into (1) we find

$$\frac{8S \sin \theta \cos^2 \frac{\gamma}{2} \cos^2 \frac{\delta}{2}}{\prod \sin \alpha} = -a^2 \cot \frac{\alpha+\beta}{2} \sum \cot \frac{\alpha}{2} + 2ap \frac{\cos \frac{\alpha+\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}} - 4p^2 \frac{\sin^2 \frac{\theta}{2}}{\sin \alpha \sin \beta}, \tag{3}$$

where $\prod \sin \alpha$ is the product of the sines of the four angles and $\sum \cot \alpha/2$ is the

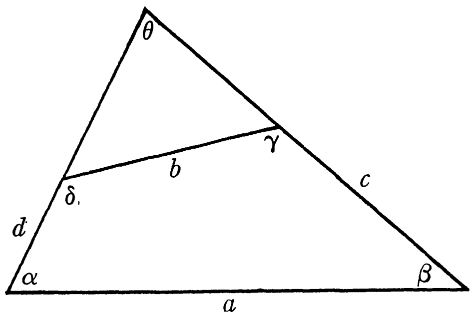


FIG. 1

sum of the cotangents of the four half-angles. From (3) we see that S has its maximum value for

$$a = p \frac{\sin \frac{\alpha+\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sum \cot \frac{\alpha}{2}} = p \frac{\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}}{\sum \cot \frac{\alpha}{2}} \cdot \tag{4}$$

Changing the notation so that the angle at vertex i is α_i and the side joining the vertices i and j is a_{ij}, a little calculation shows that

$$a_{ij} = p \frac{\cot \frac{\alpha_i}{2} + \cot \frac{\alpha_j}{2}}{\sum \cot \frac{\alpha}{2}} \cdot \tag{5}$$

Now if we construct a circle tangent to d, b, and c, we find its radius r to be $p/\sum \cot \alpha/2$, and the circle tangent to d, a, c, has the same radius. Therefore, since the sides d and c are not parallel, it must be the same circle and our quadrangle circumscribes a circle. This proves Theorem 2 for $n=4$.

For general n we use induction; we suppose the theorem true for a given n and consider a polygon of $n+1$ angles $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$. Let $p_1, p_2, \cdots, p_{n+1}$ be its sides; we keep the sides, say, p_1, p_2, p_3 fixed and consider the polygon* of n angles formed by $p_1, p_3, \cdots, p_{n+1}$ (Fig. 2). The area of this polygon will

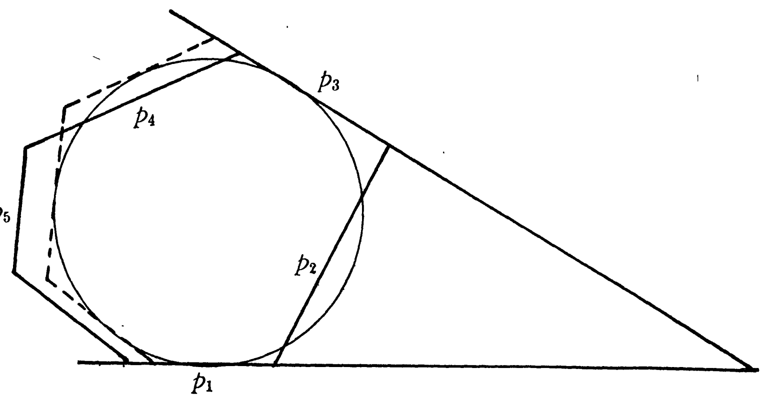


FIG. 2

be increased by making it circumscribe a circle, and therefore the area of the original polygon will also be increased. If p_2 is not tangent to this circle we keep, say, $p_1p_{n+1}p_n$ fixed and again increase the area of our polygon by making $p_1p_2 \cdots p_n$ circumscribe a circle so that our theorem is true for $n+1$ if it is true for n. But being true for $n=4$, it is true for $n=5$, *etc.*, and so it is true for any n.

The second part of Theorem 2 is proved by the fact that the interchange of any two angles may be accomplished by the interchange of consecutive angles, and the area of a triangle (of fixed perimeter) is not altered by the interchange of two angles. This completes the proof of Theorem 2.

In this case we can give an explicit expression for the maximum area, radius of the circle, and each side:

$$a_{ij} = p\,\frac{\cot\dfrac{\alpha_i}{2} + \cot\dfrac{\alpha_j}{2}}{\sum \cot\dfrac{\alpha}{2}}, \qquad r = \frac{p}{\sum \cot\dfrac{\alpha}{2}}, \qquad S = \frac{p^2}{\sum \cot\dfrac{\alpha}{2}}. \tag{6}$$

* If p_1 is parallel to p_3 we choose another set of three sides for $p_1p_2p_3$ so that p_1 and p_3 are not parallel. This is always possible for $n>4$, for the quadrangle is the only convex polygon which may have more than one set of adjacent sides $p_1p_2p_3$ such that p_1 is parallel to p_3. This interesting property of the quadrangle follows from the fact that $\sum\alpha = (n-2)\pi$, and if there were two sets of sides of the required sort, four of the angles would add up to 2π and the remaining $n-4$ would then have to add up to $(n-4)\pi$, so that at least one of them would have to be $\geqq \pi$ and the polygon would not be convex.

An interesting consequence of this theorem may be obtained by letting the number of angles tend to infinity, and therefore letting each of a sub-set of the angles tend to a straight angle. The greatest area is then given by a convex figure consisting of arcs of a circle and tangents to this circle. The explicit expression for each arc and each tangent is given by (6). Thus, in case the figure is to have n angles α_i, we have

$$r = \frac{p}{\sum \cot \frac{\alpha}{2} + \frac{1}{2} \sum \alpha - \frac{n-2}{2} \pi}, \qquad S = pr,$$

and each of the tangents forming the angle α_i is of length $r \cot \alpha_i/2$. The only additional fact needed to establish this last result is that $\lim_{\theta \to 0} \sin \theta/\theta = 1$.

In conclusion we may state an interesting problem in elimination. According to Theorem 1, the area of the greatest polygon with n given sides depends only on their magnitude but not on their order. Hence S must be a root of an algebraic equation whose coefficients are functions of the elementary symmetric functions of the sides a_i. The problem is to determine this equation. Thus in case $n=4$ and

$$x^4 - 2px^3 + qx^2 - rx + s = 0$$

is the equation whose roots are the four sides, we have

$$S_4^2 = -p^4 + qp^2 - rp + s,$$

and for $n=3$ we need only make the length of one side zero, which makes $s=0$, and obtain

$$S_3^2 = -p^4 + qp^2 - rp.$$

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Algebra. A Text-Book on Determinants, Matrices, and Algebraic Forms. By W. L. Ferrar. Oxford, Clarendon Press, 1941. 7+202 pages. $3.50.

College Algebra. Second edition. By H. P. Pettit and P. Luteyn. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1941. 14+247 pages. $1.90.

The Second Yearbook of Research and Statistical Methodology and Reviews. By O. K. Buros, Editor. Highland Park, New Jersey, The Gryphon Press, 1941. 20+383 pages. $5.00.

Galois Lectures. By J. Douglas, P. Franklin, C. J. Keyser, and L. Infeld. (Addresses delivered at the Galois Institute of Mathematics, Long Island University, Brooklyn, New York.) (The Scripta Mathematica Library, No. 5.) New York, Scripta Mathematica, Yeshiva College, 1941. 124 pages.

The Calculus of Extension, including examples by Robert Genese. By H. G. Forder. Cambridge, University Press; New York, The Macmillan Company, 1941. 16+490 pages. $6.75.

REVIEWS

Pandiagonal Magic Squares of Composite Order. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1941. 10+155 pages. $1.00.

This book may very appropriately be considered a complement to the author's book, *Pandiagonal Magic Squares of Prime Order*, reviewed in this MONTHLY, vol. 47, 1940, p. 563.

Two principal methods of construction are used in the present volume. One consists of first making a simple transformation on the natural square and then applying a process analogous to the uniform-step method. The other method depends upon the construction of a pandiagonal square whose order is a factor of the order of the square to be constructed. Both methods are easy to apply and readily kept in mind.

Most readers interested in the subject of magic squares are familiar with the so-called uniform-step method of constructing pandiagonal magic squares, and with the fact that this method fails if the order of the square to be constructed is divisible by three. This has been pointed out in various places in the literature on magic squares but apparently no one, so far, has given simple, easily applied rules for constructing squares of this type. In the opinion of the reviewer, one of the most interesting features of the present volume is that it does furnish very simple methods for obtaining pandiagonal squares whose order n is of the form $3m$. As an indication of the great number of squares of this type we mention

that the author finds over 17 billion pandiagonal squares of order 9, and a single basic type of order 27 leads to over 2 quadrillion squares.

In addition to squares whose order is of the form $3m$, squares whose orders are of the forms $4m$, m^2, m^3, and pq are treated in detail. In most cases the author finds the total number of squares coming under each of his types, but in some cases the enumeration is incomplete and leaves room for further investigation.

G. E. RAYNOR

Factor Analysis to 1940. By Dael Wolfle. (Psychometric Monographs, No. 3.) Chicago, University of Chicago Press, 1940. 7+69 pages. $1.25.

Factor analysis is a statistical technique employed by psychologists for studying interrelationships of human abilities and personality traits. The theorems underlying this analysis are based upon matrix algebra, multidimensional geometry, and vectorial representation. Wolfle's monograph is written, however, for the psychologist untrained in mathematics. Consequently, the mathematical reader will not find here those aspects of factor analysis in which he is most interested.

For those who wish to be informed on the aims, assumptions, and results of the various factorial methods, Wolfle's work will serve as an excellent and clearly written introduction. The bibliography will furnish complete references to other works in which the mathematical aspects of the problem are fully considered. In order to understand the various opposing methods, it is necessary to follow clearly the psychological assumptions involved. The mathematical differences between the various methods are the natural results of the differences in initial assumptions. For this reason, Wolfle's monograph may be useful even to mathematicians who wish to become acquainted with this young branch of applied mathematics.

T. A. RYAN

The Stereographic Projection. By F. W. Sohon. Brooklyn, Chemical Publishing Company, 1941. 10+210 pages. $4.00.

This extensive treatise on the stereographic projection of the sphere on the plane is motivated by the use of this projection by the cartographer, particularly in the field of seismology. The author, who is Director of the Graduate School of Georgetown University, states in the preface that he decided that "an extensive examination of the literature would be a waste of time and that the only way to present the matter at hand was to write a fairly complete exposition developing the subject from its very foundation." As there is no bibliography nor any references to other works, it seems difficult to ascertain how much of the author's presentation is original.

Stereographic projection is well known as a central projection of a sphere from a point of its surface, on a plane perpendicular to the diameter through the point. (It is usual to take the plane either tangent to the sphere or through its

center; this book adopts the latter.) The map has two significant properties: it is conformal, and circles map into circles. Great circles map into circles which cut the projection of the equator in diametric points. A set of meridians and parallels on the sphere are mapped into two conjugate coaxal sets of circles. These properties, and many others, are developed at length and with much ingenuity, mainly by analytic and vector methods. The geometry of circles in the plane, including inversion, is studied. The solutions of various problems of construction in the plane corresponding to standard problems of the sphere are given at length. The last chapter is "The Problem of the Seismologist," which is that of determining the epicenter of an earthquake from a knowledge of its great circle distances from several stations, and is thus a problem of spherical trigonometry. The problem is solved in practice by drawing the appropriate circles on the stereographic map and locating their intersection. For aid in this work several tables are given, notably the "Weston-Woodstock Table " giving geographic data for 384 seismologic stations all over the world. This table was computed by students of Weston and Woodstock Colleges. There is a brief chapter devoted to corrections for the ellipticity of the earth.

The presentation impressed the reader as somewhat obscure and lacking in effectiveness. The nomenclature is complicated and there is much repetition. One may quote the opening sentence: "The stereographic projection proposes to represent the points and the circles of the surface of a sphere as points, circles, and straight lines on a plane." Some two pages later the very attentive reader will be able to find the definition of the projection obscurely indicated. The proof that circles map into circles, of course, appears in a later chapter. This lack of straightforward clarity is characteristic of the whole work. However, the reader who has the patience will find a great deal that will amply repay the effort of working it out.

Of the two misprints noted, one is curiously obvious; in the proof on page 31 that circles map into circles, the exponents are omitted from the equation of the circle in the plane. On page 139 a radical sign is omitted from the equation in the middle of the page.

R. A. JOHNSON

Mathematical Tables. By H. B. Dwight. New York, McGraw-Hill Book Company, 1941. 8+231 pages. $2.50.

This is an unusual set of tables when judged as to content, accuracy, and ease of reading. The print is rather fine but the arrangement on the pages makes for speed in the use of the tables.

This book contains tables of the trigonometric functions and of their logarithms to hundredths of degrees; also of sines, cosines, and tangents to thousandths of radians, as well as tables of $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The rest of the tables included in this book are those of exponential and hyperbolic functions (direct functions with x to 1/1000 as well as indirect functions), binomial coefficients, $(a^2+b^2)^{1/2}/a$, factorials, Gregory-Newton as well as Lagrangean inter-

polation coefficients, surface zonal harmonics and their first derivatives, complete elliptic integrals of the first and second kinds, Bernoulli numbers, Euler's numbers, Gamma functions, probability integrals, Bessel functions, the Riemann Zeta function, and common logarithms.

After each table there are excellent references to other larger tables. The completeness and wide range of these tables should appeal to a variety of scientists and engineers, and should make this book a very valuable one to own and to use.

A. D. CAMPBELL

Mathematik für Ingenieure und Techniker. By Richard Doerfling. Berlin, R. Oldenbourg, 1940. 533 pages.

In the preface to this book the author states that it is his aim to provide an adequate text-book of mathematics for those whose talent is technical rather than mathematical. As an addition to the series of texts for engineers and physicists, he claims to provide in this text a more thorough consideration of elementary mathematics.

The author points out wisely that the mathematical knowledge required in technical applications is more often that of algebra, trigonometry, and geometry than it is of calculus or more advanced mathematics. In recognition of this fact the author devotes the first half of the book to the topics of algebra, trigonometry, and analytical geometry. In the last half, three sections are devoted respectively to differential and integral calculus, differential equations, and vector analysis.

The text is exceptionally well organized and very comprehensive. The principal techniques in algebra and numerical methods required in practical work are covered more adequately than is the rule in texts of applied mathematics. Teachers of technical students will find this book useful in supplementing similar American texts.

The chief lack from the point of view of engineering usefulness is that of examples and practical illustrations of the mathematical developments. There is very little attempt made to indicate the applications of various mathematical methods, so the technical reader will have little help in distinguishing which parts are useful to him. The book would, however, provide an adequate reference in general mathematics for the engineer who had previous knowledge of the material.

N. A. HALL

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 491. *Proposed by E. T. Frankel, Albany, N. Y.*

Prove that $\sqrt[\sqrt{-1}]{-1}=23\frac{1}{7}$, approximately.

E 492. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find a four-digit square which remains a square when two zeros are intercalated between the thousands digit and the hundreds digit.

E 493. *Proposed by N. A. Court, University of Oklahoma.*

Given a tetrahedron $ABCD$ and a point M, prove that the tangent planes, at M, to the four spheres $MBCD$, $MCDA$, $MDAB$, $MABC$, meet the respective faces BCD, CDA, DAB, ABC in four coplanar lines.

E 494. *Proposed by Henry Scheffé, Reed College.*

Show that, for every positive integer N, the equation

$$x^2 + y^2 + 2xy - 3x - y + 2 = 2N$$

has a unique solution in positive integers x, y. Give a method for finding it, without successive trials. Generalize the problem to n unknowns $x_1, x_2, \cdots, x_n$, satisfying

$$F(x_1, x_2, \cdots, x_n) = 2^{\alpha-1}N,$$

where F is a special polynomial of degree $\alpha=2^{n-1}$, with integral coefficients.

E 495. *Proposed by Daniel Arany, Budapest, Hungary.*

If x, y, z are the barycentric coördinates of a point Q with respect to a triangle ABC, show that, for any point P in the same plane,

$$xAP^2 + yBP^2 + zCP^2 = xAQ^2 + yBQ^2 + zCQ^2 + (x + y + z)PQ^2.$$

SOLUTIONS

E 456 [1941, 65]. *Proposed by Leopold Infeld, University of Toronto.*

What is the smallest popular vote by which a President can be elected in the U.S.A. under the present electoral system? Assumptions: N is the total popular vote; the popular vote in each state is proportional to the electoral vote (which you will have to look up); there are just two candidates.

Solution by R. K. Allen, Montpelier, Vermont.

There are 531 electoral votes, and so 266 necessary to elect. There exists a set of states possessing a total of exactly 266 votes, namely:

State	Electoral votes (1940 election)	Electoral votes (1944 election)
California	22	25
Illinois	29	28
Louisiana	11	11
Maine	5	5
Massachusetts	17	16
Michigan	19	20
Missouri	15	15
New Jersey	16	16
New York	47	47
Ohio	26	25
Pennsylvania	36	35
Texas	23	23
	266	266

No smaller number of states has this property. Since the popular vote is proportional to the electoral vote, we let $531\,n=N$, so that a state with k electoral votes has kn popular votes. Since kn must be an integer, and since the various electoral votes and their total are relatively prime, it follows that n is an integer. Further, N is odd or even according as n is odd or even. A successful candidate might just carry the above listed states and receive no votes in any of the others. In each state he needs $[\frac{1}{2}kn+1]$ votes. (In the above tabulation, Louisiana and Maine were chosen because the electoral vote of each is odd, this reducing by one the number of necessary votes when N is odd.)

By actually adding up the popular votes, we find that, under either apportionment, the smallest number of votes necessary is

$$\frac{133}{531}N+\epsilon,$$

where $\epsilon=8$ or 12 according as N is odd or even. This is approximately 25.05% of the total popular vote.

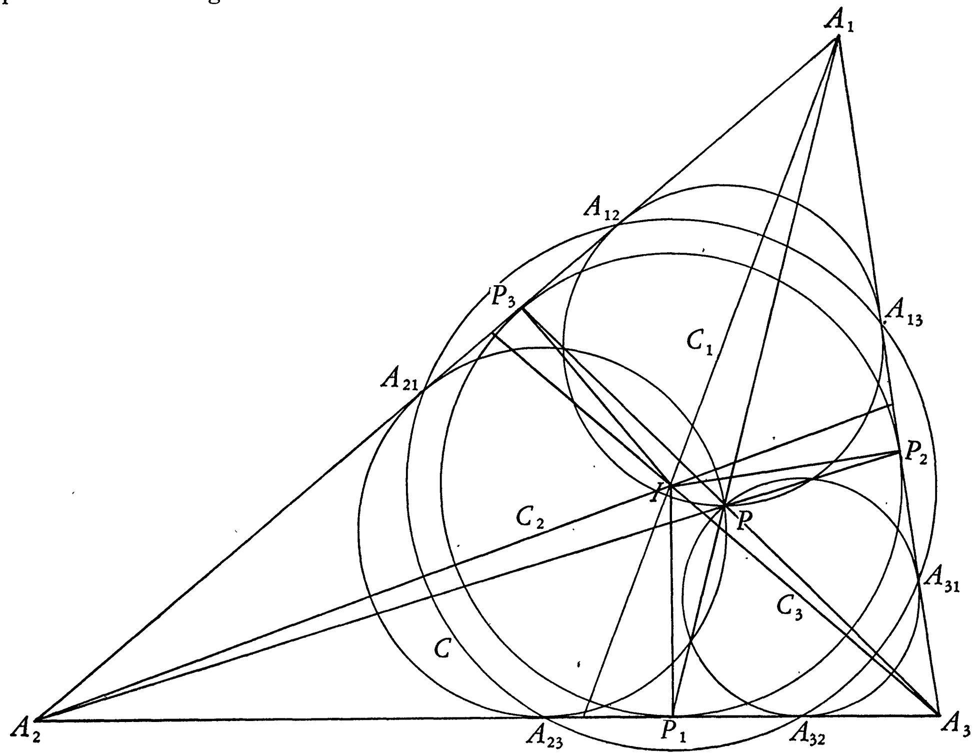
E 457 [1941, 148]. *Proposed by V. Thébault, San Sebastián, Spain.*

Construct three circles which have one common point and which are such that each touches two sides of a given triangle, the six points of contact being concyclic.

Solution by Howard Eves, Lindy's Lake, N. J.

Let $A_1A_2A_3$ be the given triangle, and let the three required circles be C_1, C_2, C_3; C_i lying in the angle A_i and touching the side A_iA_j at the point A_{ij}.

Let I be the incenter of the given triangle, and let P_1, P_2, P_3 be the feet of the perpendiculars from I on the sides of the triangle, P_i lying opposite A_i. Then, since the six A_{ij} are on a circle C, it is clear that I is the center of C, and therefore that all the segments P_iA_{jk}, $(i\neq j\neq k\neq i)$, are equal. Hence P_i and A_i lie on the radical axis of C_j and C_k. Thus the common point of the three circles C_i must be the point of concurrence of the three lines A_iP_i, namely the Gergonne point of the triangle. The construction is now evident.



Also solved by D. H. Browne, W. B. Clarke, William Douglas, and the proposer. Clarke remarks that the construction (for a circle to pass through a given point and touch two given lines) leads to two distinct sets of three circles, though only one of these sets will touch the sides internally.

E 458 [1941, 148]. *Proposed by J. L. Brenner, University of Minnesota.*

Prove that in any power of the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

two elements in the main diagonal will be the same. Show that the same result holds for any matrix (a_{rs}) in which $a_{r1}=a_{2r}$, $a_{1r}=a_{r2}$, $(r>2)$, and $a_{11}=a_{22}$.

Solution by E. P. Starke, Rutgers University.

For n-rowed square matrices we have the familiar rule

$$(a_{rs})(b_{rs}) = (p_{rs}), \qquad (b_{rs})(a_{rs}) = (q_{rs}),$$

where

$$p_{rs} = \sum_{k=1}^{n} a_{rk}b_{ks}, \qquad q_{rs} = \sum_{k=1}^{n} b_{rk}a_{ks}.$$

Let the given relations hold for both (a_{rs}) and (b_{rs}), so that

$$a_{r1} = a_{2r}, \quad b_{r1} = b_{2r}, \quad a_{1r} = a_{r2}, \quad b_{1r} = b_{r2}, \quad (r > 2), \quad a_{11} = a_{22}, \quad b_{11} = b_{22}.$$

Then

$$\begin{aligned} p_{22} &= a_{21}b_{12} + a_{22}b_{22} + \sum_{k=3}^{n} a_{2k}b_{k2} \\ &= a_{21}b_{12} + a_{11}b_{11} + \sum_{k=3}^{n} a_{k1}b_{1k} = q_{11}. \end{aligned} \tag{1}$$

If now $(b_{rs}) = (a_{rs})^t$, we must have $(p_{rs}) = (q_{rs}) = (a_{rs})^{t+1}$. This result, together with (1), implies $p_{22} = q_{11} = p_{11}$, which shows that $(a_{rs})^{t+1}$ has equal numbers for the first two elements of its main diagonal. Since $a_{11} = a_{22}$, this is a complete induction.

Also solved by the proposer, who remarks that the special matrix cited in the problem has the peculiarity that its sixth power is the identity mod 13, while its seventh power is the identity mod 2.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4009. *Proposed by J. H. M. Wedderburn, Princeton University.*

If the roots of $x^n - c_1x^{n-1} + \cdots + (-1)^n c_n$ are the variables $x_1, x_2, \cdots, x_n$, find the Jacobian of $c_n, c_{n-1}, \cdots, c_1$ with respect to $x_1, x_2, \cdots, x_n$.

4010. *Proposed by F. A. Lewis, University of Alabama.*

Let G be an Abelian group of order n^m and type $(1, 1, \cdots, 1)$. Find the number of sub-groups of G of order n^r and type $(1, 1, \cdots, 1)$; and show that this number is the same as that for order n^{m-r}.

4011. *Proposed by N. A. Court, University of Oklahoma.*

The pairs of straight lines a, a'; b, b'; c, c'; d, d' are isogonal conjugates for the trihedral angles A, B, C, D of the tetrahedron $ABCD$. Prove that: (1) If the lines a, b, c, d are concurrent, so also are the remaining four lines (the proposer's *Modern Pure Solid Geometry*, p. 242). (2) If the four lines a, b, c, d form a hyperbolic group, so also do the remaining four lines.

4012. *Proposed by V. Thébault, Le Mans, France.*

Find a number of n digits $N_0=a_1a_2 \cdots a_n$, $(a_1 \neq 0)$, such that if we transpose the first k digits from left to right, $(k=1, 2, 3, \cdots, n-1)$, the $n-1$ numbers thus obtained $N_1=a_2a_3 \cdots a_na_1$, $N_2=a_3a_4 \cdots a_na_1a_2, \cdots, N_{n-1}=a_na_1a_2 \cdots a_{n-1}$ are each multiples of N_0.

4013. *Proposed by V. Thébault, Le Mans, France.*

Determine the kind of tetrahedron for which: (1) The circumcenter (or incenter) is on one of the medians. (2) The straight line joining the circumcenter to the centroid is perpendicular (or parallel) to one of the faces.

SOLUTIONS

3933 [1939, 656]. *Proposed by T. E. Naish, Major, Royal Engineers (retired), Penticton, Lake Okanagan, B. C.*

If $f(x)=(x-1)(x-2) \cdots (x-p+1)$, where p is a prime, show that: (1) each coefficient of $f(x)$ is divisible by p except the first and the last, and the sum of these two is also divisible by p; (2) the sum of the even coefficients is equal to the sum of the odd, and this sum is $p!/2$.

Editorial Note. Since the proof of part (1) is obvious from the algebra of the residues, mod p, the following extension is offered. Let $f_m(x)$ denote the polynomial, the coefficient of whose highest power of x is unity and which vanishes for and only for the positive residues, mod p^m, where m is a positive integer, and the residues are prime to p and less than p^m. Then

$$\begin{aligned} f_m(x) &\equiv (x^{p-1}-1)^{p^{m-1}}, \text{ mod } p^m, \text{ where } p \text{ is an odd prime;} \\ &\equiv (x^2-1)^{2^{m-2}}, \text{ mod } 2^m. \end{aligned}$$

Solution by Otto Dunkel, Washington University.

The residues $0, 1, 2, \cdots, p-1$, where p is a prime, are such that the sum, or product, of any two reduces mod p to one of this set of p elements. These two operations satisfy the fundamental laws of ordinary algebra for the corresponding operations. In the algebra of these p elements, the exclusion of 0 gives the elements of an abelian group of order $p-1$. If x is any one of these group elements, it generates a cyclic group of order m and m is a divisor of $p-1$. Hence the equation

$$x^{p-1}-1 \equiv 0, \text{ mod } p,$$

has the $p-1$ elements of this group as roots.

As in ordinary algebra it may be shown that if $g(x)\equiv 0$, mod p, where $g(x)$ is a polynomial with integral coefficients and of degree less than $p-1$, has $p-1$ roots which are distinct mod p, then all of its coefficients are congruent to zero. If we set

$$f(x) = (x-1)(x-2)\cdots(x-p+1), \tag{1}$$

then $f(x)-(x^{p-1}-1)=g(x)$ is such a polynomial. Hence we have

$$f(x) \equiv x^{p-1}-1, \text{ mod } p. \tag{2}$$

This means that, if we write (1) in the form

$$f(x) = x^{p-1} - c_1x^{p-2} + c_2x^{p-3} - \cdots + (p-1)!, \qquad (p \geqq 3),$$

then c_i is divisible by p where $i=1, 2, \cdots, p-2$, and $c_{p-1}=(p-1)!$. Hence $c_{p-1}\equiv -1$, mod p; that is, $c_{p-1}+1$ is divisible by p. We also have

$$f(1) = 1 - c_1 + c_2 - \cdots + c_{p-1} = 0,$$

$$f(-1) = 1 + c_1 + c_2 + \cdots + c_{p-1} = p!,$$

$$1 + c_2 + c_4 + \cdots + c_{p-1} = c_1 + c_3 + \cdots + c_{p-2} = p!/2.$$

This completes the proof of the first part.

In order to prove the last part, where p is a prime not less than 3, we shall use induction. The positive integers less than p^m and prime to p^m are given by

$$\begin{gathered} k_0 + k_1p + k_2p^2 + \cdots + k_{m-1}p^{m-1}, \\ (k_0 = 1, 2, \cdots, p-1;\ k_j = 0, 1, 2, \cdots, p-1;\ j > 0,\ m \geqq 2); \end{gathered} \tag{3}$$

and there are p^m-p^{m-1} such integers. Set

$$f_m(x) = \prod (x - r_i), \tag{4}$$

where r_i runs through the above p^m-p^{m-1} values. Then we have

$$f_m(x) = f_{m-1}(x)\prod_{j=1}^{p-1} f_{m-1}(x - jp^{m-1}), \tag{5}$$

since the roots of $f_{m-1}(x)$ are the above integers between unity and p^{m-1}, and those of the second factor of the right member are the above integers between p^{m-1} and p^m. In order to carry out reductions, we write

$$f_{m-1}(x - jp^{m-1}) \equiv f_{m-1}(x) - jp^{m-1}f'_{m-1}(x), \text{ mod } p^m, \qquad (m \geqq 2),$$

$$\prod_{j=1}^{p-1} f_{m-1}(x - jp^{m-1}) \equiv [f_{m-1}(x)]^{p-1} - \frac{p(p-1)}{2}\, p^{m-1}f'_{m-1}(x)\,[f_{m-1}(x)]^{p-2}, \text{ mod } p^m,$$

$$\equiv [f_{m-1}(x)]^{p-1}, \text{ mod } p^m.$$

Hence

$$f_m(x) \equiv [f_{m-1}(x)]^p, \text{ mod } p^m. \tag{6}$$

If we assume that $f_{m-1}(x)\equiv(x^{p-1}-1)^{p^{m-2}}$, mod p^{m-1}, then it follows from (6) that

$$f_m(x) \equiv (x^{p-1} - 1)^{p^{m-1}}, \text{ mod } p^m. \tag{7}$$

We have already shown that (7) is true for $m=1$, and it now follows that it must be true for all positive integers m.

We now consider the case $p=2$. The positive odd integers less than 2^m, $m>2$, fall into two classes:

$$\text{(a)}\quad 1 + 2j; \qquad \text{(b)}\quad 2^m - (1 + 2j); \qquad (j = 0, 1, 2, \cdots, 2^{m-2} - 1). \tag{8}$$

In this case, (4) becomes

$$\begin{aligned} f_m(x) &= f_{m-1}(x)\prod [x + (1 + 2j) - 2^m], \ (j = 0, 1, 2, \cdots, 2^{m-2} - 1),\\ f_{m-1}(x) &= \prod [x - (1 + 2j)]; \ f_{m-1}(x + 2^m) = \prod [x + 2^m - (1 + 2j)],\\ f_{m-1}(-x + 2^m) &= \prod [-x + 2^m - (1 + 2j)] = \prod [x + (1 + 2j) - 2^m],\\ f_m(x) &= f_{m-1}(x)f_{m-1}(-x + 2^m) \equiv f_{m-1}(x)f_{m-1}(-x), \text{ mod } 2^m, \end{aligned}$$

where the congruence easily follows by expansion.

For $m=2$, we have

$$f_2(x) = (x - 1)(x - 3) = (x^2 - 1) - 4(x - 1) \equiv (x^2 - 1), \text{ mod } 2^2,$$

and the theorem that we are to prove is true for $m=2$. Let us assume that it is true for $m-1$, which means that

$$\begin{aligned} f_{m-1}(x) &\equiv (x^2 - 1)^{2^{m-3}}, \text{ mod } 2^{m-1}, \qquad (m \geqq 3),\\ f_{m-1}(x) &= (x^2 - 1)^{2^{m-3}} + 2^{m-1}r(x),\\ f_{m-1}(-x) &= (x^2 - 1)^{2^{m-3}} + 2^{m-1}r(-x),\\ f_{m-1}(x)f_{m-1}(-x) &= (x^2 - 1)^{2^{m-2}} + 2^{m-1}[r(x) + r(-x)](x^2 - 1)^{2^{m-3}}\\ &\quad + 2^{2m-2}r(x)r(-x). \end{aligned}$$

Since $r(x)+r(-x)$ is even, and $2m-2>m$, we have finally

$$f_m(x) \equiv f_{m-1}(x)f_{m-1}(-x) \equiv (x^2 - 1)^{2^{m-2}}, \text{ mod } 2^m,$$

and it now follows that

$$f_m(x) \equiv (x^2 - 1)^{2^{m-2}}, \text{ mod } 2^m, \qquad (m \geqq 2),$$

and the proof is complete.

3940 [1940, 53]. *Proposed by Otto Dunkel, Washington University.*

Denote by $\sigma_r(n)$ the elementary symmetric function of the consecutive integers $1, 2, \cdots, (n-1)$, the sum of the products of these numbers r at a time. Show that

$$\sigma_r(n) = n(n - 1) \cdots (n - r)P_r(n),$$

where $P_r(n)$ is a polynomial in n of degree $r-1$, and that $P_r(x)$ vanishes for $x=0, 1$, if r is odd and greater than unity.

Develop a method for obtaining consecutively the explicit expressions for $\sigma_r(x)$, or $P_r(x)$.

I. *Solution by G. B. Lang, Emory University.*

Let the numbers be $1^m, 2^m, \cdots, (n-1)^m$, and define $\sigma_r(n)$ as the sum of the products of these numbers r at a time. Then

(1) $s_r - \sigma_1(n)s_{r-1} + \cdots + (-1)^{r-1}\sigma_{r-1}(n)s_1 + (-1)^r r\sigma_r(n) \equiv 0, \ (r = 1, 2, \cdots),$

where s_v is the sum of the vth powers of the numbers $1^m, \cdots, (n-1)^m$, *i.e.*,

$$s_v = 1^{mv} + 2^{mv} + \cdots + (n-1)^{mv}.$$

(See Bôcher, *Higher Algebra*, p. 244.) Now if $\phi_k(n)$ is the Bernoulli polynomial of degree k, and if n is a positive integer, we have (Whittaker and Watson, *Modern Analysis*, p. 127),

$$\frac{\phi_k(n)}{k} = 1^{k-1} + 2^{k-1} + \cdots + (n-1)^{k-1},$$

so that

$$s_v = \phi_{mv+1}(n)/(mv+1).$$

Hence we have the relations

$$\text{(2)} \qquad \frac{\phi_{mr+1}(n)}{mr+1} - \frac{\sigma_1(n)\phi_{mr-m+1}(n)}{mr-m+1} + \cdots + \frac{(-1)^{r-1}\sigma_{r-1}(n)\phi_{m+1}(n)}{m+1} + (-1)^r r\sigma_r(n) \equiv 0, \qquad (r = 1, 2, \cdots).$$

Equations (2) can be used to determine successively the functions $\sigma_r(n)$. It is easily proved by induction that $\sigma_r(n)$ is a polynomial of degree $r(m+1)$ in n. Putting $m=1$ we have the case proposed in the problem. The fact that $\sigma_r(n)=0$ for $n=0, 1, \cdots, r$ can be seen from the definition. The fact that $\sigma_r(n)$ has 0, 1 as double zeros when n is odd and exceeds one is shown as follows. The Bernoulli polynomials vanish at 0 and 1, and have double zeros there if the degree is even and exceeds 2. Hence if r is odd and larger than 2, $\phi_{r+1}(n)$ has double zeros, and each following term of (2) has a double zero, since it contains as factors both a $\sigma_i(n)$ and a $\phi_i(n)$.

II. *Solution by the Proposer.*

The following method uses the difference equation

$$\text{(1)} \qquad \sigma_r(n+1) - \sigma_r(n) = \Delta\sigma_r(n) = n\sigma_{r-1}(n),$$

and the formula

$$\text{(2)} \qquad \Delta x^{(r)} = rx^{(r-1)}, \ x^{(r)} = x(x-1)\cdots(x-r+1), \ x^{(0)} = 1.$$

Obviously $\sigma_r(r+1) = r!$, and then (1) gives $\sigma_r(r)=0$; also $\sigma_1(n) = n^{(2)}/2$, and then (1) gives $\sigma_0(n)=1$. We shall give the results of computations leading up to that for $\sigma_5(n)$ and the full details of the computation of the latter:

$$\sigma_0(n) = 1, \qquad \sigma_1(n) = \frac{n^{(2)}}{2}, \qquad \sigma_2(n) = \frac{n^{(4)}}{2\cdot 4} + \frac{n^{(3)}}{3},$$

$$\sigma_3(n) = \frac{n^{(6)}}{2\cdot 4\cdot 6} + \frac{n^{(5)}}{2\cdot 3} + \frac{n^{(4)}}{4} = \frac{n^{(4)}}{2\cdot 4\cdot 6} n(n-1), \tag{3}$$

$$\sigma_4(n) = \frac{n^{(8)}}{2\cdot 4\cdot 6\cdot 8} + \frac{n^{(7)}}{2\cdot 3\cdot 4} + \frac{13n^{(6)}}{3\cdot 4\cdot 6} + \frac{n^{(5)}}{5}.$$

We now compute $\sigma_5(n)$ using (2) in the manner by which the above may be obtained. Since $\Delta\sigma_5(n) = n\sigma_4(n)$, we have

$$\begin{aligned} \Delta\sigma_5(n) &= \frac{n^{(9)}}{2\cdot 4\cdot 6\cdot 8} + \frac{8n^{(8)}}{2\cdot 4\cdot 6\cdot 8} + \frac{n^{(8)}}{2\cdot 3\cdot 4} + \frac{7n^{(7)}}{2\cdot 3\cdot 4} + \frac{13n^{(7)}}{3\cdot 4\cdot 6} \\ &\quad + \frac{13\cdot 6n^{(6)}}{3\cdot 4\cdot 6} + \frac{n^{(6)}}{5} + \frac{5n^{(5)}}{5}, \\ &= \frac{n^{(9)}}{2\cdot 4\cdot 6\cdot 8} + \frac{n^{(8)}}{16} + \frac{17n^{(7)}}{36} + \frac{77n^{(6)}}{60} + n^{(5)}. \end{aligned} \tag{4}$$

Hence, after using (2), we have

$$\sigma_5(n) = \frac{n^{(10)}}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \frac{n^{(9)}}{9\cdot 16} + \frac{17n^{(8)}}{8\cdot 36} + \frac{11n^{(7)}}{60} + \frac{n^{(6)}}{6}. \tag{5}$$

Since for $n=5$ both right and left members of the equation vanish, no constant is to be added. Without going through the formal induction steps it will be seen that

$$\begin{aligned} \sigma_r(n) &= \frac{n^{(2r)}}{2\cdot 4\cdots(2r)} + \cdots + \frac{n^{(r+1)}}{r+1} \\ &= n^{(r+1)}\left[\frac{(n-r-1)(n-r-2)\cdots(n-2r+1)}{2\cdot 4\cdots(2r)} + \cdots + \frac{1}{r+1}\right], \quad r \geqq 1. \end{aligned}$$

This disposes of the requirements of the problem except for the proof of the vanishing of $P_r(n)$ in certain cases; and we now consider this more difficult part.

For the polynomial $P_r(n)$ we have the difference equation

$$(n+1)P_r(n+1) - (n-r)P_r(n) = nP_{r-1}(n), \quad r \geqq 2. \tag{6}$$

Hence for $n=0$ we have

$$P_r(1) + rP_r(0) = 0; \tag{7}$$

and, if $P_r(0)=0$, we must have also $P_r(1)=0$. If r is odd and not less than 3, we shall show that $P_r(0)$ is zero and it then follows that $P_r(1)$ is also zero.

We may write

$$\sigma_r(n) = \sum (n - a_{i_1})(n - a_{i_2}) \cdots (n - a_{i_r}),$$
$$= \sum \sum_{u=0}^{r} (-1)^u n^{r-u} a_{i_1} a_{i_2} \cdots a_{i_u},$$

where $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ runs through all distinct sets of r integers chosen from $1, 2, \cdots, n-1$, there being ${}_{n-1}C_r$ such selections. The product $a_{i_1}a_{i_2} \cdots a_{i_u}$ will occur in ${}_{n-1-u}C_{r-u} = (n-1-u)^{(r-u)}/(r-u)!$ ways. Hence we have

$$\sigma_r(n) = \sum_{u=0}^{r} (-1)^u n^{r-u} \frac{(n-1-u)^{(r-u)}}{(r-u)!} \sigma_u(n),$$

$$[1 - (-1)^r]\sigma_r(n) = \frac{n^r(n-1)^{(r)}}{r!} + \sum_{u=1}^{r-1} (-1)^u n^{r-u} \frac{(n-1-u)^{(r-u)}}{(r-u)!} \sigma_u(n),$$

$$2P_r(n) = \frac{n^{r-1}}{r!} + \sum_{u=1}^{r-1} (-1)^u n^{r-u} \frac{P_u(n)}{(r-u)!}, \quad r \text{ odd and } r \geqq 3,$$

where in the last equation the factor $n^{(r+1)}$ has been removed from both members of the equation. We now see that $2P_r(0) = 0$, and the proof is complete. The expression in (5) for $\sigma_5(n)$ after collecting terms is

$$\sigma_5(n) = \frac{n^{(6)}}{2\cdot 4\cdot 6\cdot 8\cdot 10} \frac{n(n-1)(3n^2 - 7n - 2)}{3}.$$

There are other ways of computing in succession $\sigma_r(n)$, or $P_r(n)$, and for the latter the equation (6) may also be used.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Applications for Benjamin Peirce Instructorships at Harvard University for the academic year 1942–43 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

The Departments of Mathematics in the University of California are coöperating with the Engineering, Science, and Management Defense Training Program sponsored by the United States Office of Education. Professor E. C. Goldsworthy is supervising mathematics courses in the San Francisco Bay region. Professor W. M. Whyburn is in educational charge of such courses in the Los Angeles and San Diego areas. Members of the Los Angeles Mathematics Department who are teaching these courses are P. H. Daus, W. E. Mason, P. G. Hoel, A. E. Taylor, Ralph Byrne, W. T. Puckett, and F. A. Valentine. Other

instructors for the courses in the southern area come from the various aircraft plants, the San Diego State College, and the Santa Monica Junior College.

The summer session arranged at Brown University for Advanced Instruction and Research in Mechanics, and for which sixty students were accepted, met with such general approval that the program is being continued during the academic year 1941–42. Courses are being given in (1) Advanced Dynamics, (2) Elasticity, (3) Fluid Dynamics, (4) Theory of Airflight, (5) Graphical and Numerical Methods in Applied Mathematics, (6) Partial Differential Equations; in addition there are two seminars. More than thirty persons are enrolled. On the staff are Professors J. L. Synge, Richard von Mises, J. D. Tamarkin, Willy Feller, Stefan Bergman, and Willi Prager. The instruction is supported by the Engineering, Science, and Management Defense Training Program of the United States Office of Education and by the Carnegie Corporation of New York. A few small fellowships are still available for the second semester. Those desiring information should communicate with the Dean of the Graduate School, Brown University, Providence, R. I.

Assistant Professor C. J. Blackall of the College of St. Thomas has been appointed to a professorship at De Sales College, Toledo, Ohio.

Associate Professor R. S. Burington of the Case School of Applied Science is on leave of absence and is serving as Research and Consulting Mathematician in the Bureau of Ordnance, Navy Department, Washington, D. C.

Associate Professor C. M. Cleveland of the University of Texas has been promoted to a professorship in applied mathematics.

Associate Professor H. V. Craig of the University of Texas is on leave of absence for the first semester of 1941–42.

Associate Professor H. B. Curry of Pennsylvania State College has been promoted to a professorship.

Mrs. R. B. Eide of River Falls State Teachers College is on leave for the first semester of 1941–42, and will be at Austin, Texas.

Dr. J. S. Georges of Wright Junior College has been elected president of the Central Association of Science and Mathematics Teachers.

Dr. J. W. Green of the University of Rochester has been promoted to an assistant professorship.

T. J. Higgins of Purdue University has been appointed an assistant professor at Tulane University.

Dr. E. E. Ingalls of Albion College has been promoted to an assistant professorship.

Professor T. S. Jacobsen, chairman of the department of astronomy at the University of Washington, has been promoted to an associate professorship of astronomy and mathematics.

Dr. F. B. Jones of the University of Texas has been promoted to an assistant professorship in pure mathematics.

Neil Little of Purdue University has been promoted to an assistant professorship.

Assistant Professor G. A. Lyle of the U. S. Naval Academy has been promoted to an associate professorship.

Associate Professor H. B. MacDougal of South Dakota State College, acting head of the department since 1939, has been promoted to a professorship.

Associate Professor H. M. MacNeille of Kenyon College has been promoted to a professorship.

Reverend P. H. McGrath of St. Peter's College, Jersey City, has been promoted to a professorship.

Associate Professor Anna E. Many of Sophie Newcomb College has been promoted to a professorship.

Associate Professor R. H. Marquis of Ohio University has been promoted to a professorship.

Dr. C. A. Messick of Oakland City College, Indiana, has been made professor and head of the department.

Professor Eugenie M. Morenus of Sweet Briar College is on leave for 1941–42 and is spending the winter at Berkeley, California.

Assistant Professor J. E. Powell of Michigan State College has been promoted to an associate professorship.

Professor H. W. Reddick of Cooper Union Institute of Technology has been granted leave of absence for the current year to become Director of the Defense Training Institute in Brooklyn, N. Y. Assistant Professor F. H. Miller is acting head of the department at Cooper Union during Professor Reddick's absence.

Dr. E. N. Shawhan of the University of Minnesota is on leave of absence for the year 1941–42 to do necessary defense work in the Naval Ordnance Laboratory at Washington.

Dr. M. F. Smiley of Lehigh University has been promoted to an assistant professorship.

Assistant Professor A. H. Taub has returned to the University of Washington after a year's leave of absence spent at the Institute for Advanced Study.

Dr. E. W. Titt of the University of Maryland has been appointed associate professor of applied mathematics at the University of Texas.

Margaret C. Weeber of the Teachers College of Connecticut has been promoted to an assistant professorship.

E. J. Hirschler, professor of mathematics and astronomy at Bluffton College for thirty-eight years, died May 22, 1941, at the age of sixty-five. He was a charter member of the Mathematical Association.

Dr. C. M. Sparrow, professor of physics at the University of Virginia, died August 30, 1941, at the age of sixty-one. He had served on the faculty there for thirty years. He had been a member of the Mathematical Association since 1925.

Professor K. D. Swartzel, of the University of Pittsburgh, died October 30, 1941, at the age of seventy-two. He had taught at Ohio State University from 1895 to 1922 and was professor of mathematics and head of the department at the University of Pittsburgh from 1922 to 1939. He was a charter member of the Mathematical Association.

Dr. T. H. Taliaferro, professor of mathematics and dean of the faculty of the University of Maryland, died September 25, 1941, at the age of seventy. He had been a member of the Mathematical Association since 1925.

THE FIFTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fifth annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 7, 1942. This Competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from the Secretary of the Association, W. D. Cairns, 97 Elm Street, Oberlin, Ohio, by a postcard request. All applications must be filed with the Secretary not later than February 17, 1942. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made by the

Secretary in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in ample time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination the books are to be sent by registered mail or by express to the Secretary of the Association, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200, and $100 in the order of their rank. In addition, there will be prizes of $40, $30, $20, and $10 awarded to the members of these teams according to the rank of the team, and a prize of $50 to each of the five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to the five individuals next in order after the five individual winners. For further encouragement of the Competition, there will be awarded at Harvard University* an annual $1000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

More complete details of the general regulations under which the plan is operated are given in the announcement which is being mailed to colleges and universities in the United States and Canada. Reports on the four previous competitions will be found in the MONTHLY for May 1938, 1939, 1940, and 1941.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942

Twenty-fifth Summer Meeting, Ithaca, New York, September 7–9, 1942

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, May 1–2, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS, Hays, March 27–28, 1942
KENTUCKY
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 6–7, 1942
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., Dec. 6, 1941
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN, Detroit, Nov. 15, 1941
MINNESOTA
MISSOURI, Kansas City, April 17, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, Berkeley, Jan. 31, 1942
OHIO, Columbus, April 2, 1942
OKLAHOMA, Oklahoma City, Feb. 13, 1942
PHILADELPHIA, Swarthmore, Nov. 29, 1941
ROCKY MOUNTAIN, Golden, Colo., April 17–18, 1942
SOUTHEASTERN, University, Ga., March 26–27, 1942
SOUTHERN CALIFORNIA, Los Angeles, March 14, 1942
SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, Rochester, May 2, 1942
WISCONSIN, Oshkosh, May 2, 1942

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

* Or at Radcliffe College, in the case of a woman.

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By R. G. Sanger, The University of Chicago

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Edited by J. R. Musselman, Western Reserve University

MEETINGS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

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NEW BOOKS RECEIVED

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Names of the authors are in ordinary type, those of the reviewers in capitals.

CLUBS AND ALLIED ACTIVITIES

Edited by E. H. C. HILDEBRANDT, New Jersey State Teachers College

TOPICS

ACTIVITIES

PROBLEMS AND SOLUTIONS

Edited by OTTO DUNKEL, H. S. M. COXETER, and ORRIN FRINK, Jr.

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Numbers refer to pages, black face type indicating a problem solved and solution published; italics, a problem solved, but the solution not published; ordinary type, a problem proposed.

SOLUTIONS

Numbers in black face type refer to problems, those in light face to pages.

NEWS AND NOTICES

Edited by R. G. SANGER, University of Chicago

PERSONAL MENTION

This section contains the names of persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY

CORRIGENDA

Vol. 47, 1940, p. 182. Problem 3949 should read: "Given the angles $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$ with the common initial line Ox, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{n(n+1)/2+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$, or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$."

Vol. 47, 1940, p. 652. In the review of *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, the statement that the actual work of computation was undertaken by a staff of six computors should be replaced by the statement that the six mentioned were the supervisors of the actual computors, who included more than two hundred and fifty persons.

Vol. 48, 1941, p. 148. In problem E-458, the last sentence should read: "Show that the same result holds for any matrix (a_{rs}) in which $a_{r1} = a_{2r}$, $a_{1r} = a_{r2}$, $(r > 2)$, and $a_{11} = a_{22}$."

Vol. 48, 1941, p. 204. In the list of New Books Received, after *A Survey of Methods of Apportionment in Congress*, insert "By E. V. Huntington."

Vol. 48, 1941, p. 258 and p. 262. In the list of New Books Received and in the book review, the title of the book should read *Displacement, Velocity, and Acceleration Factors for Reciprocating Motion*.

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 48 DECEMBER 1941 NUMBER 10

CONTENTS

The AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 450 Ahnaip Street, Menasha, Wisconsin, or 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him five weeks before the change becomes effective.

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THE TWENTY-SECOND ANNUAL MEETING OF THE ILLINOIS SECTION

The twenty-second meeting of the Illinois Section of the Mathematical Association of America was held at Bradley Polytechnic Institute on Friday and Saturday, May 9–10, 1941. Professor Mildred Hunt, chairman of the Section, presided at all sessions.

The meeting was attended by fifty persons, including the following thirty-eight members of the Association: Edith I. Atkin, H. G. Ayre, S. F. Bibb, G. A. Bliss, A. O. Boatman, C. E. Comstock, D. R. Curtiss, J. E. Davis, W. M. Davis, Edna M. Feltges, Elinor B. Flagg, L. R. Ford, A. E. Gault, G. D. Gore, M. R. Hestenes, W. N. Huff, Mildred Hunt, R. N. Johanson, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, J. R. Mayor, C. N. Mills, G. E. Moore, Mary W. Newson, I. E. Perlin, J. W. Peters, E. W. Ploenges, Ruth B. Rasmusen, W. T. Reid, J. M. Sachs, R. G. Sanger, H. A. Simmons, F. C. Smith, N. W. Wells, F. E. Wood, Alice K. Wright.

The Section accepted the invitation of James Millikin University, Decatur, Illinois, for the meeting on May 8–9, 1942. The following officers were elected: Chairman, R. N. Johanson, Bradley Polytechnic Institute; Vice-Chairman, E. W. Ploenges, James Millikin University; Secretary, C. N. Mills, Illinois State Normal University.

The Section passed a resolution to coöperate with the Regional Governors in any plan worked out for financial aid to the Sections.

The following fourteen papers were read:

1. "Solution of a class of Diophantine problems including an unsolved problem of Tanzo Takenouchi" by Professor H. A. Simmons, Northwestern University.
2. "The generalized hypergeometric equation" by Professor F. C. Smith, College of St. Francis.
3. "Predicting class quality by means of orientation test" by Professor W. C. Krathwohl, Illinois Institute of Technology.
4. "Evaluation of mathematical instruction" by Dr. W. H. Erskine, Wright Junior College, introduced by Professor Hunt.
5. "Some aspects of junior college mathematics" by N. W. Wells, Springfield Junior College.
6. "Sturm's theorem for functions of two variables" by Professor D. R. Curtiss, Northwestern University.
7. "Complex numbers and wing profiles of airplanes" by Dr. J. M. Dobbie, Northwestern University, introduced by Professor Curtiss.
8. "Mathematical aspects of meteorology" by Professor C. G. Rossby, Assistant Chief of U. S. Weather Bureau, introduced by Professor Hunt.
9. "Mathematics south of the border" by Professor Rufus Oldenburger, Illinois Institute of Technology.
10. "When to teach theory of equations" by Professor J. E. Davis, Central Y.M.C.A. College.

11. "Mathematics in exterior ballistics" by Professor G. A. Bliss, University of Chicago.

12. "The theorem of Morley" by Dr. J. W. Peters, University of Illinois.

13. "Boundary value problems" by Professor W. T. Reid, University of Chicago.

14. "A classification of classifications" by Professor F. E. Wood, Northwestern University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. By modifying a set of inequalities of Curtiss, a sieve was developed for catching the individual maximum number that exists in any E-solution (defined in *Trans. of Am. Math. Soc.*, Nov., 1932) of many equations of the form $\sum(1/x_1)=b/a$, $(a, b)=1$, $a\geqq b$. Professor Simmons showed that the sieve functions in the case of the equation $\sum(1/x_1)=5/11$, suggested by Tanzo Takenouchi. The process is found to work equally well for any equation in the form $\sum(1/x_1)=(5+9t)/(11+20t)$, $(t=0, 1, 2, \cdots)$.

2. Professor Smith reviewed the well known solutions of Gauss's hypergeometric equation, and then outlined the methods used to obtain the corresponding solutions of a $q+1$ by $q+1$ hypergeometric equation in both non-logarithmic and logarithmic cases.

3. Professor Krathwohl showed by means of correlation coefficients and equations of lines of regression that it is sometimes possible to predict from a 45-minute mathematics aptitude test what the average final grade of a class in mathematics will be as late as two years after the test has been taken. The degree of prediction depends on the consistency of grading and the standards of the instructor. The average correlation coefficients lie between 0.45 and 0.65. For individual instructors, correlation coefficients have been found as high as 0.90. The exceptional cases are those of instructors who grade on a normal frequency curve and hence have a correlation coefficient of zero. The correlation coefficients seem to increase with the teaching experience of the instructor.

4. Dr. Erskine reviewed a study in the evaluation of the instruction in mathematics at Wright Junior College, based upon the index numbers on individual questions in a series of tests given in the first-year mathematics course over a period of several semesters. The index numbers were calculated by the formula $(C-I)/(C+I)$, where C and I are respectively the number of correct and incorrect answers. The study showed that although the index numbers of different classes vary considerably (one-half the possible range), the index numbers of the classes of each instructor are remarkably stable, with an average variation of from 1/10 to 2/10 of the possible range, and indeed comparable to the stability of the index numbers for the whole department (400 students). It was pointed out that, because of the degree of stability, index numbers may be used as a guide in improving tests and also in improving instruction.

5. Mr. Wells stated that the effectiveness of junior college mathematics is dependent upon thoroughness and effectiveness in the teaching of high school

mathematics. This can be achieved partially by a large number of problems to develop technic, and also by a good analytical reading ability on the part of the student. Another desideratum is an effective vigorous facility in arithmetic. The junior college can achieve similar effectiveness by numerous problems and also by showing the possibilities of the application of mathematics to some of the sciences. The aim should be a teaching of fundamental principles with the development in the student of confidence in his ability to use mathematics as a tool.

6. If F and Φ are power series in x and y which are convergent in a neighborhood of the origin and vanish for $x=0$, $y=0$, then Φ may not be a divisor of F in the sense used by Weierstrass and others. In this case there may be various definitions of quotient and remainder. Professor Curtiss used one such definition to set up a greatest common divisor process which, applied to F and its derivative with respect to y, produces a set of functions having properties analogous to those of the Sturm's functions of the elementary theory of equations. They can be used to locate branches of the curve $F=0$ that pass through the origin, and to determine whether F has a maximum or minimum at the origin.

7. Dr. Dobbie gave a brief introduction to the theory of conformal mapping as applied to the shaping of wing profiles, including a discussion of a class of profiles developed recently. A review of the potential theory for the flow past a circular cylinder of infinite span was included.

9. In Mexico the emphasis of university learning in the past has been on the arts. It was only three years ago that mathematics was taken from the department of philosophy in the National University and made a separate department. Since then there has been definite encouragement for the faculty to engage in research. Professor Oldenburger stated that a journal of science to inspire research in both physics and mathematics will soon be started. The National University gives master's degrees but not doctor's degrees in mathematics. The library facilities are poor, and there is a great need for journals and advanced books. This year the first Mexican scientist joined the American Mathematical Society, and it is likely that others will follow.

10. Professor Davis suggested that a first course in theory of equations may profitably be given to fourth-semester college students, concurrently with the second semester of calculus. The advantages claimed are that greater coherence, systematization, and economy of effort in subsequent undergraduate courses in mathematics are thereby made possible. The remarks were based upon nine years' experience with such practice at the Central Y.M.C.A. College.

11. Professor Bliss described some of the mathematics used in the control of artillery fire, beginning with a brief description of military maps, the construction of which is a highly technical problem of the theory of surfaces. The determination of the position of a battery on the map, of the map range from the battery to a target, and of the azimuth of the line of fire, require only relatively elementary applications of the mathematics of surveying. The use of a range table to find the corrections to the map range due to various types of abnormal

conditions, and to find the elevation corresponding to the corrected range, is also mathematically simple. The mastery and rapid coördination of all these operations in the field is the fine art of the fire control officer. Perhaps the most serious mathematical problem connected with artillery fire is the construction of range tables. Various methods of computation of trajectories, by the approximation of Siacci, by short arc methods of computation, and mechanically by means of the differential analyzer of Bush, together with methods of computing differential corrections, were discussed.

12. "If the trisectors of the interior angles of a triangle A, B, C be drawn, and if those trisectors adjacent to BC meet at P, those adjacent to AC meet at Q, and those adjacent to AB meet at R, then P, Q, R is an equilateral triangle." This is known as the theorem of Morley. Dr. Peters discussed the history of this theorem from its discovery by Frank Morley until the present.

13. Professor Reid was concerned with fundamental relationships and analogs that exist between certain problems for pencils of quadratic forms and the principal results for definitely self-adjoint and H-definitely self-adjoint differential systems as developed by Bliss and the speaker.

14. Professor Wood discussed the desirable properties which a classification should have, and gave illustrations of classifications with those properties and of classifications which did not possess certain stated properties.

C. N. Mills, *Secretary*

THE TWENTY-FIFTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-fifth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado College, Colorado Springs, Colorado, April 18–19, 1941. There were three sessions. Professor W. V. Lovitt, chairman of the Section, presided at each. The Saturday morning session was a joint meeting with the mathematics section of the Eastern Division of the Colorado Education Association.

There were thirty-four present, including the following twenty-five members of the Association: C. F. Barr, M. T. Bird, Jack Britton, I. M. DeLong, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, A. E. Mallory, W. K. Nelson, Greta Neubauer, M. G. Pawley, G. B. Price, O. H. Rechard, A. W. Recht, C. H. Sisam, V. J. Varineau, G. A. Whetstone.

At the business meeting the following officers were elected for next year: Chairman, J. C. Fitterer, Colorado School of Mines; Vice-Chairman, A. E. Mallory, Colorado State College of Education; Regional Governor for Region 12, 1942–43, O. H. Rechard, University of Wyoming.

The joint session held on Saturday morning consisted of a discussion of the two following reports: (1) "The place of mathematics in secondary education"

by the Joint Commission of the M. A. A. and the N. C. T. M.; (2) "Mathematics in general education" by a commission of the Progressive Education Association. The discussion was led by Dr. H. R. Douglass, Director of the College of Education, University of Colorado.

The following papers were presented:

1. "Various types of singular points of differential equations of the first order" by Professor J. R. Everett, Colorado School of Mines.

2. "A note on Klein's determinant approach to the line integral of area" by Professor C. F. Barr, University of Wyoming.

3. "Generalized euclidean rings" by Dr. V. J. Varineau, University of Wyoming.

4. "Specification of elastic strain" by Dr. G. A. Whetstone, Amarillo College.

5. "Interpolation with the calculating machine" by Professor A. W. Recht, University of Denver.

6. "Excursions from the beaten path of undergraduate mathematics" by Professor M. T. Bird, Utah State Agricultural College.

7. "The inherent error in extension of Newton's method for approximating real roots" by Professor M. G. Pawley, Colorado School of Mines.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Everett discussed (a) nodal points, (b) vortex points, (c) spiral points, and (d) saddle points, arising in differential equations of the first order.

2. Klein develops a special statement of Green's theorem in a plane. His approach is by a determinant of triangular area. The suggestion of generality is obvious. Professor Barr extended this approach to a surface integral of volume.

3. Dr. Varineau defined a class of generalized euclidean rings. He showed that the class of euclidean rings, as defined in the literature, is included in this class of generalized euclidean rings. He also demonstrated that the ring of matrices with elements in a proper euclidean ring is a generalized euclidean ring.

4. In an elastic body in space account must be taken of six components of strain, not all of which can be independent since they are defined as linear combinations of the first order partial derivatives of the three displacements. Dr. Whetstone proved that with the exception of the three sets (e_x, e_y, e_{xy}), (e_y, e_z, e_{yz}), and (e_z, e_x, e_{zx}) we may select any three strains arbitrarily and may then determine which of the coefficients in the Taylor expansions of the other three are arbitrary. These results were obtained by the methods of Riquier.

5. Professor Recht demonstrated the use of the calculator in ordinary interpolation using first and second differences; also, a special method of subdivision of tables to fifths of intervals using up to fourth differences.

6. The results of these excursions as given by Professor Bird are probably not new, but they may be found in novel settings. The values of $\log_{10}2$, $\log_{10}3$, and $\log_{10}7$ were found to four digits directly from the definition of logarithm. The relations between the law of sines, law of cosines, *etc.*, were made explicit. A construction for the axes of the ellipse $Ax^2-2Bxy+Cy^2=D$ was related to the

lines $Ax=By$ and $Bx=Cy$. A construction for the hyperbola $b^2x^2-a^2y^2=a^2b^2$ was related to the parametric form $x=a(1+t^2)/(1-t^2)$, $y=2bt/(1-t^2)$. The series for $\log_e N$ was exhibited as the result of certain rearrangements of the series $1-1/2+1/3-1/4+\cdots$. Unusual weighted sums were considered as approximations for a definite integral and contrasted with the usual "rules." Finally, π was computed by the use of the inverse sine.

7. Professor Pawley described extensions of Newton's method for approximating real roots in which the desired root is approximated by x intercepts of curves of higher order of contact than the tangent. He derived an upper bound to the error involved in these approximations. In particular, he simplified the well known parabolic approximation by expanding an x intercept of the parabola into a convergent alternating series. An upper limit to the error involved in this approximation was derived and illustrated by an example.

A. J. LEWIS, *Secretary*

EQUATIONS IN QUATERNIONS

IVAN NIVEN, University of Illinois

1. Introduction. We prove the existence of a quaternion root of the equation

$$a(x) = x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m = 0, \qquad a_m \neq 0, \tag{1}$$

with coefficients from the algebra of real quaternions. The writer had proved this result when m is odd, but the proof was rendered obsolete when Nathan Jacobson pointed out that the result (without restriction on m) can be obtained as a simple consequence of some work of Ore [1]. This is given in detail in §2.

In §3 we give a method for obtaining the roots of (1), which is not very practical in the sense that it involves the simultaneous solving of two real equations of degree $2m-1$. The method used is a generalization of Sylvester's treatment [2] of the quadratic equation corresponding to (1). Sylvester's conclusion that a quadratic equation has six roots is incorrect because he neglects to show that they exist, and also overlooks the possibility of an infinite number of roots; a complete analysis is given in §4 (Theorem 2). The number of roots of (1) is discussed in §5 (Theorem 3), necessary and sufficient conditions being given for an infinite number of roots.

The proof given here of the existence of a root of (1) is stated for the general case where the coefficients of the equation are quaternions over any real-closed field R (*i.e.*, no sum of squares in R is equal to -1, and no algebraic extension of R has this property).

Reinhold Baer, on hearing of this existence proof, proved the converse, so that we have the following strong result:

THEOREM 1. *Let D be a non-commutative division algebra with centrum C. Then every equation* (1) *with coefficients from D has a solution in D if and only if C is a real-closed field, and D is the algebra of real quaternions over C.*

The necessity of these conditions is shown in §6; the proof gives a slightly stronger result than stated in the theorem above, since only those equations with coefficients from C are used. The writer is indebted to Jacobson and Baer for permission to give their proofs here.

Note that (1) is a special equation. The most general quadratic term, for example, would have the form $bxcxd$, involving three coefficients. However, the results are valid for equations similar to (1) having all coefficients to the right of the powers of the unknown.

2. The existence of a root. Let the coefficients of (1) be quaternions over a real-closed field R. By replacing the quaternions a_r by their conjugates $\bar{a}_r$, we obtain a polynomial $\bar{a}(x)$. We multiply this on the right by $a(x)$, and allow x to be commutative with the coefficients. Thus we obtain a polynomial $\alpha(x)$ with coefficients in R, which is, by the fundamental theorem of algebra, factorable into linear factors in $R(i, x)$, and hence *a fortiori* in $R(i, j, x)$. Theorem 1 on p. 494 of Ore's paper [1] states that any other factorization of $\alpha(x)$ in $R(i, j, x)$ must also have linear factors. Now $a(x)$ can be factored into irreducible factors, and these are factors of $\alpha(x)$. Hence by Ore's theorem they are linear. Taking $x-c$ as the right linear factor, we can write

$$a(x) = (x^{m-1} + b_1x^{m-2} + \cdots + b_{m-1})(x - c).$$

That $x=c$ is a root of $a(x)=0$ is not apparent from this equation, since we have assumed that x commutes with the coefficients. However, upon rewriting the above equation in a form analogous to (1),

$$a(x) = x^m + (b_1 - c)x^{m-1} + (b_2 - b_1c)x^{m-2} + \cdots + (b_{m-1} - b_{m-2}c)x - b_{m-1}c,$$

we verify immediately that c is a root.

3. A right-division algorithm. The norm n of any quaternion x is defined as the product of x and its conjugate $\bar{x}$; and the addition of x and $\bar{x}$ gives t, the trace of x. It is well known that x satisfies the equation

$$x^2 - tx + n = 0. \tag{2}$$

We now divide $a(x)$ on the right by the expression in this equation, and obtain the algorithm

$$a(x) = q(x^2 - tx + n) + f(a_r, t, n)x + g(a_r, t, n), \tag{3}$$

the remainder being comprised of the last two terms, polynomials in t, n and the coefficients of $a(x)$. The nature of the quotient q does not interest us. Note that the remainder vanishes for any root of equation (1), and conversely.

When $f\neq 0$, the vanishing of this remainder can be expressed in the form

$$x = -\frac{1}{f}\cdot g = -\frac{1}{\bar{f}f}(\bar{f}g), \tag{4}$$

where $\bar{f}$, the conjugate of f, is obtained by replacing each a_r in f by its conjugate

$\bar{a}_r$. Since the conjugate of a product equals the product of the conjugates in reverse order, we have

$$\bar{x} = -\frac{1}{\bar{f}f}(\bar{g}f).$$

By multiplication and addition of the last two equations we get the norm and trace of x; thus

$$(5) \qquad n = \frac{1}{\bar{f}f}(\bar{g}g), \qquad t = -\frac{1}{\bar{f}f}(\bar{f}g + \bar{g}f),$$

since $\bar{f}f$ and $\bar{g}g$ have real coefficients and are commutative with the other polynomials. These equations may be written in the forms

$$(6) \qquad N(t, n) = n\bar{f}f - \bar{g}g = 0, \qquad T(t, n) = t\bar{f}f + \bar{f}g + \bar{g}f = 0,$$

where $N(t, n)$ and $T(t, n)$ are polynomials in t and n with real coefficients.

First we note that any root x_0 of (1) has a trace t_0 and a norm n_0 which satisfy equations (6). This is apparent except when $f(a_r, t_0, n_0) = 0$, in which case equation (4) is meaningless. But in this case equation (3) implies that $g(a_r, t_0, n_0)$ vanishes, and equations (6) are satisfied.

Conversely, any simultaneous real solution (t_0, n_0) of (6) gives one or more roots of (1). First suppose that $f(a_r, t_0, n_0) \neq 0$. The values t_0 and n_0 can be substituted in (4) to give a quaternion x_0, and since these quantities satisfy (2), equation (3) indicates that x_0 is a root of $a(x) = 0$. It is important to note that in this case one solution of equations (6) gives exactly one solution of (1).

On the other hand, if $f(a_r, t_0, n_0) = 0$, then the first equation (6) gives

$$\bar{g}(a_r, t_0, n_0)g(a_r, t_0, n_0) = 0.$$

But the product of a quaternion and its conjugate is zero only if the quaternion is zero, and hence $g(a_r, t_0, n_0) = 0$. Returning to (3), we see that any solution of

$$(7) \qquad x^2 - t_0 x + n_0 = 0$$

is also a solution of (1). The above analysis enables us to inquire into the number of roots of (1), but first we must know the number of solutions in quaternions of equation (7).

4. **Quadratic equations.** Consider the equation

$$(8) \qquad x^2 + bx + c = 0, \qquad c \neq 0,$$

b and c being real quaternions. We assume that $t(b)$, the trace of b, is zero, for otherwise the substitution $x = y - \frac{1}{4}t(b)$ gives a quadratic equation with the required property. For example, the substitution $x = y + \frac{1}{2}t_0$ in equation (7) gives

$$(9) \qquad y^2 - d = 0, \qquad d = \tfrac{1}{4}t_0^2 - n_0.$$

We shall need the treatment of this equation to complete the discussion of (8). Suppose that $y=y_0+y_1i+y_2j+y_3ij$, each y with a subscript being real. We substitute in (9) and separate the result with respect to the linearly independent units 1, i, j, and ij, to obtain

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 = d, \qquad y_0y_1 = y_0y_2 = y_0y_3 = 0.$$

If $d \geqq 0$, then $y_1=y_2=y_3=0$, and the roots of (9) are $\pm\sqrt{d}$. If $d<0$, then $y_0=0$, and we obtain an infinitude of quaternion solutions of (9), corresponding to the real solutions of $y_1^2+y_2^2+y_3^2=-d$. Henceforth we take b and c to be not both real.

Applying the division algorithm of §3 to (8), we obtain the following values for the functions f and g:

$$f = b + t, \qquad g = c - n. \tag{10}$$

Hence equations (6) become

$$nt^2 + nb\bar{b} - c\bar{c} + n(c + \bar{c}) - n^2 = 0, \tag{11}$$

and

$$t^3 + tb\bar{b} - 2nt + t(c + \bar{c}) + \bar{b}c + \bar{c}b = 0, \tag{12}$$

since $b+\bar{b}$ vanishes. Following the theory of §3, we see that if a real solution $t=t_0$, $n=n_0$ of these equations satisfies $f=0$ and $g=0$, then we have $b=-t_0$ and $c=n_0$. But t_0 is real, and b has zero trace, so that both are zero; also, c must be real. Hence equation (8) reduces to one of type (9), contrary to hypothesis. Consequently the solutions of (11) and (12) do not satisfy $f=0$, and this, by §3, implies that each of these real solutions gives exactly one solution of (8); the solution is given by the substitution of the functions (10) in equation (4).

We introduce the notation

$$B = b\bar{b} + c + \bar{c}, \qquad C = c\bar{c}, \qquad D = \bar{b}c + \bar{c}b, \tag{13}$$

noting that B, C, and D are real. First we consider solutions of (11) and (12) with $t=0$, so that $D=0$, by (12). Then the possible values of n are given by (11), which reduces to

$$n^2 - Bn + C = 0. \tag{14}$$

We want real roots; any real root will be positive because of the manner in which equations (11) and (12) were set up. Thus we obtain 0, 1, or 2 roots of (8) according as B^2-4C is negative, zero, or positive.

Finally, we search for solutions of (11) and (12) with $t\neq 0$. We solve (12) for n; thus

$$n = (t^3 + tB + D)/2t, \tag{15}$$

and we substitute this value in (11) to obtain

$$t^6 + 2Bt^4 + (B^2 - 4C)t^2 - D^2 = 0. \tag{16}$$

Each distinct real root of this equation gives us, by use of (15), a root of (8). In order to find the number of real roots of (16), we prove the following:

LEMMA 1. *If B is negative, so is B^2-4C.*

Proof. Since $b\bar{b}$ is not negative, $c+\bar{c}$ must be negative by the hypothesis. We can write

$$B^2 - 4C = b\bar{b}B + b\bar{b}(c + \bar{c}) + (c - \bar{c})^2.$$

If c has the form $c_0+c_1i+c_2j+c_3ij$, then the last term on the right side of this equation equals $-4(c_1^2+c_2^2+c_3^2)$. Hence the three terms on the right are real, and none of them is positive. They cannot all be zero, for that would imply that $b=0$ and $c=\bar{c}$, contrary to hypothesis, and this proves the lemma.

We consider (16) as a cubic in t^2, and look for positive roots. If $D\neq 0$, the number of positive roots is one by Descartes' rule of signs and Lemma 1. Thus equation (16), considered as a sextic, has two real roots when $D\neq 0$.

If $D=0$, we divide the obvious zero roots out of (16), and have

$$t^4 + 2Bt^2 + B^2 - 4C = 0. \tag{17}$$

Considering this as a quadratic equation, we see that the discriminant is not negative, so that the roots are real. If B^2-4C is positive, B is positive by Lemma 1, and the quadratic (17) has negative roots. Hence the quartic (17) has no real root. Similarly, if B^2-4C is zero, we find that the quartic (17) has no real roots other than zeros. In both these cases, all roots of (8) are obtained from (14). Finally, if B^2-4C is negative, the quartic (17) has exactly two real roots, giving two solutions of (8). Note that in this case no solutions of (8) result from (14).

We summarize these results in the following:

THEOREM 2. *Consider the quaternion equation* (8), *the trace of b being zero. If b and c are real* (*so that $b=0$*), *the equation has an infinite number of roots or just two roots according as c is positive or negative. Otherwise, the equation has one or two roots according as the quantities defined in* (13) *satisfy the relations $D=B^2-4C=0$ or not.*

5. The number of roots of (1). We suppose first that no real solution of equations (6) satisfies $f=g=0$, so that there is a one-to-one correspondence between the roots of (1) and the real solutions of (6). We now need some information about the nature of the functions f and g of equation (3).

LEMMA 2. *The functions f and g of equation* (3) *are of degree $m-1$ in n and t; moreover, f has only one term of this degree, namely t^{m-1}. Also, every term of g is divisible by n, with the exception of a_m.*

PROOF. The proof is by induction on m. Equations (10) indicate the truth of the lemma in case $m=2$. We now obtain recurrence relations for f and g. The polynomial of degree $m+1$ analogous to $a(x)$ can be written in the form

$a(x)\cdot x+a_{m+1}$, and corresponding to the algorithm (3) we have

$$\begin{aligned} a(x)\cdot x + a_{m+1} &= qx(x^2 - tx + n) + fx^2 + gx + a_{m+1} \\ &= (qx + f)(x^2 - tx + n) + ftx - fn + gx + a_{m+1}. \end{aligned}$$

Calling the remainder in the last expression above $Fx+G$, we have the relations $F=ft+g$ and $G=-fn+a_{m+1}$. The induction is completed by noting that if the functions f and g have the properties stated in the lemma, so do F and G, m being replaced by $m+1$.

It is a consequence of the above lemma that equations (6) are of degree $2m-1$ in n and t. Now it is known [3] that the curves represented by (6) cannot have more than $(2m-1)^2$ intersections provided that the polynomials N and T are relatively prime; and this is the case when neither of the two resultants of N and T vanishes identically. We now show that this is the case.

By Lemma 2, equations (6) can be written in the forms

$$\begin{aligned} N(t, n) &= c_0 + c_1 t + \cdots + c_{2m-3}t^{2m-3} + c_{2m-2}t^{2m-2}, \\ T(t, n) &= d_0 + d_1 t + \cdots + d_{2m-2}t^{2m-2} + t^{2m-1}, \end{aligned}$$

the coefficients c_r and d_r being polynomials in n. Then the resultant obtained by eliminating t is

$$\left|\begin{array}{ccccccccccc} c_0 & c_1 & \cdot & \cdot & \cdot & c_{2m-2} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & c_0 & c_1 & \cdot & \cdot & c_{2m-3} & c_{2m-2} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & c_0 & c_1 & \cdot & \cdot & \cdot & c_{2m-2} \\ d_0 & d_1 & \cdot & \cdot & \cdot & d_{2m-2} & 1 & 0 & \cdot & \cdot & 0 \\ 0 & d_0 & d_1 & \cdot & \cdot & \cdot & d_{2m-2} & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & d_0 & d_1 & \cdot & \cdot & \cdot & d_{2m-2} & 1 \end{array}\right| \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 2m-1 \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 2m-2 \text{ rows} \end{array}$$

This is a polynomial in n; in order to show that it does not vanish identically, we prove that it has a non-zero constant term. When we set $n=0$, Lemma 2 and equation (6) show that each $c_r=0$ for $r=1, 2, \cdots, 2m-2$, whereas c_0 assumes the value $-\bar{a}_m a_m$; that is, all elements of the determinant above the principal diagonal vanish. Hence the value of the constant term of this resultant is $(-\bar{a}_m a_m)^{2m-1}$, which is not zero.

Having shown that the polynomials f and g have no common factor involving t, we now eliminate the possibility that they have a polynomial in n alone as a common factor. We could show that this is not possible by proving that the resultant which eliminates n does not vanish identically. But it is easier to proceed directly. If a polynomial in n divides T, it must divide the coefficient of the highest power of t. But this coefficient is unity.

Having shown that equation (1) cannot have more than $(2m-1)^2$ roots when no real solution of (6) satisfies $f=g=0$, we turn to the case where these equations are satisfied by certain solutions of (6). Let there be s such real solutions. Then we have s equations of type (7), each having either two roots or an infinite number of roots (by Theorem 2). If the number of roots is finite, that is, if these equations have two roots each, we divide them out of equation (1). Thus we obtain an equation of degree $m-2s$, which has no factors of the form (7), and hence has at most $(2m-4s-1)^2$ quaternion roots. Adding $2s$ to account for the roots of the quadratic equations, we note that

$$2s + (2m - 4s - 1)^2 < (2m - 1)^2,$$

when $m \geqq 2s$ and $s > 0$. Hence we have shown that if the number of roots of (1) is finite, it cannot exceed $(2m-1)^2$. Theorem 2, and in particular equation (9), can be used now to give the following result:

THEOREM 3. *Equation* (1) *has an infinite number of quaternion roots if and only if $a(x)$ is divisible by an expression of type* (7), *with the real values t_0 and n_0 satisfying the inequality $t_0^2 < 4n_0$. If the number is finite, it cannot exceed* $(2m-1)^2$.

6. The necessity of the conditions of Theorem 1. Denote by u the order of D over its centrum C. Thus there exist u elements in D which are linearly independent over C, but any $u+1$ elements in D are linearly dependent over C. Given any element x in D, there exist therefore elements c_i in C such that $x^u + \sum_{i=0}^{u-1} c_i x^i = 0$. Since D is a division algebra, it follows now that the sub-field $C(x)$ of D which is generated by adjoining the element x of D to C, is a commutative field, finite over C, and the irreducible equation in C which is satisfied by x has a degree not exceeding u. Since every equation in C has a solution in D, this implies that the degrees of irreducible equations in C do not exceed u.

LEMMA 3. *Let A denote the essentially uniquely determined algebraically closed commutative field which contains C and is algebraic over C. Then A is finite over C.*

Proof. Suppose first that C is of characteristic $p \neq 0$. We prove that there is no element t in C such that the equation $z^p - t = 0$ has no solution in C. For, if there were such an element, then none of the equations $z^{p^i} - t = 0$ would have a solution in C. Since this last equation has the form $(z-t_i)^{p^i} = 0$ in the field A, it has one and only one solution in A; since the p^{i-1}th power of this solution is a solution of $z^p - t = 0$, it follows that each of these equations is irreducible in C. But this is impossible, since some p^i is larger than u. Hence every element in C is the pth power of an element in C. Consequently [4], A is separable over C; and this result is also true when the characteristic of C is zero.

If B is some field between A and C, and if B is finite over C, then B is a simple extension of C since it is separable over C. Thus the degree of B over C is equal to the degree of the irreducible equation in C whose solution generates B over C. Since the degrees of irreducible equations in C do not exceed u, it follows that the degrees of finite extensions of C do not exceed u. Consequently,

there exists a field M between A and C which is finite of maximal degree over C. If w is any element of A, then $M(w)$ is finite over C. Since the degree of M over C is as large as possible, it follows that M and $M(w)$ have the same degree over C. Hence w is in M, and $M=A$, and the lemma is proved.

Now it follows from a theorem by Artin-Schreier [5] that either C is a real-closed field, or A equals C. The latter is impossible since C, the centrum of a non-commutative division algebra, cannot be algebraically closed. Hence C is a real-closed field; but the only non-commutative division algebra over C is the algebra of real quaternions [6], and this completes the proof.

References

1. Oystein Ore, Theory of non-commutative polynomials, Annals of Math. (II), vol. 34, 1933, pp. 480–508.

2. Outlined by Cayley in P. G. Tait's Quaternions, third edition, Cambridge Press, 1890, pp. 157–159.

3. *Cf.*, Bôcher's Introduction to Higher Algebra, Macmillan, 1927, problem 4 on p. 239, and the theorem on p. 202. The theorem on p. 202 is not precisely what we want here, since we need a proposition about *polynomials* in two variables. Two polynomials $f(x, y)$, $g(x, y)$ have two resultants, R_x obtained by eliminating x, and R_y by eliminating y. Bôcher's proof shows that the vanishing of R_x identically is a necessary and sufficient condition for the two polynomials to have a common factor involving x. But it is possible for $f(x, y)$ and $g(x, y)$ to have a common factor in y alone, with R_x not identically zero; in such a case R vanishes (*e.g.*, consider the polynomials $xy-4x-3y+12$, $xy+y^2-4x-4y$).

4. See E. Steinitz, Algebraische Theorie der Körper, 1930, p. 50.

5. Artin-Schreier, Hamburger Abhandlung, vol. 5, 1927, p. 230, Theorem 4.

6. See A. A. Albert, Structure of Algebras, A.M.S. Coll. Publ. 24, Chap. 9, Theorem 28, p. 146.

A NOTE ON THE LINEAR DIOPHANTINE EQUATION

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The purpose of this paper is to propose an alternative to the methods presented in a recent paper by Lehmer.* The principal advantage of the alternative is that it leads to a general solution with much smaller coefficients. The value of having small coefficients will be illustrated by using such a solution as an aid in solving an approximation problem.

While the method of this paper applies to any number of variables, we shall confine our attention to the case of one equation in four unknowns:

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = K. \tag{1}$$

We shall assume throughout that K and the a's are integers and that the greatest common factor of the a's is unity. We ask for a general solution in integers of (1).

THEOREM 1. *Let α_{ij} be integers such that*

$$|\alpha_{ij}| = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix} = \pm 1,$$

and

$$\begin{aligned} &a_1\alpha_{11} + a_2\alpha_{12} + a_3\alpha_{13} + a_4\alpha_{14} = 1, \\ &a_1\alpha_{i1} + a_2\alpha_{i2} + a_3\alpha_{i3} + a_4\alpha_{i4} = 0, \qquad (i = 2, 3, 4). \end{aligned} \tag{2}$$

Then

$$x_i = K\alpha_{1i} + L\alpha_{2i} + M\alpha_{3i} + N\alpha_{4i}, \qquad (i = 1, 2, 3, 4), \tag{3}$$

is a general solution of (1).

Proof. The conditions (2) clearly insure that (3) makes (1) identically true in K, L, M, and N. So (3) is a solution of (1). To show that (3) is a general solution of (1), it suffices to show that any set of integral x_i's which satisfies (1) can be expressed by means of (3). Suppose we have such a set of x_i's. Consider (3) as four equations in K, L, M, and N. If we solve them for K, L, M, and N, we will get integral values because $|\alpha_{ij}| = \pm 1$. That the K which we obtain this way is identical with the K of (1), follows from the fact that (3) is a solution of (1).

We might note in passing that, because $|\alpha_{ij}| \neq 0$, there is just one set of K, L, M, N for each set of x_i's, and conversely.

We now indicate an algorithm for finding a set of α_{ij}'s such as described in Theorem 1. We proceed to describe the basic step of the algorithm. We warn the reader not to confuse $|\beta_{ij}|$, which indicates a determinant, with $|W_i|$, which de-

* D. H. Lehmer, A note on the linear Diophantine equation, this MONTHLY, vol. 48, 1941, pp. 240–246.

notes an absolute value.

THEOREM 2. *Suppose there is a set of integers* β_{ij} *with* $|\beta_{ij}| = \pm 1$ *and*

$$a_1\beta_{i1} + a_2\beta_{i2} + a_3\beta_{i3} + a_4\beta_{i4} = W_i, \qquad (i = 1, 2, 3, 4). \tag{4}$$

If $|W_1| + |W_2| + |W_3| + |W_4| > 1$, *then there is a set of integers* γ_{ij} *with* $|\gamma_{ij}| = \pm 1$ *and*

$$a_1\gamma_{i1} + a_2\gamma_{i2} + a_3\gamma_{i3} + a_4\gamma_{i4} = V_i, \qquad (i = 1, 2, 3, 4),$$

such that $|V_1| + |V_2| + |V_3| + |V_4| < |W_1| + |W_2| + |W_3| + |W_4|$.

Proof. Let $|W_1| + |W_2| + |W_3| + |W_4| > 1$. Suppose that all but one of the W_i's are zero. Then by solving equations (4) for the a_i's (remembering that $|\beta_{ij}| = \pm 1$), we see that the non-zero W_i must be a common factor of the a_i's. As the a_i's were assumed coprime, the non-zero W_i must be ± 1. However, this contradicts $\sum |W_i| > 1$. So at least two W_i's are not zero. Let us order the β_{ij}'s and W_i's so that $W_1 \neq 0$, $W_2 \neq 0$, $|W_1| \geqq |W_2|$. This reordering clearly leaves $|\beta_{ij}| = \pm 1$. Now divide W_1 by W_2, obtaining

$$W_1 = QW_2 + R, \qquad |R| < |W_2|.$$

Take

$$\gamma_{ij} = \beta_{ij}, \qquad (j = 1, 2, 3, 4;\ i = 2, 3, 4),$$

$$\gamma_{1j} = \beta_{1j} - Q\beta_{2j}, \qquad (j = 1, 2, 3, 4).$$

In the determinant of the β_{ij}'s, we have merely subtracted Q times the second row from the first, so that $|\gamma_{ij}| = \pm 1$. Also, referring to (4), we see that $V_2 = W_2$, $V_3 = W_3$, $V_4 = W_4$, $V_1 = W_1 - QW_2 = R$. As $|R| < |W_2| \leqq |W_1|$, we have

$$\sum |V_i| < \sum |W_i|.$$

It is evident from Theorem 2 that, to find a set of α_{ij}'s satisfying the conditions of Theorem 1, one can proceed as follows:

I. Find a set of β_{ij}'s such that $|\beta_{ij}| = \pm 1$. It will usually be convenient to start with the unit matrix.

II. Compute the W_i's by means of (4).

III. By applying Theorem 2, successively reduce the value of $|W_1| + |W_2| + |W_3| + |W_4|$ until it has the value unity. Then three of the W_i's are zero and the fourth is ± 1. By reordering and changing signs (if necessary), a set of α_{ij}'s is exhibited.

In applying the algorithm, it is more convenient not to rename the β_{2j}, β_{3j}, β_{4j}, W_2, W_3, and W_4, and also to denote $\beta_{1j} - Q\beta_{2j}$ by β_{5j} and $W_1 - QW_2$ by W_5, and then apply the algorithm the next time to W_2, W_3, W_4, W_5. We shall illustrate by applying the algorithm to the solution of

$$99x + 79y + 55z + 33w = K. \tag{5}$$

(To facilitate comparisons, we have chosen the same equation which Lehmer used to illustrate his method.) We arrange the computations in the following table:

	W_i	β_{i1}	β_{i2}	β_{i3}	β_{i4}	i
	99	1	0	0	0	1
	79	0	1	0	0	2
	55	0	0	1	0	3
	33	0	0	0	1	4
W_1-W_2	20	1	-1	0	0	5
W_2-W_3	24	0	1	-1	0	6
W_3-2W_4	-11	0	0	1	-2	7
W_4-W_6	9	0	-1	1	1	8
W_6-W_5	4	-1	2	-1	0	9
W_5+2W_7	-2	1	-1	2	-4	10
W_7+W_8	-2	0	-1	2	-1	11
W_8-2W_9	1	2	-5	3	1	12
W_9+2W_{11}	0	-1	0	3	-2	13
$W_{10}-W_{11}$	0	1	0	0	-3	14
$W_{11}+2W_{12}$	0	4	-11	8	1	15

This gives the general solution

$$
\begin{aligned}
x &= 2K - L + M + 4N, \\
y &= -5K - 11N, \\
z &= 3K + 3L + 8N, \\
w &= K - 2L - 3M + N.
\end{aligned} \tag{6}
$$

At each step of the algorithm we divided the numerically largest W_i by the next largest. This made a rather long algorithm, but in compensation gave very small coefficients in the general solution. We could shorten the algorithm very much if we are willing to have larger coefficients in the general solution. For instance, taking our cue from Lehmer's very short algorithm, we could take W_1, W_2, W_3, W_4 as above, and then put $W_5=W_1-W_2-W_4$, $W_6=W_2+6W_5$, $W_7=W_3-55W_6$, $W_8=W_4-33W_6$, $W_9=W_5+13W_6$. This leads to exactly the same solution that Lehmer gives in his equation (21). To compare the amount of computation in the two methods, note that our algorithm contains nine steps, and Lehmer's algorithm contains six steps plus the computation of sixteen cofactors. In Lehmer's algorithm, only three sequences are computed, namely his A_k, B_k, C_k, as against four sequences in our algorithm, namely β_{i1}, β_{i2}, β_{i3}, β_{i4}. However, we also only need to compute three sequences, namely β_{i1}, β_{i2}, and β_{i3}, because at any time in the computation, one can compute β_{i4} by means of (4).

We now consider an approximation problem. Let us ask for integers x, y, and z with $x>1000$ such that simultaneously

$$\left| \sqrt{2} - \frac{y}{x} \right| < x^{-3/2}, \qquad \left| \sqrt{3} - \frac{z}{x} \right| < x^{-3/2}.$$

Multiplying through by x, we see that we are asking that

$$(7) \qquad | x\sqrt{2} - y | < 1/\sqrt{x}, \qquad | x\sqrt{3} - z | < 1/\sqrt{x}.$$

Using continued fractions, we find 577/408 and 1393/985 to be two convergents to $\sqrt{2}$, and 989/571 and 1351/780 to be two convergents to $\sqrt{3}$. So

$$\begin{aligned} &985\sqrt{2} - 1393, \qquad && 780\sqrt{3} - 1351, \\ &408\sqrt{2} - 577, && 571\sqrt{3} - 989 \end{aligned}$$

are all less than 0.002 in absolute value. Thus if x, y, z, and w are small integers, then both

$$(8) \qquad (985x + 408y)\sqrt{2} - (1393x + 577y),$$

$$(9) \qquad (780z + 571w)\sqrt{3} - (1351z + 989w)$$

will be small. If at the same time

$$985x + 408y = 780z + 571w,$$

then (8) and (9) may perhaps be a solution of (7). Therefore we would like to find a solution in small integers of

$$(10) \qquad 985x + 408y - 780z - 571w = 0.$$

As a start, we apply the algorithm of Theorem 2 to find a general solution in integers of

$$(11) \qquad 985x + 408y - 780z - 571w = K.$$

We obtain the following for the last four steps:

W_i	β_{i1}	β_{i2}	β_{i3}	β_{i4}	i
0	8	11	10	8	13
1	5	5	6	4	14
0	3	10	20	-15	15
0	9	5	3	15	16

This will give a general solution of (11) which has small coefficients. Hence, from this solution we can readily deduce small integer solutions of (10). With this in mind, let us try to reduce the size of the coefficients still further. To see how this can be done, let W_i symbolize the vector $(\beta_{i1}, \beta_{i2}, \beta_{i3}, \beta_{i4})$ as well as the number W_i. It is obvious that the size of the components of the vector W_{16} will be reduced if we subtract the vector W_{13}. So we replace W_{16} by $W_{17} = W_{16} - W_{13}$. Clearly, the

conditions of Theorem 1 are still satisfied. A further obvious reduction will result if we replace W_{15} by $W_{18}=W_{15}+2W_{17}$.

No other obvious way of reducing the sizes of the coefficients is now apparent, so we write down the solution which we have obtained, namely,

$$\begin{aligned} x &= 5K + 5L + M + 8N, \\ y &= 5K - 2L - 6M + 11N, \\ z &= 6K + 6L - 7M + 10N, \\ w &= 4K - L + 7M + 8N. \end{aligned} \tag{12}$$

Putting $L=1$, $K=M=N=0$, and substituting in (8) and (9), we have

$$\begin{aligned} 4109\sqrt{2} - 5811 &= 0.00353, \\ 4109\sqrt{3} - 7117 &= -0.00323. \end{aligned}$$

Note that $0.00353<1/4\sqrt{4109}$, so that we have a solution of (7) with a factor of 4 thrown in. Solutions of (7) with a factor of 2 thrown in are

$$\begin{aligned} 1463\sqrt{2} - 2069 &= -0.00556, \\ 1463\sqrt{3} - 2534 &= -0.00967 \end{aligned}$$

(found by putting $M=-1$, $K=L=N=0$), and

$$\begin{aligned} 2646\sqrt{2} - 3742 &= 0.00909, \\ 2646\sqrt{3} - 4583 &= 0.00644 \end{aligned}$$

(found by putting $L=M=1$, $K=N=0$). We even obtain a solution of (7) by putting $N=1$, $K=L=M=0$, namely,

$$\begin{aligned} 12368\sqrt{2} - 17491 &= -0.00666, \\ 12368\sqrt{3} - 21422 &= 0.00438. \end{aligned}$$

One might wonder whether the general solution (12) of (11) has the smallest possible coefficients or not. This depends on the definition of "smallest possible." If one interprets "smallest possible" as meaning that the sum of the squares of the coefficients is a minimum, then

$$\begin{aligned} x &= 5L + M + 3N, \\ y &= 7K - 2L - 6M + 13N, \\ z &= 6L - 7M + 4N, \\ w &= 5K - L + 7M + 9N \end{aligned} \tag{13}$$

is a solution with the "smallest possible" coefficients. A very sophisticated algorithm* was used to derive (13). Nevertheless, (12) compares very favorably with (13), in spite of the simplicity of the methods used to derive (12).

* Barkley Rosser, A generalization of the euclidean algorithm to several dimensions, Duke Mathematical Journal, March, 1942.

THE POLYGONAL REGIONS INTO WHICH A PLANE IS DIVIDED BY n STRAIGHT LINES*

W. B. CARVER, Cornell University

It is well known that in certain branches of mathematics, notably in number theory, there are unsolved and certainly difficult problems which can, however, be stated so simply that anyone can understand the problems themselves. It is my purpose to call attention to such a problem in geometry.

Consider a set of n lines, $n \geqq 3$, lying in a plane, no three of them concurrent in a point. One sees that the lines separate the plane into polygonal regions. If the plane is euclidean, some of the regions will be completely bounded, while others will extend out indefinitely between two of the lines. I prefer, however, to consider the problem in the projective plane, in which case each region will be completely bounded. In Figure 1, for instance, there is the 5-sided region R_5

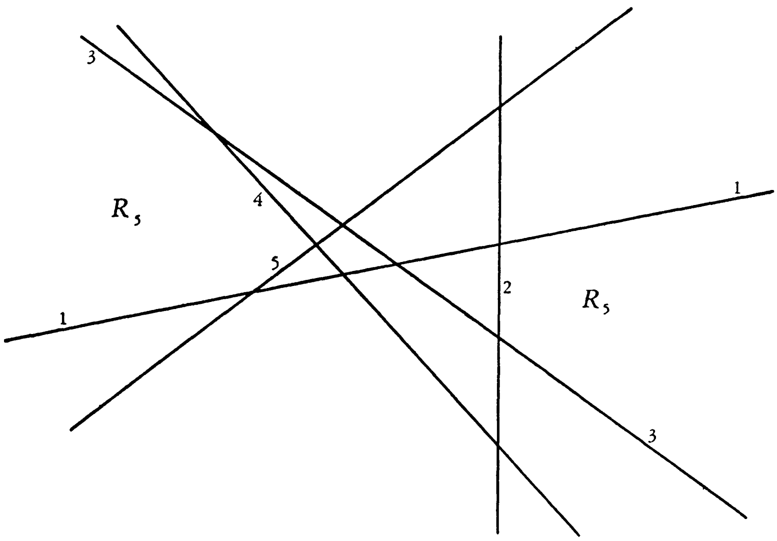

FIG. 1

whose boundary is made up of segments of the lines 1, 2, 3, 4, 5 in that order. Such a region is always the projection of a polygonal region bounded in the ordinary euclidean sense. In addition to this pentagonal region in this figure, one observes 5 triangular regions and 5 quadrilateral regions.

The problem, in its simplest form, may be stated as follows:

If n lines, no three concurrent, lie in a projective plane, separating it into

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* Retiring presidential address presented in Chicago, September 3, 1941, at the joint meeting of the Mathematical Association of America and the American Mathematical Society.

polygonal regions, how many triangular regions will there be, how many quadrilateral, $\cdots$, how many n-sided regions?

Of course it is not intended to imply that there is a unique answer for a given value of n.

I should like first to talk about the problem in an intuitive way, using certain terms (such as "polygonal region") which will be defined loosely or not at all, but whose meaning will be roughly understood. Later I shall attempt to put the whole matter on a more sharply defined basis.

The notation F_n will be used to denote the entire figure of n lines (no three concurrent) in a plane, and R_m to denote any region of m sides ($m \geqq 3$) which is part of such a figure.

For a given n, let α_i, ($i=3, 4, \cdots, n$), be the number of i-sided regions R_i in a figure F_n. For $n=3$, we see that the figure F_3 consists always of 4 triangular regions, $\alpha_3=4$. We see also that when we have a figure F_n and add one more line, the new figure F_{n+1} will have n more regions than the figure F_n. It follows inductively that the total number of regions in a figure F_n is

$$\alpha_3 + \alpha_4 + \alpha_5 + \cdots + \alpha_n = (n^2 - n + 2)/2. \tag{1}$$

It is also easily seen that the total number of all the sides of all the regions in a figure F_n is

$$3\alpha_3 + 4\alpha_4 + 5\alpha_5 + \cdots + n\alpha_n = 2n^2 - 2n. \tag{2}$$

For any given n this gives us two Diophantine equations in the α's which must be satisfied by non-negative integers. For $n=4$ we have the unique solution $\alpha_3=4$, $\alpha_4=3$, indicating that the figure F_4 must always consist of 4 triangular and 3 quadrilateral regions. But for $n>4$, these equations have many solutions in non-negative integers which do not correspond to any possible figure F_n. Thus for $n=5$ we have the four solutions

$$\begin{aligned} \alpha_3 &= 4 \quad 5 \quad 6 \quad 7, \\ \alpha_4 &= 7 \quad 5 \quad 3 \quad 1, \\ \alpha_5 &= 0 \quad 1 \quad 2 \quad 3, \end{aligned}$$

only one of which corresponds to an actual figure F_5.

If from (1) and (2) we eliminate α_4, we obtain

$$\alpha_3 = \alpha_5 + 2\alpha_6 + 3\alpha_7 + \cdots + (n-4)\alpha_n + 4, \tag{3}$$

which shows that in any F_n there must always be at least 4 triangular regions.

Also, we note that in any F_n the average number of sides for a region is

$$\frac{2n^2 - 2n}{(n^2 - n + 2)/2} = 4 - \frac{8}{n^2 - n + 2},$$

which is slightly less than 4 and approaches 4 as n increases.

I believe that the only known results on the problem are those given by

H. S. White and Louise D. Cummings of Vassar College in four papers in the *Bulletin* in 1932 and 1933. They give complete results for values of n up through $n=7$ and partial results for $n=8$, as tabulated below. Their method of attack was (to quote White) "observational plus very little logic."

n	α_3	α_4	α_5	α_6	α_7	α_8
3	3					
4	4	3				
5	5	5	1			
6	6	9	0	1		
	6	8	2	0		
	7	6	3	0		
	10	0	6	0		
7	7	14	0	0	1	
	11	5	5	1	0	
	9	9	3	1	0	
	8	11	2	1	0	
	7	13	1	1	0	
	10	6	6	0	0	
	9	8	5	0	0	
	8	10	4	0	0	
	8	10	4	0	0	
	7	12	3	0	0	
	7	12	3	0	0	
8	8	20	0	0	0	1
	14	7	7	0	1	0
	12	11	5	0	1	0
	12	11	5	0	1	0
	12	11	5	0	1	0
	10	15	3	0	1	0
	9	17	2	0	1	0
	9	17	2	0	1	0
	8	19	1	0	1	0
	11	12	5	1	0	0
	10	14	4	1	0	0
	10	14	4	1	0	0
	9	16	3	1	0	0
	9	16	3	1	0	0
	8	18	2	1	0	0

More important than their results for these special values of n was a feature of the problem which was brought out in their work. They found that there could be two figures F_7 in each of which there were 8 triangles, 10 quadrilaterals, and 4 pentagons, but these regions arranged quite differently in the two figures. In the one figure, for instance, two of the pentagonal regions have a side in common while in the other figure no two of the four pentagons have such a common side (Figs. 2 and 3). This is the reason for the repeated lines in the table, and it necessitates an extension of the problem. It is not enough to ask for each n how many regions of each kind we may have, but we want to know how many kinds of non-equivalent figures F_n exist, equivalence to be defined in some definite way. Professor Cummings in her final paper uses a test for equivalence to show that certain of her figures are not equivalent; but it seems rather unlikely that her test is a sufficient test for the equivalence defined later in this paper.

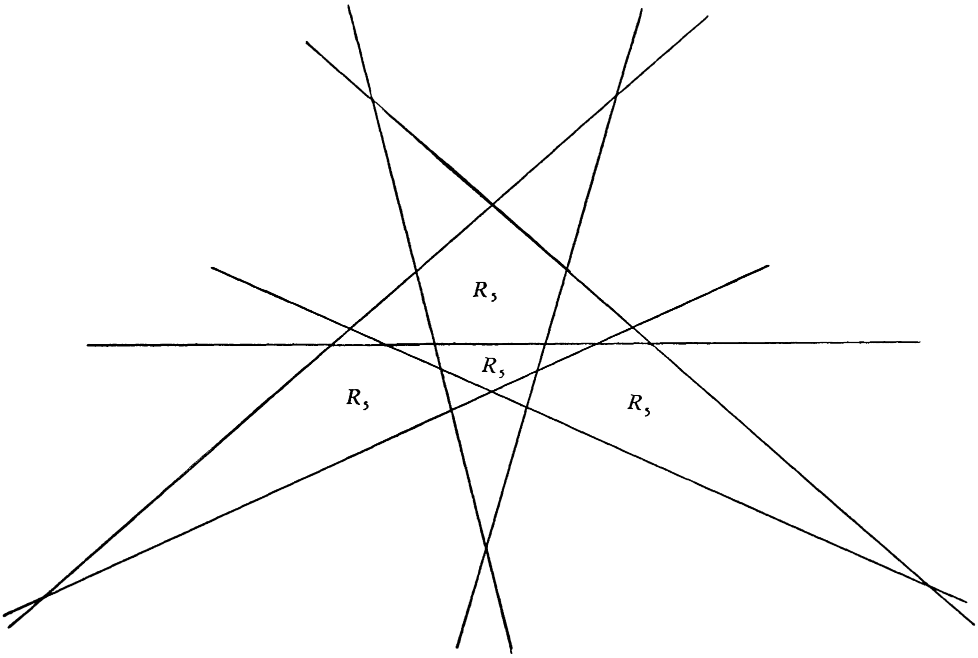


FIG. 2

It is clear that one can hardly hope to obtain any general results in the problem on the basis of intuitive and observational facts, and that accurate definitions of the ideas involved are necessary. It is my purpose now to attempt to lay such a foundation for the problem, and to give a few rather trivial general results.

Our geometry is that of the real projective plane, points being in one-to-one correspondence with ordered homogeneous triads of real numbers $(x_1:x_2:x_3)$ and lines with similar triads of real numbers $[u_1:u_2:u_3]$, point and line being incident when $u_1x_1+u_2x_2+u_3x_3=0$.

Let p_i, $(i=1, 2, 3, 4, 5)$, be five lines with coördinates $[u_1^{(i)}:u_2^{(i)}:u_3^{(i)}]$, and let $|ijk|$ denote the determinant

$$\begin{vmatrix} u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \\ u_1^{(j)} & u_2^{(j)} & u_3^{(j)} \\ u_1^{(k)} & u_2^{(k)} & u_3^{(k)} \end{vmatrix}.$$

Then the 5-line symbol $[p_i \cdot p_j p_k \cdot p_l p_m]$, or more simply $[i \cdot jk \cdot lm]$, will denote

$$\operatorname{sgn} \{ |ijl| \cdot |ijm| \cdot |ikl| \cdot |ikm| \},$$

where sgn x is the signum function, equal to 1 when x is positive, to -1 when x is negative, and to 0 when $x=0$. It is easily seen that the product of the four

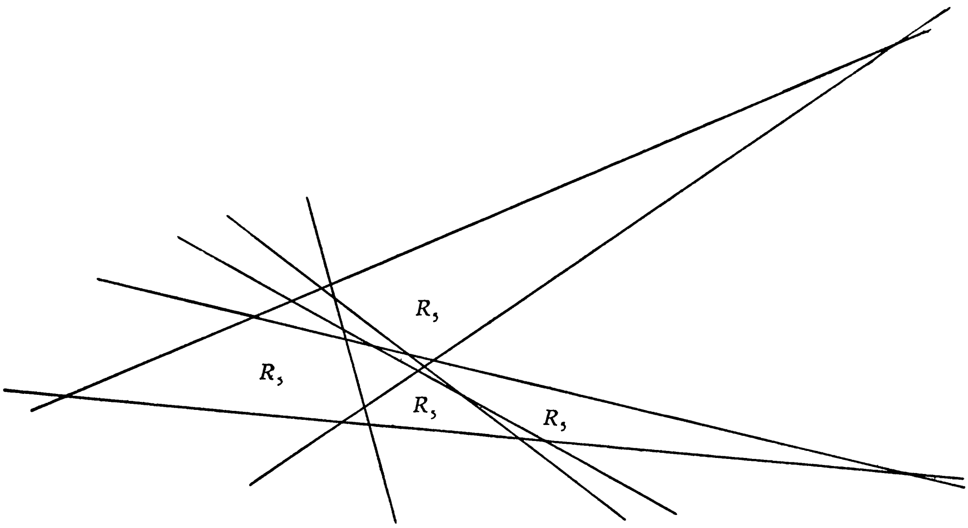


FIG. 3

determinants has the same sign as the cross-ratio of the four points where the line p_i is cut by the lines p_j, p_k, p_l, p_m. If no three of the lines are concurrent, none of the determinants will vanish; and we will have $[i \cdot jk \cdot lm] = -1$ when on the line p_i the points of intersection with p_j and p_k separate the points of intersection with p_l and p_m, and $[i \cdot jk \cdot lm] = 1$ when these pairs of points do not separate each other. Referring to Figure 1, for example, we see that for the five lines of that figure, $[1 \cdot 24 \cdot 35] = -1$, $[4 \cdot 52 \cdot 13] = -1$, $[3 \cdot 45 \cdot 12] = 1$.

Dually, we define a 5-point symbol $(P_i \cdot P_j P_k \cdot P_l P_m)$, or simply $(i \cdot jk \cdot lm)$, and we have $(i \cdot jk \cdot lm) = \pm 1$ according as the lines P_iP_j and P_iP_k do or do not separate the lines P_iP_l and P_iP_m at the point P_i.

For a set of five distinct elements, the 120 such symbols are obviously equal in sets of eight, such as

$$\begin{aligned}(i \cdot jk \cdot lm) &= (i \cdot kj \cdot lm) = (i \cdot jk \cdot ml) = (i \cdot kj \cdot ml) = (i \cdot lm \cdot jk) \\ &= (i \cdot lm \cdot kj) = (i \cdot ml \cdot jk) = (i \cdot ml \cdot kj),\end{aligned}$$

and hence there are essentially 15 symbols to be considered for 5 elements, and 15_nC_5 symbols for n elements.

The notation $(i\cdot jk\cdot lm)=1$ may be regarded as an algebraic equation or as a symbolic method of stating a certain geometric fact. Without great difficulty one may prove algebraically certain relations between these symbols, and these relations may then be regarded as a symbolic statement of geometric theorems. A rather obvious case is the statement:

If $(i\cdot jk\cdot lm)=-1$, then $(i\cdot jl\cdot km)=(i\cdot jm\cdot lk)=1$.

A less obvious relation is the identity

$$(i\cdot jk\cdot lm)(i\cdot jk\cdot ln) = (i\cdot jk\cdot mn);$$

and there are other useful relations.

It is important to note that these 5-point and 5-line symbols are invariant under a linear homogeneous transformation with real coefficients and non-vanishing determinant, that is, under real projective transformations.

By an ordered set of n lines (or points) we shall mean n lines in a specified *reversible cyclic* order (like n keys on a key-ring). A set of n things, $n\geqq 3$, may be ordered in this sense in any one of $(n-1)!/2$ different ways,—three things in only one way, four things in three ways.

We may now define convexity for an ordered set of n lines. Any ordered set of 3 lines not concurrent will be a convex set; and any ordered set of 4 lines, no three of them concurrent, will be a convex set. A set of 5 lines in the order $p_1p_2p_3p_4p_5$ will be convex if and only if

$$[1\cdot 24\cdot 35] = [2\cdot 35\cdot 41] = [3\cdot 41\cdot 52] = [4\cdot 52\cdot 13] = [5\cdot 13\cdot 24] = -1.$$

An ordered set of n lines, $n>5$, will be convex if and only if every sub-set of five lines *with the order preserved* is convex.

The five conditions above for the convexity of five lines are not independent. From the two conditions

$$[1\cdot 24\cdot 35] = [4\cdot 52\cdot 13] = -1, \tag{4}$$

the other three may be deduced, and these two conditions are then necessary and sufficient for the convexity of five lines. Moreover, they determine the values (± 1) of all the other symbols for the five lines. Similarly, the necessary and sufficient conditions that n lines in the order $p_1p_2p_3\cdots p_n$ be convex are the $2(n-4)$ conditions

$$[1\cdot 2(s+3)\cdot(s+2)(s+4)] = [(s+3)\cdot(s+4)2\cdot 1(s+2)] = -1, \qquad (s=1, 2, \cdots, n-4). \tag{5}$$

On the basis of this definition of convexity, we can prove rather simply that any set of five lines no three of which are concurrent can be ordered in one and only one way such that the set will be convex in that order. Also, for $n>5$, if a set of lines is convex in one order it is not convex in any other order. Moreover, when a set of n lines is convex in the order $1\ 2\ \cdots\ n$, any one line, say p_i, will be

cut by the other $n-1$ lines in points in the order $1\ 2 \cdots (i-1)\ (i+1)\ (i+2) \cdots n$. On the line p_i, then, there will be a segment terminated by the intersections with p_{i-1} and p_{i+1} and having on it none of the other points of intersection.

Dualizing all of the above one may define convex sets of n points in terms of the 5-point symbols. Then one proves the theorem that given a convex ordered set of n lines, the ordered set of n points which are intersections of adjacent lines is a convex set of points; and dually we have the converse theorem. Thus with any convex set of n lines there is the associated convex set of n points. and *vice versa*.

We are now prepared to define a region of m sides, R_m. There are special difficulties with the case $m=3$, so we shall first define the region R_m for $m \geqq 4$.

A region R_m, $m \geqq 4$, is uniquely determined by a convex ordered set of m lines, $p_1p_2 \cdots p_m$ and the associated convex ordered set of m points, $P_1P_2 \cdots P_m$ (P_i being the point of intersection of p_i and p_{i+1}, and all subscripts for such ordered sets being reduced mod m). We shall call p_i the bounding lines and P_i the bounding points (or vertices) of the region. There will always exist points X satisfying the $m\ (m-3)$ conditions

$$(P_i \cdot P_{i-1}P_{i+1} \cdot P_kX) = 1, \qquad (i = 1, 2, \cdots, m), \tag{6}$$

k being any subscript other than i, $i-1$, $i+1$. We shall call these points X the points of the region, or the points *inside* the region. Similarly, there will be lines u satisfying the dual conditions

$$[p_i \cdot p_{i-1}p_{i+1} \cdot p_ku] = 1; \tag{7}$$

and these lines u will be called the lines of the region, or the lines *outside* the region. The segment of the bounding line p_i terminated by the intersections with p_{i-1} and p_{i+1} and containing none of the other intersections may be called the *side* s_i of the region.

A convex set of 3 lines and the associated set of 3 points do not determine a unique triangular region, but four such regions. In order to specify a particular triangular region we must make use of its bounding lines and points and an arbitrary fixed point in the plane not on any of the bounding lines. In terms of the bounding lines and points and this arbitrary fixed point, one may define the points of a unique R_3, its lines, and its sides, in a way similar to the definitions for an R_m with $m>3$.

A set of 4 lines is convex in any one of the three possible orders, and hence there are three different regions with the same set of bounding lines; but this cannot occur for larger values of m because for $m>4$, m lines, if convex at all, will be so in only one order.

On the basis of these definitions we can now prove the following theorem of fundamental importance:

THEOREM I. *Let there be given a region R_m with bounding lines in the order $p_1p_2 \cdots p_m$ and a line p which is not outside R_m and not through any vertex of R_m;*

then p will cut two and only two sides of R_m, say S_i and S_{i+k} with $i<i+k\leqq m$, and p will divide R_m into two regions R_{k+2} and R_{m-k+2} in the sense that every point of R_m and not on the line p will be a point of one and only one of these two regions, and the bounding lines of R_{k+2} and R_{m-k+2} in order will be respectively

$$p_i p_{i+1} \cdots p_{i+k} p,$$

and

$$p_{1+k} p_{i+k+1} \cdots p_m p_1 p_2 \cdots p_i p.$$

Note that if $m=3$ in this theorem, $k=1$ or 2, and R_{k+2} and R_{m-k+2} must be R_3 and R_4.

Note also that

$$3 \leqq \left\{ \begin{matrix} k+2 \\ m-k+2 \end{matrix} \right\} \leqq m+1.$$

We now define a figure F_n. Let there be given n lines in the plane, no three of them concurrent. If we can pick out a sub-set of m of these lines (including the possible case $m=n$) which is convex in some order and such that all the other $n-m$ lines of the set are outside of the region R_m determined by the m lines of the sub-set,* we will say that R_m is a region of the figure F_n and that F_n consists of all such regions.

Then we prove the following:

THEOREM II. *There will be exactly $(n^2-n+2)/2$ regions of the figure F_n. If α_i is the number of regions R_i in F_n, then*

$$\sum_{i=3}^{n} \alpha_i = (n^2 - n + 2)/2,$$

and

$$\sum_{i=3}^{n} i\alpha_i = 2n^2 - 2n.$$

THEOREM III. *Each point in the plane which is not on any of the lines of F_n lies in one and only one of the $(n^2-n+2)/2$ regions of F_n.*

DEFINITION. *Two figures F_n and F_n' are equivalent if a one-to-one correspondence exists between the lines of F_n and those of F_n' such that whenever m ordered lines of F_n bound a region R_m of F_n the corresponding ordered m lines of F_n' bound a region R_m' of F_n'.*

All of the foregoing merely establishes a sound logical basis for the problem, and states facts more or less obvious intuitively. The problem of finding how many non-equivalent figures F_n exist for a given n is still unanswered, even for $n=8$.

* For $m=3$ we must say that the other $n-3$ lines of the set are all outside *some one* of the four R_3's defined by the three lines.

In the line of general theorems I can offer only three, the last and most important of which is due to N. G. Gunderson, a graduate student at Cornell, who has interested himself in the problem.

THEOREM IV. *Every figure F_n for which $n \geqq 5$ must contain a region R_m for which $m \geqq 5$.*

THEOREM V. *For any $n \geqq 5$ there exists a figure F_n consisting of one R_n, $(n^2-3n)/2$ R_4's, and n R_3's. This is the F_n when the n lines are convex in some order. All figures F_n containing a region R_n are equivalent.*

THEOREM VI. *For any even $n \geqq 6$ there exists a figure F_n consisting of one R_{m-1}, $n-1$ R_5's, $(n^2-7n+6)/2$ R_4's, and $2(n-1)$ R_3's. A special example of such a figure is the regular polygon of $n-1$ sides together with the line at infinity.*

THEOREM VII. (Gunderson's theorem.) *If in a figure F_n there exist two different regions R_m and R_s, then $m+s \leqq n+4$.*

The consequences of this theorem are observable in the table for the values of the α's. For $n=7$ we could not have two R_6's or two R_7's, or an R_6 and an R_7. For $n=8$, we must, in each row, have at least two zeros in the last three columns. However, we may have two R_6's although this does not occur in the incomplete table given.

ON A TRIANGLE INSCRIBED IN A RECTANGULAR HYPERBOLA

E. F. ALLEN, Oklahoma A. and M. College

1. Introduction. The formulas of inversive geometry* for the equations of a circle, a line through two points, and the polar line of a point with respect to a circle, are expressed in terms of $z=x+iy$, and its conjugate $\bar{z}=x-iy$, where z is a complex number and i is a symbol defined by the equation $i^2=-1$. These formulas admit of a wider interpretation which depends upon the definition of the symbol i. Using the symbol r instead of the symbol i, it is the purpose of this paper to show, by defining the symbol r in a different way, that these formulas altered by the introduction of a suitable constant hold for any central conic, and to apply them to two theorems concerning triangles inscribed in rectangular hyperbolas.

Let a point $Z(x, y)$ in the xy-plane be designated by $z=x+ry$, where $r\neq 0$, and its conjugate point by $\bar{z}=x-ry$; then the equation of a central conic with center at the origin and axes along the coördinate axes is

$$z\bar{z} = a^2. \tag{1}$$

This conic is either an ellipse or hyperbola depending on the definition of the symbol r. If r is defined by the equation $r^2=-k^2$, where k is a real positive number, equation (1) represents the ellipse $x^2+k^2y^2=a^2$, and if $r^2=+k^2$, the equation rep-

* Morley and Morley, Inversive Geometry.

resents the hyperbola $x^2-k^2y^2=a^2$. In particular, if $k=1$ equation (1) represents a circle or a rectangular hyperbola.

2. The equation of a line through two points. It will be assumed that the equation of a line through the two points t_1 and t_2 in determinant form is

$$\begin{vmatrix} z & \bar{z} & 1 \\ t_1 & \bar{t}_1 & 1 \\ t_2 & \bar{t}_2 & 1 \end{vmatrix} = 0. \tag{2}$$

If the two points lie on the conic $z\bar{z}=a^2$, the determinant may be written

$$\begin{vmatrix} z & \bar{z} & 1 \\ t_1 & a^2/t_1 & 1 \\ t_2 & a^2/t_2 & 1 \end{vmatrix} = 0. \tag{3}$$

Upon expanding the determinant and removing the factor t_2-t_1, since $t_1t_2\neq 0$, this becomes

$$a^2z + t_1t_2\bar{z} = a^2(t_1+t_2). \tag{4}$$

If $a=1$ and $r^2=-1$, this is the equation of a line through two points on the base circle as found in inversive geometry.

It is easy to reduce the determinant (2) to the ordinary equation of a line through two points. Since by definition $z=x+ry$, $\bar{z}=x-ry$, $t_i=x_i+ry_i$, and $\bar{t}_i=x_i-ry_i$, where $i=1, 2$, the determinant becomes

$$\begin{vmatrix} x+ry & x-ry & 1 \\ x_1+ry_1 & x_1-ry_1 & 1 \\ x_2+ry_2 & x_2-ry_2 & 1 \end{vmatrix} = 0. \tag{5}$$

Writing this as the sum of four determinants, we have

$$\begin{vmatrix} x & x & 1 \\ x_1 & x_1 & 1 \\ x_2 & x_2 & 1 \end{vmatrix} + r^2 \begin{vmatrix} y & -y & 1 \\ y_1 & -y_1 & 1 \\ y_2 & -y_2 & 1 \end{vmatrix} + r \begin{vmatrix} y & x & 1 \\ y_1 & x_1 & 1 \\ y_2 & x_2 & 1 \end{vmatrix} - r \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{6}$$

The first and second determinants vanish identically and the third and fourth may be combined, giving

$$-2r \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{7}$$

Since $r\neq 0$, we obtain

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \tag{8}$$

which is the two point form of the equation of a line. In this derivation it was not necessary to specify whether r^2 equals -1, $+1$, $-k^2$, or $+k^2$, and therefore equation (4) is the equation of a line through two points of the conic (1).

In a similar manner it can be proved that

$$a^2 z + t_1^2 \bar{z} = 2a^2 t_1 \tag{9}$$

is the equation of the tangent line to the conic (1) at the point t_1. Upon dividing by t_1, equation (9) becomes

$$\bar{t}_1 z + t_1 \bar{z} = 2a^2. \tag{10}$$

Also, the equation of the polar line of the point p with respect to the conic $z\bar{z} = a^2$ can be proved to be

$$\bar{p}z + p\bar{z} = 2a^2. \tag{11}$$

If the conic is given by the equation

$$(z - c)(\bar{z} - \bar{c}) = a^2, \tag{12}$$

the equation of the polar line of the point p becomes

$$(\bar{p} - \bar{c})(z - c) + (p - c)(\bar{z} - \bar{c}) = 2a^2. \tag{13}$$

3. A nine-point hyperbola. Every triangle inscribed in a rectangular hyperbola has a nine-point hyperbola having asymptotes parallel to the asymptotes of the circumscribing hyperbola, which has some of the characteristics of the nine-point circle of a triangle. The reader will recall that the nine-point circle of a triangle passes through the midpoints of the sides of the triangle, the feet of the altitudes, and the midpoints of the line segments joining the vertices to the orthocenter. A nine-point hyperbola of a triangle inscribed in a rectangular hyperbola is a rectangular hyperbola, having asymptotes parallel to the asymptotes of the given hyperbola, and a transverse axis equal to one-half the transverse axis of that hyperbola. The nine-point hyperbola passes through the following sets of points:

(*a*) The midpoints D_1, D_2, D_3 of the sides of the triangle.

(*b*) Three points, one on each side of the triangle, determined as follows: From the center O of the hyperbola draw a line to the centroid G of the triangle and produce the line to the point H so that the segment OH is three times as long as the segment OG. Lines drawn through each vertex of the triangle and the point H intersect the opposite sides in points A_1, A_2, A_3 which lie on the nine-point hyperbola.

(*c*) The midpoints B_1, B_2, B_3 of the line segments joining each vertex to the point H.

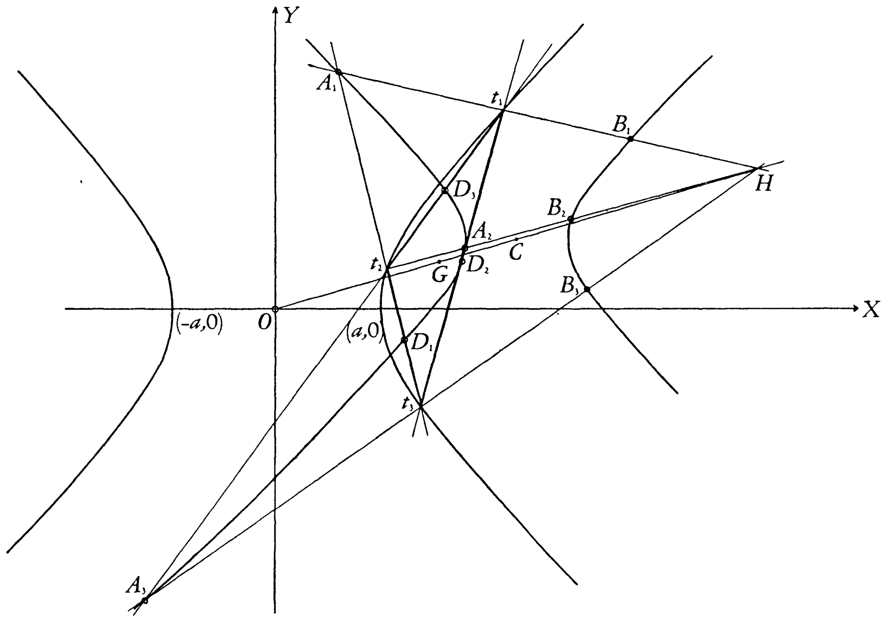
Let the triangle t_1, t_2, t_3 be inscribed in the hyperbola

$$z\bar{z} = a^2. \tag{14}$$

(In the remainder of this paper it is assumed that $r^2=+1$.) The centroid G of this triangle is given by $g=s_1/3$, where $s_1=t_1+t_2+t_3$ and $\bar{s}_1=\bar{t}_1+\bar{t}_2+\bar{t}_3$, which is equivalent to $\bar{s}_1=a^2(t_1t_2+t_2t_3+t_3t_1)/(t_1t_2t_3)$ since $t_i\bar{t}_i=a^2$. From this it follows that the point H has the coördinates $h=s_1$ and $\bar{h}=\bar{s}_1$. The center C of the nine-point hyperbola is midway between O and H; hence, $c=s_1/2$ and $\bar{c}=\bar{s}_1/2$ are the coördinates of C. According to the above statements, the equation of the nine-point hyperbola is

$$(z - s_1/2)(\bar{z} - \bar{s}_1/2) = a^2/4. \tag{15}$$

We prove that equation (15) represents the nine-point hyperbola of the triangle t_1, t_2, t_3 inscribed in the hyperbola $z\bar{z}=a^2$ by showing that the coördinates of the nine points listed above satisfy the equation.



The nine-point hyperbola of triangle $t_1\ t_2\ t_3$.

(*a*) By using the methods of the first part of this paper it can be proved that the midpoint of the side t_1t_2 is given by

$$z = \tfrac{1}{2}(t_1 + t_2), \qquad \bar{z} = a^2(t_1 + t_2)/(2t_1t_2). \tag{16}$$

The last expression is obtained from $\bar{z}=\frac{1}{2}(\bar{t}_1+\bar{t}_2)$ by using the relations $\bar{t}_1=a^2/t_1$ and $\bar{t}_1=a^2/t_2$. Substituting the expressions for z and $\bar{z}$ and reducing, it is seen that equation (15) is satisfied identically. Therefore the point D_3 lies on the nine-point hyperbola. A similar proof holds for D_1 and D_2.

(*b*) The equation

$$a^2z - t_1t_2\bar{z} = a^2t_3 - t_1t_2\bar{t}_3 \tag{17}$$

is satisfied by $z=t_3$ and $\bar{z}=\bar{t}_3$, and also by the coördinates of H. Hence, (17) is the equation of the line t_3H. To find the coördinates of A_3, we solve equation (17) and

$$a^2z + t_1t_2\bar{z} = a^2(t_1 + t_2), \tag{18}$$

the equation of the line t_1t_2. This gives

$$z = s_1/2 - (t_1t_2)/(2t_3), \qquad \bar{z} = \bar{s}_1/2 - (a^2t_3)/(2t_1t_2). \tag{19}$$

These values, and likewise the coördinates of A_1 and A_2, satisfy equation (15). Hence A_1, A_2, and A_3 lie on the nine-point hyperbola.

By transforming to rectangular coördinates, the line t_3H is found to be the anti-parallel of the altitude of the triangle from the vertex t_3 with respect to the asymptotes of $z\bar{z}=a^2$. It follows at once that the lines t_1H and t_2H are anti-parallel to the altitudes of the triangle from the vertices t_1 and t_2, respectively, with respect to the asymptotes of the hyperbola $z\bar{z}=a^2$. The form of the equation of t_3H shows that the angle between the line t_3H and the side t_1t_2 is bisected by a line parallel to an asymptote. Due to this relation, the point A_3 can be found geometrically by observing that it is the midpoint of the line segment intercepted on the line t_1t_2 by two lines through t_3 parallel to the asymptotes of the hyperbola $z\bar{z}=a^2$. By virtue of the above relations, I am calling the point H the anti-orthocenter of the triangle $t_1t_2t_3$ with respect to the given hyperbola.

(c) The coördinates of B_3, the midpoint of the line segment t_3H, are

$$b_3 = \tfrac{1}{2}(t_3 + s_1), \qquad \bar{b}_3 = \tfrac{1}{2}(\bar{t}_3 + \bar{s}_1), \tag{20}$$

or in terms of t_1, t_2, t_3,

$$b_3 = \tfrac{1}{2}(t_1 + t_2 + 2t_3), \qquad \bar{b}_3 = \tfrac{1}{2}a^2(2t_1t_2 + t_2t_3 + t_3t_1)/(t_1t_2t_3). \tag{21}$$

These values are readily seen to satisfy equation (15).

The line segment D_3B_3 is bisected by the center C of the nine-point hyperbola, since the expression $\frac{1}{2}(d_3+b_3)$ reduces to $s_1/2$ when the values above are substituted for d_3 and b_3.

Thus we have found nine points on the hyperbola (15). The theory of the nine-point hyperbola and that of the nine-point circle differ mainly in the interpretation of the line t_3H. If r in the expression $z=x+ry$ is defined by the equation $r^2=-1$, then equation (17) represents the altitude of the triangle from the vertex t_3; but if $r^2=+1$, (17) represents the anti-parallel to the altitude from t_3 with respect to the asymptotes of the hyperbola $z\bar{z}=a^2$. For the general case, it can be proved by transforming to rectangular coördinates that the slope of the line t_3H multiplied by $-r^2$ equals the slope of the altitude from t_3.

It should be pointed out that a given triangle z_1, z_2, z_3 does not have a unique nine-point hyperbola, since the triangle can be inscribed in a rectangular hyperbola having any two perpendicular directions for its asymptotes. But if we specify that the circumscribing rectangular hyperbola shall have a given orientation, then the nine-point hyperbola has the same orientation. In this case we

are studying a set of rectangular hyperbolas having parallel asymptotes. It is obvious that the direction of an asymptote must not be parallel to one of the sides of the triangle. For every orientation of the circumscribing hyperbola, except as noted in the last sentence, the nine-point hyperbola passes through the midpoints of the sides of the given triangle, but the sets A_1, A_2, A_3 and B_1, B_2, B_3 depend upon the particular orientation.

The point H has been called the anti-orthocenter of the triangle $t_1t_2t_3$ with respect to the given hyperbola. Four points such that any one of them is the anti-orthocenter of the triangle determined by the other three form an anti-orthocentric group of points, and the four triangles obtained by taking three points at a time is called an anti-orthocentric group of triangles. The four points $t_1t_2t_3H$ form an anti-orthocentric group of triangles having a common nine-point hyperbola. Consider the triangle t_1t_2H. It may be inscribed in a rectangular hyperbola whose asymptotes are parallel to those of the nine-point hyperbola (15). Since a rectangular hyperbola, with given directions for its asymptotes, is determined by three points and since the hyperbola (15) passes through the midpoints of the sides of the triangle t_1t_2H, this triangle and triangle $t_1t_2t_3$ have a common nine-point hyperbola. The transverse axis of the nine-point hyperbola has been proved to be one-half the transverse axis of the circumscribing hyperbola; therefore the rectangular hyperbolas circumscribing the triangles of an anti-orthocentric group of triangles are congruent. Moreover, their centers can be constructed. For example, to construct the center of the hyperbola in which t_1t_2H is inscribed, draw a line from t_3 to C and produce it to O_3 such that C is the midpoint of t_3O_3. Then O_3 is the required point. Also, it is obvious that the centroid G_3 of this triangle can be found on the line t_3C.

4. Another application of the formulas of section 2. Many theorems concerning a triangle inscribed in a rectangular hyperbola can be proved by means of the formulas of section 2. As an example consider the following:

THEOREM. *Given a triangle $t_1t_2t_3$ inscribed in the unit rectangular hyperbola and a variable point t on the hyperbola, then the parallels to tt_1, tt_2, tt_3 drawn from the anti-orthocenter H meet the corresponding sides of the triangle in three points on the polar Δ of the point t', diametrically opposite to the point t on the hyperbola, with respect to the hyperbola*

$$(z - s_1)(\bar{z} - \bar{s}_1) = \tfrac{1}{2}(s_1\bar{s}_1 - 1). \tag{22}$$

Also, this polar line passes through the midpoint of the segment tH.

Proof. The equation of the line through H parallel to the line tt_1 is

$$z + tt_1\bar{z} = s_1 + tt_1\bar{s}_1. \tag{23}$$

The line tt_1 intersects the side t_2t_3 of the triangle in the point

$$\begin{aligned} z &= s_3(s_1 + t)/(s_3 - tt_1^2), \\ \bar{z} &= t_1(t\bar{s}_1 + 1)/(tt_1 - t_2t_3), \end{aligned} \tag{24}$$

where $s_3=t_1t_2t_3$. Since $t'=-t$, the polar line of t' with respect to the hyperbola (22) is, by equation (13),

$$(ts_2+s_3)z+ts_3(s_1+t)\bar{z}=(s_1+t)(ts_2+s_3), \tag{25}$$

where $s_2=t_1t_2+t_2t_3+t_3t_1$. Since this equation is satisfied by the values (24) of z and $\bar{z}$, the first part of the theorem is proved. The midpoint of the segment tH is given by

$$z=\tfrac{1}{2}(t+s_1), \qquad \bar{z}=(ts_2+s_3)/(2ts_3), \tag{26}$$

which values also satisfy equation (25). This proves the second part of the theorem.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

A COURSE ON THE SIGNIFICANCE OF MATHEMATICS*

HARRIET F. MONTAGUE, University of Buffalo

The announcement of courses by the Mathematics Department at the University of Buffalo contains this paragraph:

"*The Significance of Mathematics.* A course designed to study the place of mathematics in the modern world by discussing its methods, some of the important topics with which it is concerned, and its relation to science, philosophy, and human experience. Prerequisite: intermediate algebra."

This course has been given for four years with sufficient success, we feel, to warrant this discussion of it. Before describing a course, however, which is supposed to be different from the usual run, it might be wise to determine how many other institutions are offering courses of a similar nature. We found in the office of the university registrar from 425 to 450 catalogs from colleges and universities in the United States. We examined the descriptions of courses in the sections on mathematics, endeavoring to find so-called cultural courses in mathematics with no prerequisite beyond intermediate algebra and no restriction as to class or field of concentration of the student registering for the course. We did not count courses in the history of mathematics. In the total number there were only ten courses comparable to the Significance of Mathematics.

Having established to our satisfaction that such a course is the exception rather than the rule, we next examined the types of students attracted by such an offering. We have had 44 students registered in the four years: 23 women and 21 men. They were distributed in the different schools as follows: 23 in the

* Presented to the Upper New York State Section of the Mathematical Association of America at Ithaca, New York, May 3, 1941.

College of Arts and Sciences, 15 in the School of Business Administration, 5 preprofessional, and 1 in the School of Education. At the time they were enrolled in the course, 8 were freshmen, 12 sophomores, 13 juniors, and 11 seniors. The mathematical preparation of the students was varied: 7 had gone no further than intermediate algebra, 7 had had no more than business mathematics, 9 had had trigonometry, 4 advanced algebra, 3 solid geometry, 10 analytic geometry, and 2 had had calculus, but none more than calculus.

If these four-year statistics indicate nothing else, they certainly point out that the groups have been non-homogeneous. The range in age and preparation has been so great as to force us to do the very thing we were desirous of doing, namely, to discuss topics that no one in the group had studied before. Consequently the lectures delivered have not been surveys of elementary mathematics such as trigonometry, analytic geometry, and college algebra. They have instead been introductions to concepts of higher mathematics. It is as if we lifted ourselves above the whole field of mathematics and then dipped down to some of the high points, discovering incidentally the influence of mathematics on other branches of knowledge. Emphasis has been placed on the concepts of modern mathematics, that is, from the 17th century to date. Topics have not always been selected in chronological order. In fact, the historical development has been a secondary consideration. Outstanding contributors to mathematics have been mentioned in connection with their contributions and in this way the historical development has been taken care of to some extent.

Lectures are delivered in the course twice weekly. Each year the content of the course has been changed, new lectures prepared and old ones revised. The first year, 1937–38, we used Dresden's *Invitation to Mathematics* as a basis for lectures. Our first topic was the last theorem of Fermat. Thus at the very beginning of the course the students were jarred out of their complacency in regarding mathematicians as people who always know all the answers.

The next topic was infinite sets. The ideas of 1-1 correspondence, equivalence, and countability were introduced. Some of the controversial points of the theory were indicated, showing that mathematicians need not always agree.

There followed several lectures on the development of our number system. Here we were able to tell the students about fields, giving them an idea of the abstract nature of mathematics. The concept of the Dedekind cut found a place in this part of the work.

The discussion of the number system led us to the graphical representation of the complex numbers. The four fundamental operations were performed graphically on real numbers and also on complex numbers. Having discussed the representation of complex numbers graphically we could bring in the concept of vectors.

The equation $a^n=b$ provided us with zero, negative, and fractional exponents, as well as a springboard to the subject of logarithms. Methods for computing logarithms by the use of infinite series were suggested.

Our next large topic was algebraic and transcendental numbers, with empha-

sis on π and e. The connection between these new types of numbers and construction problems was outlined. We sincerely hope we have done our part in nipping potential circle-squarers in the bud.

The first semester's lectures ended with the lighter topics of scales of notation and tests for divisibility.

The second semester of the first year was devoted in the main to geometry. Starting with some advanced theorems of plane euclidean geometry, we came to an examination of the general structure of a geometry and the axioms of Euclid in particular. The history of the parallel postulate controversy was traced up to the 19th century settlement of the question.

Non-euclidean geometry occupied a large part of our time. Theorems alike in the different non-euclidean geometries were pointed out, and also theorems peculiar to the particular geometries. Having acquired a proper frame of mind we proceeded to inquire what geometry best suited our universe. The geometry of the theory of relativity seemed to be the answer, so it was advisable to discuss some of that theory.

Certain other geometries then claimed our attention, namely analysis situs and projective geometry. We selected outstanding theorems and problems from these geometries to show how they differed from the geometries previously discussed.

The second semester of the first year closed with an attempt to outline the main problems of the calculus and to introduce analytic geometry as the helpmate of the calculus.

The second year in which we offered the course we thought it wise to go over essentially the same material. Some weeding out was done to prevent the lectures from spreading over too wide an area. But on the whole the same topics were included.

In the third year we made a distinct separation between the work of the two semesters. It was found that there was an increase in registration the second semester, so that it was unwise to have the material in the second semester depend in any way upon that of the first semester. Consequently we abandoned Dresden's book as a basis of lectures and devoted the first half of the course to the study of numbers using Dantzig's *Number the Language of Science* for a guide. Of course there were several of the same topics which we had discussed in previous years, but the approach was somewhat different.

The second semester was again devoted primarily to topics in the field of geometry, including elementary problems of the calculus. Again we examined the axioms of euclidean geometry, introduced non-euclidean geometry, relativity, and spent some time on analysis situs and projective geometry.

In the present year it has been necessary to shorten the course to one semester because of pressure of other duties on the department. We chose to schedule the course for the second semester, and we are now completing the fourth year. Cutting the time in half meant revising the content and material to a great extent. We have found Kasner and Newman's *Mathematics and the Imagination*

the answer to our problems. We have used this book as a text, requiring its purchase by the students. We have still supplemented the material found there with material from other sources, but we have followed their scheme of development. We still have the development of the number system, algebraic and transcendental numbers, the famous problems of antiquity, euclidean and non-euclidean geometry, and the calculus. But in addition there have been paradoxes, famous mathematical pastimes, and probability.

It is, of course, quite obvious that in the lectures we cannot fail to take account of the overlapping of mathematics with other fields of knowledge. Non-euclidean geometry and relativity have implications for philosophy that we could not possibly neglect. The dependence of the physical sciences on mathematics is apparent throughout the course. The social sciences come to mind in discussing the subject of probability.

We have said that each semester we used some particular book as the basis for lectures. Other useful sources are listed below for the convenience of any who may wish to make a similar experiment.

It is difficult to evaluate a course of this type, especially difficult for the person giving it to view it objectively. We feel that a department of mathematics owes it to the college or university as a whole to provide some such course as this, and from our point of view it has been successful. We endeavor throughout the course to arouse in the students the feeling that mathematics is not a dry and fixed subject where a thing is either right or wrong and where there is always a single answer. We show them problems which have never been solved. We point out that there may be more than one interpretation for a certain phenomenon. We try to make mathematics adventurous, with a problem to be solved, various methods of attack to be considered, and different solutions consequent to the particular choice of method.

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

THE SOLUTION OF $x^2 \equiv a \pmod{p}$

N. G. GUNDERSON, Cornell University

The solving of the congruence $x^2 \equiv a \pmod{m}$ reduces to the solving of $x^2 \equiv a \pmod{p}$ for each prime factor p of m. The case $p=2$ offers no difficulty, so assume p an odd prime.

This congruence, $x^2 \equiv a \pmod{p}$, can be solved* by the method of exclusion, using Hollerith cards.† This method has the disadvantages that several fairly long calculations must be made, and, of course, the cards must be available.

The following method is an extremely simple one. The calculations are easy, and the time required to solve a congruence compares very favorably with the time required by the exclusion method, particularly for values of the modulus less than 1000, for which it is usually much less.

Write $x^2 \equiv a \pmod{p}$ as

$$x^2 \equiv a + bp \pmod{p},$$

where $b > 0$ is the solution of

$$a + bp \equiv 0 \pmod{c^2}.$$

Since for simplicity b should be 0, 1, 2, or 3, c usually will be 2, but may occasionally be taken as 3, 10, *etc.*, if a solution is evident. Of course, p must not divide c.

Hence, upon dividing by c^2, one obtains

$$(x/c)^2 \equiv (a + bp)/c^2 \equiv a' \pmod{p}.$$

This process is repeated until a congruence of the type

$$(x/m)^2 \equiv n^2 \pmod{p}$$

is recognized, whence

$$x/m \equiv \pm n \pmod{p}, \qquad x \equiv \pm mn \pmod{p}.$$

Since there are $(p-1)/2$ possible values for the right-hand side of the successive congruences, namely, the quadratic residues of p, and about $\sqrt{p}$ perfect squares, one might expect to proceed, on the average, about $\sqrt{p}/2$ steps in order to obtain the answer. Usually fewer steps are required, for when square factors

* This assumes that a is a quadratic residue of p. In a specific example the quadratic character of a can be decided by the quadratic reciprocity law.

† R. M. Robinson, Stencils for Solving $x^2 \equiv a \pmod{m}$, University of California Press, 1940.

such as 9 or 100 are removed, perfect squares are more frequent among the smaller resulting numbers.

If the number on the right-hand side of one of the congruences turns out to be a, the procedure must be repeated with different square factors being removed.

For an illustration, to solve $x^2 \equiv 14 \pmod{31}$ we write

$$\left.\begin{aligned} x^2 &\equiv 14 \equiv 76 \\ (x/2)^2 &\equiv 19 \equiv 50 \\ (x/2\cdot 5)^2 &\equiv 2 \equiv 64 \equiv 8^2 \\ \therefore\ x/2\cdot 5 &\equiv \pm 8 \end{aligned}\right\} \pmod{31},$$

and

$$x \equiv \pm 8\cdot 2\cdot 5 \equiv \pm 13 \pmod{31}.$$

THE NUMBER OF TERMS IN A POLYNOMIAL

F. E. HOHN, University of Arizona

The following is a direct proof that a complete, homogeneous polynomial of degree n in $r+1$ variables has ${}_{n+r}C_r$ distinct terms, the method being to associate with each term a unique symbol which shows the structure of the term and then to count the symbols.

Let the $r+1$ variables be $x_0, x_1, \cdots, x_r$. Disregarding constant coefficients, we can express any term of the polynomial in the form

$$(1) \qquad x_{i_1}x_{i_2}\cdots x_{i_n},$$

where we specifically require

$$(2) \qquad i_1 \leqq i_2 \leqq \cdots \leqq i_n,$$

the subscripts i_λ being chosen otherwise arbitrarily from $0, 1, 2, \cdots, r$. Now, in the formation of a product (1), two types of operation are involved: (i) the introduction of a factor x_i into the product, and (ii) the transition from one variable to the next. An operation of the first type we denote by M, and one of the second type by T. Then the structure of any product (1), subject to condition (2), is representable uniquely by a permutation of n M's and r T's. In such a permutation, the ith symbol T will denote transition from x_{i-1} to x_i, and each M between the ith T and the $(i+1)$th T will denote the introduction of a factor x_i into the product.

An illustration will make this clear. Let $n=6$, $r=4$; then to $x_0^2x_1x_3x_4^2$ corresponds the symbol $(MMTMTTMTMM)$. The two initial M's correspond to two factors x_0, introduced before any transition is made. The two adjacent T's correspond to the transition from x_1 to x_2, the introduction of *no* factor x_2, and then the transition from x_2 to x_3. Similarly, to x_2^6 corresponds the symbol $(TTMMMMMMTT)$. The initial T's here correspond to the transition from

x_0 to x_1 to x_2 *before* a factor is introduced; the final T's correspond to the absence of x_3 and x_4.

It is clear that any permutation of n M's and r T's may be interpreted uniquely as a product (1). Since there are

$$\frac{(n+r)!}{n!r!} = \binom{n+r}{n} = \binom{n+r}{r}$$

such permutations, and since they are in one-to-one correspondence with the products (1), there are ${}_{n+r}C_r$ terms in the polynomial.

This same method of reasoning gives easy solutions to other problems; for example, "How many paths (involving no retreats) are there from one vertex to the opposite one, along the lines of a chess board?" Any path may be interpreted as 8 vertical steps V, and 8 horizontal steps H. Further, any permutation of these 16 steps is a path. But there are $16!/(8!)^2$ distinct permutations of 8 V's and 8 H's, and hence there are that many paths.

A TEST FOR THE NATURE OF THE ROOTS OF THE CUBIC EQUATION

E. E. WATSON, Iowa State Teachers College

We offer a simple proof of the following well known result:

If a cubic equation has the form

$$x^3 - cx + d = 0, \tag{1}$$

where c and d are real, then the nature of the roots depends on the value of $(c/3)^3-(d/2)^2$.

Proof. 1. If there are two equal roots, let the roots be a, a, $-2a$. Then (1) becomes

$$x^3 - 3a^2x + 2a^3 = 0.$$

Comparing coefficients, we have $c=3a^2$, $d=2a^3$. Hence $(c/3)^3-(d/2)^2=0$.

2. If the roots are real and unequal, let them be $a+b$, $a-b$, $-2a$, where $b\neq 0$ and $b\neq \pm 3a$. Equation (1) becomes

$$x^3 - (3a^2+b^2)x + 2a(a^2-b^2) = 0,$$

and we find that

$$(c/3)^3 - (d/2)^2 = b^2(9a^2-b^2)^2/27 > 0.$$

3. If two roots are imaginary, let the roots be $a+bi$, $a-bi$, $-2a$, where $b\neq 0$. For (1) we have

$$x^3 - (3a^2-b^2)x + 2a(a^2+b^2) = 0,$$

and, as above,

$$(c/3)^3 - (d/2)^2 = -b^2(9a^2+b^2)^2/27 < 0.$$

In each case the third root is determined by the condition that the sum of the three roots is zero.

THE ADDITION FORMULAS FOR THE SINE AND COSINE

E. J. McSHANE, University of Virginia

Off and on, for some years, I have tried to find a proof of the addition formulas for the sine and cosine which would have the following three properties. (1) It should be valid for all angles, and not involve any discussion of the quadrants in which the angles lie. (2) It should not require previous knowledge of the formulas for the functions of $n \cdot 90° \pm A$ in terms of functions of A. (3) It should not be too difficult for first-year students to follow. The proof below, which I have not seen published, satisfies (1) and (2), and perhaps comes as close to satisfying (3) as any other. It requires a knowledge of the distance formula, of the general definitions of the trigonometric functions, and of the equations

$$\cos^2 \theta + \sin^2 \theta = 1,$$

$$\cos 0° = \sin 90° = 1, \qquad \cos 90° = \sin 0° = 0.$$

From the definitions of the sine and cosine we deduce at once that if P is a point whose distance from the origin is r and for which the angle of OP with the positive x-axis is θ, the coördinates of P are $(r \cos \theta, r \sin \theta)$.

Let A and B be any two angles. With a vertex O and a half-line OW as a beginning we construct angles A and B, and on their terminal half-lines we choose points P and Q respectively, each at distance 1 from O. Let d denote the distance from P to Q. We shall now make two computations for d^2, using first OW and then OQ as x-axis.

First using OW as x-axis, we find that the coördinates of P and Q are $(\cos A, \sin A)$ and $(\cos B, \sin B)$ respectively, since they each have distance 1 from the origin O and the half-lines OP, OQ make the respective angles A, B with the positive x-axis. Hence

$$\begin{aligned} d^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= 2 - 2[\cos A \cos B + \sin A \sin B]. \end{aligned}$$

Next we use OQ as positive x-axis. The half-lines OP, OQ now make the respective angles $A - B$, 0 with the positive x-axis, so P and Q have coördinates $(\cos (A - B), \sin (A - B))$ and $(1, 0)$ respectively. Hence

$$\begin{aligned} d^2 &= (\cos (A - B) - 1)^2 + \sin^2 (A - B) \\ &= 2 - 2 \cos (A - B). \end{aligned}$$

Equating the two expressions for d^2 yields

$$\cos (A - B) = \cos A \cos B + \sin A \sin B. \tag{1}$$

If we wish to use the formulas for the functions of $n \cdot 90° \pm A$ in terms of functions of A, the formulas for $\cos (A + B)$ and $\sin (A \pm B)$ can quickly be deduced from (1). However, we do not need to use the formulas for the functions of $n \cdot 90° \pm A$; the equations mentioned in the introduction, together with (1), are enough.

In (1) we set $A=0$, obtaining

$$\cos(-B) = \cos B. \tag{2}$$

Again, by setting $A=90°$ in (1) we find

$$\cos(90° - B) = \sin B. \tag{3}$$

In (3) we set $B=90°-C$; this yields

$$\sin(90° - C) = \cos C. \tag{4}$$

By (3), and (4) and (1), we have

$$\begin{aligned} \sin(A+B) &= \cos[90° - (A+B)] \\ &= \cos[(90° - A) - B)] \\ &= \cos(90° - A)\cos B + \sin(90° - A)\sin B \\ &= \sin A \cos B + \cos A \sin B. \end{aligned} \tag{5}$$

In (3) we set $B=-A$, obtaining

$$\sin(-A) = \cos(90° + A). \tag{6}$$

Now from (6) and (1), we deduce

$$\begin{aligned} \sin(-A) &= \cos[A - (-90°)] \\ &= -\sin A. \end{aligned} \tag{7}$$

By (1), (2), and (7) we have

$$\begin{aligned} \cos(A+B) &= \cos[A - (-B)] \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \sin B; \end{aligned} \tag{8}$$

and likewise by (5), (2), and (7) we find

$$\sin(A-B) = \sin A \cos B - \cos A \sin B. \tag{9}$$

Since we have not used the formulas for the functions of $n \cdot 90° \pm A$ in deducing (1), (5), (8), and (9) we can use the latter, with the known values of $\sin(n \cdot 90°)$ and $\cos(n \cdot 90°)$, to deduce the former.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Mathematics of Finance. By L. W. Perkins and R. M. Perkins. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1941. 20+321 pages. $3.25.

Tables of Probability Functions. Volume I. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York. Conducted under the sponsorship of the National Bureau of Standards. New York, Work Projects Administration, 1941. 28+302 pages. $2.00.

Tables of Natural Logarithms. Volume II. Logarithms of the Integers from 50,000 to 100,000. New York, Work Projects Administration, 1941. 17+501 pages. $2.00.

Higher Chemical Calculations. By A. J. Mee. Brooklyn, Chemical Publishing Company, Inc., 1941. 8+184 pages. $2.00.

A Survey of Modern Algebra. By Garrett Birkhoff and Saunders Mac Lane. New York, The Macmillan Company, 1941. 11+450 pages. $3.75.

Topics in Elementary Algebra for Adult Students. By V. V. Lavroff. Athens, Ga., University of Georgia Press, 1940. 5+151 pages.

Mathematics; Its Magic and Mystery. By Aaron Bakst. New York, D. Van Nostrand Company, 1941. 14+790 pages. $3.95.

Fundamental Theorems of Orthographic Axonometry and their Value in Picturization. By W. H. Roever. Washington University Studies, New Series. Science and Technology, No. 12. St. Louis, Mo., Washington University Press, 1941. 47 pages. $1.00.

Higher Mathematics for Engineers and Physicists. Second edition. By I. S. Sokolnikoff and E. S. Sokolnikoff. New York and London, McGraw-Hill Book Co., Inc., 1941. 11+587 pages. $4.50.

Tools. A Mathematical Sketch and Model Book. By R. C. Yates. Baton Rouge, La., Louisiana State University Press, 1941. 193 pages. $1.60.

REVIEWS

Outline of the History of Mathematics. By R. C. Archibald. Fifth edition, revised and enlarged. Oberlin, Ohio, The Mathematical Association of America, Inc., 1941. 75 pages. $0.75.

The question will naturally arise as to whether a fifth edition of a work is worthy of review. The reviewer was privileged to see the copy of the fourth edition used by the author in making final corrections and changes, and the one to be used by the printer in setting up the fifth edition. Without this privilege,

Elementary Calculus. By V. H. Wells. New York, D. Van Nostrand Company, 1941. 13+410 pages. $3.25.

This book has been written to provide an introduction to the calculus which could be adapted to the needs of academic colleges and also to engineering schools. The simultaneous presentation of the differential and integral calculus makes possible the attainment of the author's objective: to give a continuous development of the subject and at the same time to provide a high degree of flexibility in the choice of material.

After a chapter on necessary preliminaries, differentiation is introduced for polynomials. A chapter on differentials is then followed by integration (as an inverse process) of polynomials. Applications to algebraic functions are given next. The definite integral as the limit of a sum is developed in detail at this point, after which the elementary transcendental functions are discussed together with applications. The last third of the book is devoted to topics from which any desirable selection could be made: an elementary discussion of series; treatment of problems in polar coördinates; functions of two variables and partial differentiation; multiple integrals and their applications; integration of the simplest types of differential equations.

More than 2500 problems, mostly with very brief statements not requiring figures, are divided into 107 sets scattered at intervals of about one lesson. These problems are divided into several groups in 91 of these sets, in much the same way as was done in the author's *First Year College Mathematics*.* The A groups are direct and simple exercises based on the work just presented; the B groups are somewhat harder. In 21 sets there is a C group which in most cases contains a large proportion of review problems, or problems in which topics not taken up in the text proper are suggested. Answers are given to the odd-numbered problems.

The book is written so as to be independent of other books. Thus, an appendix contains the basic formulas of trigonometry and plane analytics and a table of 83 well chosen indefinite integrals. The chapters on polar coördinates and space analytics are entirely independent of any former treatment of these subjects which the student may have had. Basic principles are constantly brought to the attention of the student by restatement when they are needed.

Throughout the book the weaknesses of the average student are recognized and anticipated. Considerations of this sort may have led the author to adopt a rather informal style. The reviewer regrets that so many of the illustrative examples are presented in the form a student would use, the author's comments being placed at the end instead of being naturally distributed as the solution progresses. It is unfortunate that the individual illustrative examples were not separately designated so that the reader could tell at a glance where one ends and the next begins.

As compared with other elementary texts on the calculus, this one gives an

* Reviewed by R. L. Jeffery, this MONTHLY, vol. 45, 1938, pp. 379–380.

unusually clear presentation of differentials. The discussion of the definite integral as the limit of a sum is quite detailed when first presented; in the subsequent applications the part it plays is carefully indicated. An unusually full discussion of attraction occurs among the applications. Although the evaluation of indeterminate forms is based on the Theorem of the Mean, its connection with Taylor's series is not mentioned. Envelopes, oblique asymptotes, change of variable in differential expressions, and mechanical quadrature are not included.

The reviewer is obliged to point out that there are numerous occasions on which the clarity of the exposition is marred by awkward phraseology. See for example p. 113, p. 175, p. 185, and the definitions on p. 2, p. 25, p. 142, p. 206, p. 255. The thoughtful student will be puzzled when he finds that dy is defined as the principal part of Δy, and it is also defined as $f'(x)dx$ (p. 51). He will be surprised to find the phrase "product of three finite numbers" on p. 254 after having been warned on p. 8 that ∞ does not represent a number. The average student might be misled by the (unproved) statement (p. 266) about the error committed by terminating an alternating series; also, by the statement (p. 373) that the sum of two solutions of a differential equation is a solution. From the comment at the foot of p. 8, the student could easily infer that all discontinuities of a single-valued function correspond to asymptotes. On p. 74 we read that "$y^2=x+1$ is a two-valued function." In Fig. 82 (2) a curve is shown which has a cusp; it is erroneously said to be discontinuous at the corresponding point. Evidently in the interest of uniformity, the principal values of arc sin, arc tan, and arc csc are defined to lie in the interval from $-\pi/2$ to $\pi/2$; for the other three inverse trigonometric functions, the interval from 0 to π is used. This is, of course, inconsistent with the familiar formulas for the derivatives of arc sec and arc csc as given on p. 159.

The arrangement of the material included results in definite advantages which may well outweigh the faults mentioned when the text is considered for certain student groups. However, the reviewer would hesitate to recommend the text for those intending to base further study on the calculus.

RANDOLPH CHURCH

Tables of Natural Logarithms. Volume II. Logarithms of the Integers from 50,000 to 100,000. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York; A. N. Lowan, Technical Director. Conducted under the sponsorship of the National Bureau of Standards. New York, 1941. 18+501 pages. $2.00.

In this second volume of the tables of natural logarithms, the argument starts at 50,000 and continues, by integers to 100,000. In other respects this volume is identical with Volume I of the same title; hence, the comments made in the review of the first volume apply to this one. See this MONTHLY, vol. 48, 1941, p. 549.

VIRGIL SNYDER

Statistical Procedures and Their Mathematical Bases. By C. C. Peters and W. R. Van Voorhis. New York and London, McGraw-Hill Book Company, 1940. 13+516 pages. $4.50.

The stated purpose of this book is "to bridge the gap between the elementary courses, in which the formulas are given purely authoritatively, and the original contributions in the monographic press, which are often highly mathematical in character." The authors appear to regard statistics as merely a collection of formulas the derivations of which are not understood by relatively untrained persons largely because of "the omission of steps which are supposed to be obvious." They seek to supply these steps.

The first 39 pages present "A Little Calculus." After this beginning, the book follows a more or less familiar course with preponderant emphasis on correlation. There are chapters on multiple factor analysis, analysis of variance, chi square, curve fitting, and controlled experimentation. The book is intended for students of psychology and education, and assumes that the reader is familiar with their jargon (*e.g.*, "samples matched on an infallible criterion"). The book is based on an earlier lithoprinted edition published privately in 1935.

Many of the techniques developed by R. A. Fisher are included. The authors "have attempted to take the magic out of them," and bring them into "synthesis with classical statistics," but they do not appear to be happy with the result or entirely successful in it. May this not be because they have been blind to the forest of statistical theory in their concentration on the trees of specific formulas?

The book is uneven in quality from page to page and chapter to chapter. In many ways it compares favorably with other text-books now in use. It has a number of general defects, however, that are serious liabilities for student and instructor alike.

The first general defect is careless use of language. This may be a byproduct of the authors' attempt to write a "chatty, leisurely volume." In trying to avoid mathematical terminology they say: "It [Δx] will necessarily drag Δy down with it $\cdots$ " (p. 4); "Wait a minute! The curve is flat at $x=4$, that is true, $\cdots$ " (p. 13); "Sure enough, the second derivative is negative." (p. 14); and "make a correction to atone for the error $\cdots$ " (p. 64).

Inaccurate statements abound in the text; a few examples follow:

"When Δx has become so small as to have approached zero as its limit let us replace $\Delta y/\Delta x$ by dy/dx. At this limit the Δx in the last term of our equation will approach zero in value and thus disappear from consideration." (p. 4). There is no definition of a limit.

"The term [moments] is also used in a different and more technical sense in statistics to designate the power to which deviations are raised before averaging them." (p. 41).

"The f in the formula is merely a symbol of operation; the formula would mean exactly the same if it were not there," (p. 47).

"Random samples for a given ρ becomes highly skew at the two ends." (p. 155).

"Substituting infinity for b and 1 for $a \cdots$" (p. 205).

"$\cdots$ included the array which was to be correlated with it." (p. 212).

"In this the N is the population of the sample $\cdots$" (p. 319).

In several places ordinates of the normal probability function are given as probabilities (pp. 299, 301, and 367). The authors find it necessary to show the reduction of 2/4 to 1/2 (p. 285), but are confident that "the reader has, of course, noticed that the degrees of freedom are additive, which fact follows from some complication of the algebraic expression $(N-k)+(k-1)=(N-1)$." (p. 350).

A second general defect is the weakness of the treatment of probability. The authors frequently confuse or obscure the relation of sample and population. They do not define or explain, except by casual and incomplete references, the meaning of independence, random sampling, the point binomial, and many other fundamental concepts.

The general concept of a probability function is not developed. The normal probability curve is presented after, not before, chapters on correlation, standard errors, and factor analysis, and then it is given primarily as a function for graduating or describing frequency distributions. The pages fairly bristle with formulas for probable errors, but almost no attention is paid to the theory of tests of significance. The authors badly misinterpret significance tests on pages 119 and 137.

A third general defect is the fondness of the authors for certain of their own contributions to statistics. (In this they are not unlike a number of other authors.) Seventeen pages are devoted to tetrachoric and biserial correlation formulas for "widespread classes." No table of z or F is given for use in analysis of variance because the authors believe ϵ^2, Kelley's modification of the correlation ratio, is superior to the analysis of variance. Instead, Snedecor's table of F is transformed into a table of ϵ^2 which is reproduced. The probabilities given for this table (p. 494) are only half the correct values since negative values of ϵ^2 are to be interpreted as arising by chance (p. 355). The statement that "The distribution is of the same shape oriented toward negative values as toward positive," (p. 355) is incorrect. It is stated erroneously (p. 334) that the class frequencies must be equal if an analysis of variance is to be valid. The assumption implicit in equation 177 that the ratio of two unbiased estimates is unbiased is false.

A fourth defect is to be found in the treatment of assumptions underlying several of the derivations. Often they are stated correctly but not in a prominent position, and frequently they are not satisfied by the examples that the authors present. The example given for biserial r is drawn from a study of 100 children that left school and 200 that did not leave. Apparently each of these numbers was determined arbitrarily by the investigator, but the authors proceed to treat them as if they fulfilled the assumptions stated on page 365.

Again, they illustrate tetrachoric correlation by the relation of teaching success to hours of courses in pedagogy, dividing the latter into "six hours or less" and "more than six hours." Hours of credit in pedagogy certainly is not a con-

tinuous normally distributed variable; the illustration could not have been much poorer in relation to the assumptions stated on page 370.

The reviewer fears that the authors may befuddle or mislead more students than they rescue from the mystery and magic of text-books that do not present the mathematical basis of statistical procedures. Their purpose is a worthy one, their conception of the difficulties perhaps too narrow, their performance too badly marred by inaccuracies and mistakes. The book would be more useful if greater care had been given to fundamentals and less to formulas.

F. F. STEPHAN

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

UNDERGRADUATE RESEARCH IN MICHIGAN

E. R. SLEIGHT, Albion College

The student of unusual ability has always been an inspiration, as well as a challenge, to any teacher. In an article which appeared in *School and Society*, Dr. G. R. Williams of Amherst College emphasized the advantages of a plan of teaching that would challenge the superior student, and in recent years this plan has taken the form of research in many of our colleges. To what extent should this spirit of research among undergraduates be encouraged? President Angell of Yale University, in one of his lectures, made this statement: "Individual initiative, resourceful ingenuity, imagination, vision, must be kept at high pitch." The modern spirit of scientific research has gripped us, and I believe that it has become something of a dominant note in our teaching.

The Michigan Section of the Mathematical Association of America has done much to create interest in undergraduate research among the colleges of that state. So far as my own interest is concerned, it dates from the spring of 1911 when a student placed on my desk a copy of an investigation of the problem concerning the number of types of normals that may be drawn to a parabola. The inspiration came to him as a result of a class discussion. In the text there appeared the statement that from any point in the plane of the parabola three normals may be drawn to the curve. The student verified this statement, first by algebraic processes, then by calculus methods. In addition he pointed out that, under similar conditions, four normals may be drawn to a central conic.

For a period of years no other attempt was made to produce anything that might be classified as research. However, in 1930 permission was obtained from Dr. Matthewson, now of Dartmouth College, to use his results on the "Normals to the Conics," and one of the students from Albion College presented the material at the Michigan Section of the Mathematical Association. This seemed to arouse interest in other students, and nearly every succeeding year some undergraduate from Albion appeared on the program. Among the topics included were: The divisibility of numbers to any base, The theory of poles and polars applied to any curve, Two summation formulas, A proof of the fundamental theorem of calculus, A pendulum problem, A new approach to the trisection problem, Computation by counters, On the normals to a cardioid, and Mechanical devices used in mathematics.

At the Section meeting in 1934–35, a committee was appointed to investigate the problem of creating a wider interest in undergraduate research. We studied the advisability of offering prizes, and many other possible incentives. We finally decided to devote a portion of our annual program

to undergraduate papers. These were mimeographed by the Association, and bound copies were sent to all institutions interested. The plan seems to have provided the necessary inspiration, as representatives from several institutions have participated in the annual programs.

Another step in developing this particular field of interest occurred two years ago when the *Albion College* chapter of *Kappa Mu Epsilon* invited all of the Michigan colleges and some from northern Ohio to participate in a meeting, comparable to the Association meetings, but all papers were presented by undergraduates. About seventy-five students, together with many faculty representatives, were present. The result is that now we have a definite organization in Michigan which, I believe, will carry on this type of work for several years. Last spring the second meeting was held, with Michigan State Normal College as host. Some very worth while papers were presented to more than one hundred students and faculty members. The Michigan Section of the Association still sponsors the organization to the extent of mimeographing and binding the papers, and distributing copies to those institutions and individuals interested.

To any teacher the greatest satisfaction comes from the inspiration received from these young people. Again referring to President Angell's statement that all education should be so geared that "individual initiative, resourceful ingenuity, imagination, vision, must be kept at high pitch," I am wondering if this aim is not best obtained by creating a spirit of research even among our undergraduates. In this way we have brought the student face to face with the need of thinking a problem through for himself, and of devising methods for its solution. Perhaps this is the ultimate aim of education, and if so what better method can be devised?

UNDERGRADUATE MATHEMATICS CONFERENCE AT MICHIGAN STATE NORMAL COLLEGE

The second annual Undergraduate Mathematics Conference of colleges and universities throughout Michigan, northern Ohio, and northern Indiana was sponsored by the *Mathematics Club* of *Michigan State Normal College* at Ypsilanti on May 3, 1941. Over 100 representatives attended from the colleges as follows: Alma College (6), University of Toledo (10), Central State Teachers College (16), University of Michigan (12), Wayne University (7), Western State Teachers College (5), Michigan State College (9), Albion College (12), Tri-State College (4), Ferris Institute (2), and Michigan State Normal College (35).

The morning session was devoted to the following seven papers by undergraduates: On the normals to the cardioid, by Paul Dunn of Albion College; Applications of mathematics in field artillery, by Russell Donovan of Central State Teachers College; Determination of volumes and surfaces of eggs, by Morris Rottenstein of Michigan State College; A summation scheme, by Sheldon Putnam of Albion College; Integraphs and planimeters, by W. G. Wadey of the University of Michigan; Factors of prime perfect numbers, by Robert R. Coveyou of Wayne University; and Duodecimal system, by Malcolm Chubb of Michigan State Normal College. (These papers are being mimeographed and made available for distribution to all colleges in the conference area.) After the luncheon a business meeting was held during which the invitation to meet at Central State Teachers College at Mount Pleasant in 1942 was accepted. Miss Margaret Hatcher was the general chairman of the conference and Dr. Theodore Lindquist is faculty adviser of the *Mathematics Club.*

THE GREATER BOSTON INTERCOLLEGIATE MATHEMATICS CLUBS ASSOCIATION

The first semester meeting was held on January 15, 1941, with the *Boston College Mathematics Club* as host. Student speakers were Patrick J. Corcoran of Northeastern University whose subject was Solution of indeterminate equations, and L. Wingurski of Tufts College discussing Curved sketches. Professor Cedrone of Boston College spoke on Determinants. Entertainment included a mathematical quiz, with E. T. Bell's two books *Men of Mathematics* and *Mathematics, Queen of the Sciences* as prizes. Oliver Smith of Massachusetts Institute of Technology and Irene Thomas of Regis College were the winners.

The *Regis College Mathematical Club* sponsored the second semester meeting on April 29, 1941. Selma Gottlieb of Wellesley College spoke on Number systems, and Francis Scheid of Boston University had as his topic Digitals. Correspondence was the subject discussed by Sister Leonarda of Regis College who was the principal speaker of the evening.

In addition to these two regular meetings each year, the member clubs receive announcements of all meetings of general interest conducted by the individual clubs, and any material found helpful or interesting is exchanged. Schools participating are: Boston University, Wellesley College, Tufts College, Northeastern University, Boston College, Regis College, and Massachusetts Institute of Technology. The first meeting for 1941–42 will be held at Massachusetts Institute of Technology.

CLUB REPORTS, 1940-41

Mathematics Club, Case School of Applied Science

Sixteen problems from Problems and Solutions departments of mathematics journals were selected and distributed to all members at the beginning of the year. From the *National Mathematics Magazine* the following were chosen: 329, 330, 332, 339; and from the MONTHLY: E387, E388, E390, E391, E396, E405, E408, E411, E414, E417, E419, 3958. Discussion of the solutions became an important part of the monthly programs. In addition, there were presentations of the following topics: Statistical methods in science and engineering, by Dr. Sidney McCuskey; Least squares, by R. C. Dorris; Differential equations, by J. E. Uher; Infinite series, by L. B. Hitchcock; Theory of equations, by Lester Hertz; Line integrals and the calculus of variations, by R. I. Strough; and Lebesgue integrals, by Dr. C. C. Torrance. Officers were: President, L. L. Foldy; Vice-President, P. H. Houser; Secretary, R. I. Strough; Faculty Adviser, Dr. Max Morris.

Pi Mu Epsilon, University of California at Berkeley

Twenty-three new members were initiated in the fall and twenty-four more at a spring banquet, bringing the total membership for the year to one hundred and twenty-five. Papers presented during the year were: A theory of reflector design, by William Whitmore; Some elementary considerations of quantum mechanics, by Dr. Phillip Morrison; Pollack models, by Dr. B. C. Wong; A mathematical analysis of taxation, by Dr. R. W. Shepard; A reduction theory in algebraic postulates, by Robert Levit; Extension fields and the fundamental theorem of algebra, by Dr. R. M. Robinson; Ramanujuan, by Dr. D. H. Lehmer; and An introduction to airfoil theory, by Dr. Hans Lewy. Officers were: Director, William Simons; Vice-Director, Sam Schaaf; Secretary, Elizabeth Scott; Treasurer, Theodore Geballe; Librarian, Sarah Hallam; Faculty Adviser, Sophia H. Levy.

Kappa Mu Epsilon, State Teachers College, Pittsburg, Kansas

During the year the chapter held four business and four open meetings, two picnics, and sent a large delegation to the national convention at Warrensburg, Missouri. Open meetings were held at the homes of faculty members. Topics discussed included: History of mathematics in secondary schools, What is mathematics, Advice to a graduate assistant, Ancient and medieval problems, History of algebra, Troublesome problems, Mathematical paradoxes, The early history of statistics, Making predictions in business, Mathematics of the American Indian, and Mathematics in art. Officers were: Sponsor, Professor J. A. G. Shirk; Corresponding Secretary, Professor W. H. Hill.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 496. *Proposed by R. V. Heath, New York, N. Y.*

What is the smallest value of n for which the n^2 triangular numbers $0, 1, 3, 6, 10, \cdots, \frac{1}{2}n^2(n^2-1)$ can be arranged to form a magic square?

E 497. *Proposed by V. Thébault, Tennie, Sarthe, France.*

The sides of a triangle $A'B'C'$, of constant size, remain parallel to those of a fixed triangle ABC, and form with it three more triangles and three pentagons. Show that the position of $A'B'C'$ which minimizes the sum of the areas of these three triangles makes the areas of the three pentagons all equal.

E 498. *Proposed by E. C. Kennedy, Texas College of Arts and Industries.*

Consider the relation

$$T_{n+1} = \sqrt{\frac{k + T_n}{2 - T_n}}\,.$$

What is the largest value of k such that the sequence $\{T_n\}$, for a suitable range of values of T_0, converges to a positive number? What is this number?

E 499. *Proposed by D. H. Browne, Buffalo, N. Y.*

Two intersecting circles (A) and (B) have centers mutually external. Two other circles (C) and (D), orthogonal to (A) and (B) respectively, are drawn through the points of intersection. Show that the two common tangents of (C) and (D) are concurrent with the two common tangents of (A) and (B).

E 500. *Proposed by S. H. Gould, University of Toronto.*

In how many ways can p gentlemen and q ladies sit at a circular table if each lady has the choice, as long as gentlemen are still available, of sitting on a chair or on a gentleman's knees?

SOLUTIONS

E 459 [1941, 148]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Show that the altitudes and ex-radii of any triangle satisfy the following relations:

$$\sum \frac{h_a^2(r_b + r_c)}{r_b r_c(h_a + 2r_a)} = 2, \quad \sum \frac{r_b r_c}{(r_b + r_c)(h_a + 2r_a)} = \frac{1}{2} \cdot$$

Solution by H. W. Bailey, University of Illinois.

From the well known formulas

$$r_a = \frac{\Delta}{s - a}, \text{ etc., and } h_a = \frac{2\Delta}{a}, \text{ etc.,}$$

where Δ is the area of the triangle and $2s = a+b+c$, we obtain directly

$$r_b + r_c = \frac{a\Delta}{(s-b)(s-c)}, \qquad r_b r_c = \frac{\Delta^2}{(s-b)(s-c)}, \qquad h_a + 2r_a = \frac{2s\Delta}{(s-a)a} \cdot$$

Hence

$$\sum \frac{h_a^2(r_b + r_c)}{r_b r_c(h_a + 2r_a)} = 2 \sum \frac{s-a}{s} = 2,$$

and

$$\sum \frac{r_b r_c}{(r_b + r_c)(h_a + 2r_a)} = \frac{1}{2} \sum \frac{s-a}{s} = \frac{1}{2} \cdot$$

Also solved by F. A. Alfieri, W. E. Buker, W. B. Clarke, Howard Eves, L. M. Kelly, E. P. Starke, C. W. Trigg, and G. A. Williams. Several of these used the slightly less familiar formula

$$\frac{1}{r_b} + \frac{1}{r_c} = \frac{2}{h_a} \cdot$$

E 460 [1941, 148]. *Proposed by Henry Scheffé, Oregon State College.*

Let $s(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n$ be the sum of the first n terms of the harmonic series. A well known expression for $s(n)$ which does not formally involve the sum of n terms is the integral

$$\int_0^1 \frac{u^n - 1}{u - 1} du.$$

It is desired to write $s(n)$ in the form $s(n) = f^{(n)}(0)$. Find an expression for $f(x)$ in terms of the integral of an elementary function.

Solution by G. W. Petrie, South Dakota State School of Mines.

The function $f(x)$ may be represented by the infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_0^1 \frac{u^n - 1}{u - 1} du,$$

since the nth derivative evaluated at $x = 0$ gives the required value of $s(n)$. Sufficient conditions are satisfied for the order of summation and integration to be interchanged, yielding

$$\begin{aligned} f(x) &= \int_0^1 \frac{1}{u-1} \sum_{n=0}^{\infty} \frac{(ux)^n - x^n}{n!} du = \int_0^1 \frac{e^{ux} - e^x}{u-1} du \\ &= e^x \int_0^1 \frac{1 - e^{(u-1)x}}{1-u} du = e^x \int_0^1 \frac{1 - e^{-vx}}{v} dv \\ &= e^x \int_0^x \frac{1 - e^{-t}}{t} dt. \end{aligned}$$

Also solved by the proposer, who remarks that, by integrating the Maclaurin series for the integrand and multiplying by the series for e^x, one easily obtains the formula

$$s(n) = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \binom{n}{j}.$$

E 461 [1941, 210]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the difference equation $\Delta^k N_1 = N_k$, $N_1 = 1$, defines the sequence

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \cdots,$$

whose kth term is $f^{(k)}(0)$, where $f(x) = e^{e^x - 1}$. (*Cf.*, G. H. Hardy, *Pure Mathematics*, seventh edition, p. 424, Ex. 9.)

Solution by H. W. Becker, Omaha, Nebraska.

From the given difference equation we deduce

$$N_{p+1} = (1 + \Delta)^p N_1 = \sum_{k=0}^{p} \binom{p}{k} \Delta^k N_1 = \sum_{k=0}^{p} \binom{p}{k} N_k,$$

or, in the notation of Blissard's "umbral calculus,"

$$N^{p+1} = (N + 1)^p. \tag{1}$$

The connection between this and the Maclaurin series for $e^{e^x - 1}$ has been established by several authors. See, for instance, E. T. Bell, *Annals of Math.*, vol. 39, 1938, pp. 539–557, where an extensive bibliography is given on the last page.

Sylvester observed that N_k is the number of rhyming schemes for a stanza of k lines. Thus a sonnet may be rhymed in 190899322 ways (including, of course,

the single case of blank verse). Similarly, the number of rhyming schemes in which every line rhymes with at least one other is $M_k=\Delta^k N_0$. Since

$$M_{k+1} = \Delta^k\Delta N_0 = \Delta^k(N_1 - N_0) = N_k - M_k,$$

we have the interesting theorem that the number of completely rhymed stanzas of $k+1$ lines is equal to the number of incompletely rhymed stanzas of k lines. Among the N_k rhyming schemes of k lines, the number that have j non-rhyming lines (*i.e.*, j different elements in the rhyming scheme) is easily seen to be $\Delta^j 0^k/j!$. These "Stirling's numbers of the second kind" have been tabulated by Cayley, *Transactions of the Cambridge Philosophical Society*, vol. 13, 1883, pp. 1–4. The formula

$$N_k = \sum_{j=1}^{k} \frac{\Delta^j 0^k}{j!}$$

has been used to compute N_k up to $N_{20}=51\ 724\ 158\ 235\ 372$. But the equation $\Delta^k N_1 = N_k$ provides a far easier algorithm. The work may be arranged as follows:

$$\begin{array}{c|lllll}
0 & 1 & & & & \\
1 & 1 & 2 & & & \\
2 & 2 & 3 & 5 & & \\
3 & 5 & 7 & 10 & 15 & \\
4 & 15 & 20 & \cdots & & \\
\cdots & \cdots & & \cdots & & N_k \\
k & N_k & N_k + N_{k-1} & \cdots & \cdots & N_{k+1}
\end{array}$$

Thus it is soon found that

$$\begin{aligned}
N_{25} &= 4\ 638\ 590\ 332\ 229\ 999\ 353,\\
N_{30} &= 846\ 749\ 014\ 511\ 809\ 332\ 450\ 147,\\
N_{35} &= 281\ 600\ 203\ 019\ 560\ 266\ 563\ 340\ 426\ 570.
\end{aligned}$$

Such a table should be checked by means of Touchard's congruences

$$N_{p+k} \equiv N_k + N_{k+1} \pmod{p},$$

where p is a prime.

We see that the order of magnitude of N_k is roughly that of $(k!)^{3/4}$ or of $(k/e)^{3k/4}$. A really close approximation (or asymptotic formula) has been given by L. F. Epstein, *Journal of Math. and Phys.*, vol. 18, 1939, p. 166.

Editorial Note. It appears that the earliest explicit solution of Browne's problem is due to E. Cesàro, *Nouvelles Annales de Math.* (3), vol. 4, 1885, pp. 36–40. This solution is so elegant as to bear repetition. Writing ν_p for N_{p+1} in (1), we obtain

$$(N + 1)^p = \nu^p.$$

When applied to each term of the exponential series, this gives

$$e^{(N+1)x} = e^{\nu x} = Ne^{Nx} = \frac{d}{dx} e^{Nx}.$$

Thus e^{Nx} satisfies the differential equation $ye^x = dy/dx$, whose general solution is evidently $y = e^{e^x - C}$. The value of C is determined by taking $x=0$, and we have

$$e^{e^x - 1} = e^{Nx} = \sum_{k=0}^{\infty} N_k x^k/k!,$$

as required.

Cesàro's paper ends with the remarkable formula

$$N_p = \frac{2}{\pi e}\int_0^{\pi} e^{e^{\cos\theta}\cos(\sin\theta)} \sin\left\{e^{\cos\theta}\sin(\sin\theta)\right\} \sin p\theta\, d\theta.$$

E 462 [1941, 210]. *Proposed by V. Thébault, San Sebastián, Spain.*

In what scale of notation can a square end with the digits 7777?

Solution by E. P. Starke, Rutgers University.

Let B be the base of the required system of enumeration. Then the condition of the problem may be put in the form

$$(1) \qquad N^2 \equiv 7(1 + B + B^2 + B^3) \pmod{B^4}.$$

An obvious necessary condition is

$$(2) \qquad N^2 \equiv 7 \pmod{B}.$$

Of course $B > 7$. Since 7 is not a quadratic residue mod 8, the next value to try is 9. In fact, 9 is a solution of the problem; for, in the nonary scale, $3434^2 = 13137777$. These numbers may be determined as follows. From $N^2 \equiv 7 \pmod 9$ we have $N = 9r \pm 4$, where r is an integer. From (1) we now have

$$(9r \pm 4)^2 \equiv 7(1+9) \pmod{9^2},$$

whence $\pm 8r \equiv 6 \pmod 9$, $r = 9s \pm 3$, and $N = 9^2 s \pm 31$. From (1) again we find

$$(9^2 s \pm 31)^2 \equiv 7(1 + 9 + 9^2) \pmod{9^3},$$

whence $\pm 62s + 4 \equiv 0 \pmod 9$, $s = 9t \pm 4$, and $N = 9^3 t \pm 355$. In the final step,

$$(9^3 t \pm 355)^2 \equiv 7(1 + 9 + 9^2 + 9^3) \pmod{9^4},$$

whence $\pm 710t + 165 \equiv 0 \pmod 9$, $t = 9u \pm 3$, and $N = 9^4 u \pm 2542$, which includes 2542, the value given above.

From (2) we see that, for each prime factor $p \neq 7$ of B, 7 must be a quadratic residue. Familiar methods in the theory of numbers show that this happens if and only if

$$(3) \qquad p \equiv \pm 3^k \pmod{28}.$$

It is clearly impossible for 7 to be a factor of B. If 2 is a factor, say $B=2h$, we have from (1),

$$N^2 \equiv 7 + 14h + 28h^2 \pmod 8,$$

which is possible if and only if $h\equiv 1 \pmod 4$. Thus the method used above can be applied whenever B is of the form $2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, where $\alpha=0$ or 1 and the p's satisfy (3), with the further restriction that, when $\alpha=1$, $B\equiv 2 \pmod 8$. The values of B less than 100 are:

9, 18, 19, 27, 29, 31, 37, 47, 53, 57, 58, 59, 74, 81, 83, 87.

In these scales we can find squares which end with any number of 7's. A simple example with five 7's is, in the scale of 18,

$$(4\ 4\ 11\ 13)^2 = 1\ 0\ 2\ 7\ 7\ 7\ 7\ 7.$$

Another example in which the number of 7's exceeds the number of digits of N is, in the scale of 27,

$$(7\ 0\ 14)^2 = 1\ 22\ 7\ 7\ 7\ 7.$$

Also solved by W. E. Buker, Daniel Finkel, G. W. Wishard, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4014. *Proposed by P. Erdös, University of Pennsylvania.*

Show that, if S_1 and S_2 are two squares contained in the unit square so that they have no point in common, the sum of their sides is less than unity.

It is very likely true that, if we have k^2+1 squares contained in the unit square so that no two of them have a point in common, the sum of their sides is less than k.

4015. *Proposed by N. A. Court, University of Oklahoma.*

If the base of a variable tetrahedron is fixed and the opposite vertex varies on a fixed sphere, the volume of the tetrahedron is numerically equal to the power of the variable vertex with respect to another fixed sphere.

4016. *Proposed by V. Thébault, Le Mans, France.*

The points D, E, F are taken on the sides BC, CA, AB of a triangle ABC, and the points α, β, γ are then taken on the straight lines AD, BE, CF so that $A\alpha/AD=B\beta/BE=C\gamma/CF=k$ and $\alpha D/AD=\beta E/BE=\gamma F/CF=\lambda$. Show that the area σ of triangle $\alpha\beta\gamma$ is given by

$$\sigma = (2S + s)k^2 + (1 - 3k)s = \lambda(2\lambda - 1)S + (1 - \lambda)^2 s,$$

where S and s denote the areas of ABC and DEF.

Deduce from this that in a complete quadrilateral the midpoints of the three diagonals are collinear.

SOLUTIONS

3948 [1940, 181]. *Proposed by Michael Goldberg, Washington, D. C.*

Suppose that n slotted discs are freely mounted on the same axis. If the portion of the circumference subtended by the slot of the ith disc is p_i, show that the probability that light, parallel to the axis, can pass through the slots is

$$p_1p_2 \cdots p_n\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}\right),$$

provided that $p_i+p_j \leqq 1$ for every i and j.

I. *Solution by W. J. Mays, Bethel Springs, Tenn.*

It is evident from the statement of the problem that nothing in generality is lost if each disc is supposed to have unit circumference and if possible paths for rays of light are restricted to the straight lines parallel to the axis, which generate the cylinder that envelops the discs. Choose one of the lines for the initial line and let the position of any other line be given by the portion of the circumference p included between the two lines and measured in a given sense from the initial line; likewise, denote the position of any disc by the portion of the circumference included between the initial line and the terminal point of the slot that is farther (when measured in the given sense) from that line.

Consider now the doubtful event whose outcome is to be classed as successful if and only if

i. light passes through all the slots in such a way that it certainly passes along all lines included between p and $p-\Delta p$, where Δp is a small decrement less than any p_i, $(i=1, 2, \cdots, n)$;

ii. light fails to pass through all the slots along any line whose coördinate is greater than p.

There are n independent ways in which the outcome is successful, namely, any one of the discs assumes a position between p and $p-\Delta p$, while each of the remaining is located so that all lines between p and $p-\Delta p$ fall within its slotted portion. The probability that one of the discs is between p and $p-\Delta p$ is Δp; while the probability that another, whose slotted portion is, say, p_j, includes

Assume that for $k-1$ discs,

$$\int_0^a P_{i_1,\cdots,i_{k-1}}(x)dx = p_{i_1}p_{i_2}\cdots p_{i_{k-1}}\left(\frac{1}{p_{i_1}}+\cdots+\frac{1}{p_{i_{k-1}}}\right). \tag{3}$$

Then

$$\begin{aligned} P &= \int_0^a xP_{i_1,\cdots,i_{k-1}}(x)dx + \int_0^a P_{i_1,\cdots,i_{k-1}}(x)p_{i_k}dx \\ &= \int_0^a xP_{i_1,\cdots,i_{k-1}}(x)dx + p_1p_2\cdots p_{k-1}p_k\left(\frac{1}{p_{i_1}}+\cdots+\frac{1}{p_{i_{k-1}}}\right). \end{aligned} \tag{4}$$

Under the assumption (3) and with a considered as a parameter, it is seen that $\int_0^a xP_{i_1,\cdots,i_{k-1}}(x)dx$ is also of integral rational form.

Now repeat the above argument with the remaining $k-1$ distinct selections of $i_1, i_2, \cdots, i_{k-1}$. The probability P is then expressed in k distinct ways as follows:

$$R_1(p_2,\cdots,p_k) + p_1p_2\cdots p_k\left(\frac{1}{p_2}+\cdots+\frac{1}{p_k}\right),$$

$$R_2(p_1, p_3,\cdots,p_k) + p_1p_2\cdots p_k\left(\frac{1}{p_1}+\frac{1}{p_3}+\cdots+\frac{1}{p_k}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_k(p_1p_2,\cdots,p_{k-1}) + p_1p_2\cdots p_k\left(\frac{1}{p_1}+\cdots+\frac{1}{p_{k-1}}\right).$$

The argument then is that P must contain exactly the terms in p_1 of row 1 above, since $R_1(p_2,\cdots,p_k)$ is a polynomial not containing p_1. Continuing down the rows it is seen that the only new term not involving p_1 is $p_2p_3\cdots p_k$. Therefore

$$P = p_1p_2\cdots p_k\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_k}\right).$$

For $k=3$, $\int_0^a P_{i_1,\cdots,i_{k-1}}(x)dx$ is obviously

$$p_{i_1}+p_{i_2} = p_{i_1}p_{i_2}\left(\frac{1}{p_{i_1}}+\frac{1}{p_{i_2}}\right)$$

if $p_{i_1}+p_{i_2}\leqq 1$. If $p_{i_1}+p_{i_2}>1$, the formula simply does not give a meaningful probability in the two-slot case upon which the induction is based.

3955 [1940, 245]. *Proposed by V. Thébault, Le Mans, France.*

A triangle with unequal sides has one angle of 60°, or 120°, and a side adjacent to that angle of length m, a prime. (1) Determine the lengths of the other two sides so that they are integers. (2) Show that to each value of m there

correspond two triangles such that the difference of their perimeters is a perfect square in one case, and in the other case the sum of the perimeters increased by unity is the sum of squares of two consecutive integers.

Solution by H. T. R. Aude, Colgate University.

It is known that all primitive integral triangles with an angle of 120° can be formed from the sides represented by the three numbers

$$2rs + s^2, \qquad r^2 - s^2, \qquad r^2 + rs + s^2,$$

where r and s are relatively prime integers ($r>s>0$). It will be seen that if the integers a, b, c represent the sides of a 120° triangle, there exist triangles with one angle of 60° which have the sides a, $b+a$, c and $a+b$, b, c.

In this problem either of the sides $2rs+s^2$ or r^2-s^2, which are adjacent to the angle of 120°, may be taken of length m, a prime. For the first case write

$$2rs + s^2 = s(2r + s) = m.$$

There result the two equations $s=1$ and $r=(m-1)/2$, where m must be greater than 3. It follows that the 120° triangle t_1 has the sides

$$m, \qquad (m^2 - 2m - 3)/4, \qquad (m^2 + 3)/4;$$

and its perimeter p_1 is $(m^2+m)/2$. To the triangle t_1 corresponds the 60° triangle t_3 with the sides

$$m, \qquad (m^2 + 2m - 3)/4, \qquad (m^2 + 3)/4.$$

Its perimeter p_3 is $(m^2+3m)/2$.

Turning to the second case, let the side r^2-s^2 be equal to the prime m, where $m>2$. There result the two conditions $r-s=1$, and $r+s=m$. Hence the sides of the 120° triangle t_2 are

$$(3m^2 - 2m - 1)/4, \qquad m, \qquad (3m^2 + 1)/4;$$

and its perimeter p_2 is $(3m^2+m)/2$. The sides and the perimeter of the corresponding 60° triangle t_4 are, respectively,

$$(3m^2 + 2m - 1)/4, \qquad m, \qquad (3m^2 + 1)/4;$$
$$p_4 = (3m^2 + 3m)/2.$$

From the values of the perimeters p_i of the four triangles t_i, ($i=1, 2, 3, 4$), it follows, but only for $m>3$, as announced by the proposer, that

$$p_2 - p_1 = p_4 - p_3 = m^2,$$
$$p_3 + p_2 + 1 = p_4 + p_1 + 1 = m^2 + (m + 1)^2.$$

Of course, other relations involving the perimeters of these four triangles could be stated.

Solved also by E. P. Starke and the proposer.

Editorial Note. A solution by Sun Nien-tseng, Yenching University, was received after the preparation of the above for printing. His solution was based on a different interpretation of the problem, and he showed that a 120° triangle with perimeter P_1 could be associated with one of 60° with perimeter P_2 so that $P_1 - P_2 + m = m^2$, and $P_1 + P_2 + 1 = m^2 + (m+1)^2$.

3956 [1940, 245]. *Proposed by V. Thébault, Le Mans, France.*

An arbitrary diameter Δ of the circumcircle of an equilateral triangle cuts the sides BC, CA, AB in the points α, β, γ. Show that the Euler lines of triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ determine a triangle T symmetrically equal to ABC with the center of symmetry on Δ.

Solution by O. J. Ramler, Catholic University of America.

We first prove the lemma: If an angle of a triangle is either 60° or 120°, the Euler line of the triangle together with the two sides adjacent to this angle will form an equilateral triangle.

The use of directed angles (*Cf.*, Johnson's *Modern Geometry*, §16) will enable us to prove both cases simultaneously. Let the triangle be BAC with $\measuredangle BAC = 60°$. Let the orthocenter and circumcenter be H and O, respectively. Then $\measuredangle BHC = \measuredangle BOC = 120°$; the reflection O' of O in the side BC lies on the circumcircle. Hence $AH = OO' = AO$. Then triangle HAO is isosceles, and since AH and AO are isogonally conjugate lines in the angle BAC, it follows at once that HO makes equal angles with AB and AC and each is equal to 60°.

Now in the proposed problem each of the triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ has an angle equal to 60° or 120°. Hence the Euler line in each case will be parallel to a side of the triangle ABC, thus forming an equilateral triangle $A'B'C'$ homothetic to ABC, where A' is the intersection of lines parallel to AB and AC, B' is the intersection of lines parallel to BC and BA, C' is the intersection of lines parallel to lines CB and CA. Thus AA', BB', CC' are lines joining corresponding vertices of two homothetic triangles. Hence AA', BB', CC' are concurrent.

To prove that ABC and $A'B'C'$ are equal, it will suffice to show that they have equal circumradii. Now it is well known that the algebraic sum of the perpendiculars dropped from any point in the plane of an equilateral triangle to the sides is constant and equal to three-halves of the circumradius. Letting H be the circumcenter of the equilateral triangle ABC, and θ $(60° < \theta \leqq 90°)$ be the angle $\measuredangle A\beta\gamma$, we readily find $\measuredangle \beta HA = 150° - \theta$; hence H by the law of sines is $R \csc \theta/2$, where R is the circumradius of ABC. Similarly, $H\gamma = R \csc (\theta + 60°)/2$. Hence

$$\beta\gamma = R \sin (30° + \theta) \csc \theta \csc (60° + \theta) \sqrt{3}/2.$$

Now calling O_a the circumcenter of $\Delta A\beta\gamma$, and H_a its orthocenter, we have as a corollary of the lemma above that $AO_a = AH_a = R_a$, the circumradius of $\Delta A\beta\gamma$. The perpendicular AK_a from A upon O_aH_a is equal to $R_a \cos (\theta - 60°)$, since AO_a

and AH are isogonal in the angle $\beta A\gamma$. Then HK_a is the distance of H from the side $B'C'$ of the $\Delta A'B'C'$, and is found to be $R\sin(2\theta-30°)\csc\theta\csc(120°-\theta)/4$, since $R_a=\beta\gamma\csc 60°/2$.

Similarly, HK_c is found to be $-R\sin(2\theta+30°)\csc\theta\csc(\theta-60°)/4$. Since we have, without loss of generality, placed the larger of the two angles β and γ of triangle $A\beta\gamma$ at β and called it θ, the triangle $B\alpha\gamma$ has $\measuredangle\gamma B\alpha=120°$, $\measuredangle B\alpha\gamma=\theta-60°$; hence $HK_b=-R\cos 2\theta\csc(\theta-60°)\csc(120°-\theta)/4$. Adding $HK_a+HK_b+HK_c$ we find, with the use of well known trigonometric reduction identities, their sum to be $3R/2$, showing that the sum of the perpendiculars upon the sides of $A'B'C'$ from H, any point of the plane $A'B'C'$, is equal to three-halves of the circumradius of ABC. Since $A'B'C'$ is equilateral, it follows that ABC and $A'B'C'$ are congruent as well as homothetic. Hence they are symmetrically equal.

Next, let the transversal Δ cut the sides $B'C'$, $C'A'$, and $A'B'$ in α', β', γ'. From the right triangle $HK_a\alpha'$ we find

$$H\alpha' = HK_a\csc(\theta-60°) = R\sin(2\theta-30°)/4\sin\theta\sin(120°-\theta)\sin(\theta-60°).$$

Similarly,

$$H\beta' = HK_b\csc\theta = -R\cos 2\theta/4\sin\theta\sin(120°-\theta)\sin(\theta-60°).$$

Adding, we get

$$\alpha'\beta' = R\sqrt{3}\sin(2\theta-60°)/4\sin\theta\sin(120°-\theta)\sin(\theta-60°).$$

From the triangles $HA\beta$ and $H\alpha B$ we find by the law of sines that

$$H\beta = R\csc\theta/2,$$
$$H\alpha = R\csc(\theta-60°)/2.$$

Adding, we get

$$\alpha\beta = R\sqrt{3}\sin(2\theta-60°)/4\sin\theta\sin(120°-\theta)\sin(\theta-60°),$$

thus proving that $\alpha\beta=\alpha'\beta'$.

Thus the triangles $\alpha'\beta'C'$ and $\alpha\beta C$ are congruent, making αC equal and parallel to $\alpha'C'$. Hence, the midpoint of $\alpha\alpha'$ is also the midpoint of CC', which is the center of symmetry. Therefore the center of symmetry also lies on the transversal Δ.

Editorial Note. The lemma results from two simple theorems. The reflection H' of the orthocenter H in the side BC of any triangle ABC lies on the circumcircle (O) of ABC. This theorem has been used before in this MONTHLY in the solution of problems.

If the reflection O' of the center O of circle (O) in the chord BC lies on (O), then $\angle BO'C=2\pi/3$; and if A is any other point of (O), then $\angle BAC=\pi/3$ or $2\pi/3$ according as A and O' are on opposite sides or the same side of BC. The converse follows easily. Hence, if the angle A of triangle ABC is $\pi/3$ or $2\pi/3$, the figure $OHH'O'$ is symmetric with respect to BC; and HO', $H'O$ are antiparallel with respect to the parallels OO' and HH'. Also, $H'O$, AO are antiparal-

lel with respect to the same pair of lines; hence HO', AO are parallel, $AHO'O$ is an isosceles parallelogram, and AO', OH are perpendicular bisectors. Since O' is the midpoint of arc BC, the line AO' is the internal or external bisector of angle A according as A is $\pi/3$ or $2\pi/3$. It then follows that HO forms an equilateral triangle with the sides AB, AC produced if necessary.

Denote by H_a, O_a the orthocenter and circumcenter of $A\beta\gamma$; then by the lemma, the straight line H_aO_a is parallel to BC. Thus AH_a, BH_b, CH_c are parallel, being perpendicular to Δ, and cut the straight line $\Delta'=[H, H_a, H_b, H_c]$ respectively in the last three points indicated in the brackets. Then by 3817 [1939, 177] the triangle $A'B'C'$ formed by H_aO_a, H_bO_b, H_cO_c is symmetrically equal to ABC, and the point of symmetry P lies on the Newton line (ABC, Δ'). The solution of 3818 [1939, 178] shows that Δ, (ABC, Δ), (ABC, Δ') meet in P; and this concludes the proof.

3959 [1940, 323]. *Proposed by N. A. Court, University of Oklahoma.*

The four pairs of reciprocal transversals a, a'; b, b'; c, c'; d, d' are situated, respectively, in the faces BCD, CDA, DAB, ABC of the tetrahedron $ABCD$. (1) If the lines a, b, c, d are coplanar, so also are the lines a', b', c', d' (the proposer's *Modern Pure Solid Geometry*, p. 121). (2) If the lines a, b, c, d form a hyperbolic group, so also do the remaining four lines.

Solution by the Proposer.

Let $X=(a, BC)$, $Y=(b, CA)$, $Z=(c, AB)$, and let X', Y', Z' be the symmetrics of X, Y, Z, respectively, with respect to the midpoints of the corresponding edges. Since the four lines a, b, c, d are hyperbolic, the points X, Y, Z lie on the same straight line, say p; therefore the points X', Y', Z' lie on the reciprocal transversal p' of p for the triangle ABC. Now the points X', Y', Z' belong, respectively, to the lines a', b', c' by assumption; hence p' is coplanar with each of the lines a', b', c', d'.

We have analogous lines q', r', s' in the faces BCD, CDA, DAB of the tetrahedron $ABCD$. Hence the proposition.

3960 [1940, 323]. *Proposed by R. E. Gaines, University of Richmond.*

If a series of triangles be constructed so that the sides of each are equal to the medians of the following one, then (1) the area of each triangle is three-fourths of the area of the one following; (2) the alternate triangles are similar; (3) excluding the case of equilateral triangles, no two consecutive triangles of the series are similar.

Solution by H. W. Bailey, University of Illinois.

Let three consecutive triangles of the series be $A_iB_iC_i$, with sides a_i, b_i, c_i, and choose the notation so that $a_i \geqq b_i \geqq c_i$, $(i=1, 2, 3)$. Then the medians $A_{i+1}\alpha_{i+1}$, $B_{i+1}\beta_{i+1}$, $C_{i+1}\gamma_{i+1}$ are equal, respectively, to c_i, b_i, a_i, $(i=1, 2)$. Combining these equalities with Appolonius's theorem, we have

$$
(1)\qquad
\begin{aligned}
\overline{A_{i+1}\alpha_{i+1}}^2 &= \frac{-a_{i+1}^2}{4}+\frac{b_{i+1}^2}{2}+\frac{c_{i+1}^2}{2} = c_i^2,\\
\overline{B_{i+1}\beta_{i+1}}^2 &= \frac{a_{i+1}^2}{2}-\frac{b_{i+1}^2}{4}+\frac{c_{i+1}^2}{2} = b_i^2,\\
\overline{C_{i+1}\gamma_{i+1}}^2 &= \frac{a_{i+1}^2}{2}+\frac{b_{i+1}^2}{2}-\frac{c_{i+1}^2}{4} = a_i^2,
\end{aligned}
\qquad (i = 1, 2).
$$

On multiplying out the radicand and squaring, Heron's formula for the area of a triangle may be put in the form

$$16K^2 = 4b^2c^2 - (a^2 - b^2 - c^2)^2.$$

If we write this formula for the first triangle, and replace the values of a_1, b_1, c_1 in the right member by their values from (1), $i=1$, we obtain

$$
\begin{aligned}
16K_1^2 &= 4b_1^2c_1^2 - (a_1^2 - b_1^2 - c_1^2)^2\\
&= \tfrac{1}{4}(2a_2^2 - b_2^2 + 2c_2^2)(-a_2^2 + 2b_2^2 + 2c_2^2) - \tfrac{1}{16}(a_2^2 + b_2^2 - 5c_2^2)^2\\
&= \tfrac{9}{16}\left[4b_2^2c_2^2 - (a_2^2 - b_2^2 - c_2^2)^2\right]\\
&= 9K_2^2.
\end{aligned}
$$

Hence $K_1 = 3K_2/4$, and each triangle has an area three-fourths of that of the triangle following.

On writing equations (1), $i=2$, and substituting the values of a_2^2, b_2^2, c_2^2 thus obtained in equations (1), $i=1$, there results on reduction

$$a_1^2 = \tfrac{9}{16}a_3^2, \qquad b_1^2 = \tfrac{9}{16}b_3^2, \qquad c_1^2 = \tfrac{9}{16}c_3^2.$$

Hence $a_1/a_3 = b_1/b_3 = c_1/c_3 = 3/4$, and alternate triangles of the series are similar, with a ratio of similitude of $3:4$.

Two consecutive triangles of the series are similar if and only if $a_1/a_2 = b_1/b_2 = c_1/c_2$. Taking the two independent relations $a_1b_2 = a_2b_1$ and $a_1c_2 = a_2c_1$, squaring, and replacing a_1^2, b_1^2, c_1^2 by their values as given by (1), $i=1$, we have

$$
(2)\qquad
\begin{aligned}
2a_2^4 - 2b_2^4 - 3a_2^2b_2^2 + b_2^2c_2^2 + 2a_2^2c_2^2 &= (2a_2^2 + b_2^2)(a_2^2 - 2b_2^2 + c_2^2) = 0,\\
a_2^4 - c_2^4 - 2a_2^2b_2^2 + 2b_2^2c_2^2 &= (a_2^2 - c_2^2)(a_2^2 - 2b_2^2 + c_2^2) = 0.
\end{aligned}
$$

Since $2a_2^2+b_2^2 \neq 0$ for real triangles, $a_2^2 - 2b_2^2 + c_2^2 = 0$ is a necessary condition for similarity of consecutive triangles; since it implies both of the equations (2), it is also a sufficient condition. On computing $a_i^2 - 2b_i^2 + c_i^2$ from (1), we find that

$$a_i^2 - 2b_i^2 + c_i^2 = -\tfrac{3}{4}(a_{i+1}^2 - 2b_{i+1}^2 + c_{i+1}^2).$$

Hence, a necessary and sufficient condition that two consecutive triangles of the series be similar is that for either of them,

$$a^2 - 2b^2 + c^2 = 0. \tag{3}$$

It follows immediately that if any two consecutive triangles are similar, all triangles of the series are similar.

If $a=5$, $b=4$, $c=\sqrt{7}$, the condition (3) is satisfied and all triangles of a series including this one are similar. But this triangle is not equilateral, and hence the third statement of the problem is in error and should be replaced by condition (3).

Solved also by R. A. Johnson, D. L. MacKay, and the proposer.

Editorial Note. The problem proposition (3) was altered in Johnson's solution to a form similar to that in the above solution; and in his proof of this revised proposition he used the Appolonian theorem in the above solution. He stated that propositions (1) and (2) are well known in a slightly modified form; *cf.*, Johnson's *Modern Geometry*, p. 283, and Court's *College Geometry*, p. 60. His proof of (1) and (2) is as follows: Let $A_1A_2A_3$ be the given triangle with the median point M and with O_1, O_2, O_3 the midpoints of its sides. Through A_2, A_3 draw parallels to A_1O_1, A_2O_2, respectively, meeting in B_3; let the extension of A_3O_3 cut B_3A_2 in B_2, and let the extension of A_1O_1 cut A_3B_3 in X. He proves then that $A_3B_2B_3$ is the required triangle derived from the given triangle.

For, the following are parallelograms: MA_2XA_3, MA_2B_3X, MB_2A_2X, $MA_1B_2A_2$. Therefore the midpoints of the sides of $A_3B_2B_3$ are M, A_2, X. Since $A_1B_2XA_3$ and $A_1A_2B_3M$ are also parallelograms, the medians B_2X and B_3M of $A_3B_2B_3$ are equal and parallel, respectively, to A_1A_3 and A_1A_2; while A_2A_3 is the third median. It follows at once from the figure that each side of $A_3B_2B_3$ is 4/3 the corresponding median of $A_1A_2A_3$; hence, if the construction is repeated, the alternate triangles are similar in the ratio 4/3. Also, since $A_2B_3=(2/3)A_1O_1$, area $A_2A_3B_2=(2/3)$ area $A_1A_2A_3$, and area $A_3B_2B_3=(4/3)$ area $A_1A_2A_3$. Except in the trivial case of equilateral triangles, two successive triangles can be similar only when the sides a_1, a_2, a_3 and b_3, b_2, b_1 correspond in this order. The modification of (3) then follows.

The proposer's proofs of (1) and (2) are somewhat similar to those of Bailey. MacKay's proofs of (1) and (2) are somewhat similar to those of Johnson.

A geometric solution by Sun Nien-tseng was received after the preparation of the above for printing; and the condition (3) above was also obtained.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A fifth issue of the 1941 *Annals of Mathematics* has been published, made possible by a grant from the American Philosophical Society.

A year ago the Board of Governors of the Mathematical Association notified the *Annals of Mathematics* and *Duke Mathematical Journal* that, because of the many demands upon its income, the Association would not be able to continue the subvention to these two journals after 1942. The Duke University Press has agreed to continue to allow the half rate after 1942 to those whose membership in the Association and whose subscription to the *Duke Mathematical Journal* are unbroken from 1942 to the year in question. The arrangement is expected to be permanent, but the Press reserves the right to modify or withdraw it after five years and to change the basic four dollar subscription if this should become necessary.

Professor A. W. Tucker represented the Mathematical Association at the 175th Anniversary celebration of Rutgers University, October 9, 10, 11.

Professor Harris Mac Neish attended as delegate for the Mathematical Association the Second Conference on Science, Philosophy, and Religion and their Relation to the Democratic Way of Life held at New York, September 1941.

Professor H. N. Hubbs of Hobart College represented the Mathematical Association at the inauguration of President H. E. Allen of Keuka College November 7.

The following appointments to instructorships are announced:

Antioch College: Margaret Finnigan

Brown University: J. H. Van Lonkhuyzen, Dr. K. L. Nielsen

Case School of Applied Science: Dr. R. H. Sorgenfrey

University of Illinois: Dr. H. M. Schwartz

University of Minnesota: N. R. Amundson, J. D. Novak, R. B. Saunders, Sigward Stavnes, L. W. Swanson

University of Missouri: W. E. Ferguson, L. M. Kelly

Newcomb College: Helen P. Beard

Northeastern University: S. A. Stone

Ohio State University: H. D. Huskey, C. G. Maple, Dr. E. J. Mickle

Princeton University: Dr. H. A. Arnold, Dr. John Giese, Dr. M. S. Macphail, Dr. Henry Scheffé

River Falls, Wis., State Teachers College: C. J. Kirchen

South Dakota State School of Mines: C. L. Harbison, E. L. Swanson

University of Texas: R. M. Adams, R. D. Anderson, C. E. Burgess, H. L. Cates, Jr., J. P. LaSalle

University of Utah: Dr. F. C. Biesele
West Virginia Institute of Technology: J. K. Reckzeh
Winthrop College: Dr. H. C. Miller
University of Wisconsin: Dr. R. L. Swain, Dr. Henry Wallman

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H. W. Bailey; Walter Bartky; G. A. Bliss; Henry Blumberg; R. W. Brink; B. H. Brown; W. H. Bussey; W. D. Cairns; W. B. Carver; A. B. Coble; N. A. Court; H. S. M. Coxeter; D. R. Curtiss; H. T. Davis; L. E. Dickson; Churchill Eisenhart; H. P. Evans;

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R. E. Langer; Lincoln La Paz; N. H. McCoy; J. R. Musselman; Rufus Oldenburger; E. G. Olds; H. B. Phillips; W. T. Reid; P. R. Rider; I. S. Sokolnikoff; W. J. Trjitzinsky; R. J. Walker; H. S. Wall; Marie J. Weiss; M. E. Wescott; L. R. Wilcox; F. E. Wood.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Annual Meeting, Bethlehem, Pennsylvania, December 29, 1941–January 1, 1942

Twenty-fifth Summer Meeting, Ithaca, New York, September 7–9, 1942

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain
Illinois, Decatur, May 8–9, 1942
Indiana, Crawfordsville, May 1–2, 1942
Iowa, Mt. Pleasant, April 17–18, 1942
Kansas, Hays, March 27–28, 1942
Kentucky
Louisiana-Mississippi, Jackson, Miss., March 6–7, 1942
Maryland-District of Columbia-Virginia, Washington, D. C., Dec. 6, 1941
Metropolitan New York, New York, April 18, 1942
Michigan, Detroit, Nov. 15, 1941
Minnesota
Missouri, Kansas City, April 17, 1942
Nebraska, Omaha, May 9, 1942
Northern California, Berkeley, Jan. 31, 1942
Ohio, Columbus, April 2, 1942
Oklahoma, Oklahoma City, Feb. 13, 1942
Philadelphia, Philadelphia, Nov. 28, 1942
Rocky Mountain, Golden, Colo., April 17–18, 1942
Southeastern, Emory University, Ga., March 26–27, 1942
Southern California, Los Angeles, March 14, 1942
Southwestern, State College, N. M., April 27–28, 1942
Texas, Lubbock, April 3–4, 1942
Upper New York State, Rochester, May 2, 1942
Wisconsin, Oshkosh, May 2, 1942

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics

NECROLOGY

CORRIGENDA

Vol. 47, 1940, p. 182. Problem 3949 should read: "Given the angles $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$ with the common initial line Ox, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{n(n+1)/2+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$, or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$."

Vol. 47, 1940, p. 652. In the review of *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, the statement that the actual work of computation was undertaken by a staff of six computors should be replaced by the statement that the six mentioned were the supervisors of the actual computors, who included more than two hundred and fifty persons.

Vol. 48, 1941, p. 148. In problem E-458, the last sentence should read: "Show that the same result holds for any matrix (a_{rs}) in which $a_{r1} = a_{2r}$, $a_{1r} = a_{r2}$, $(r > 2)$, and $a_{11} = a_{22}$."

Vol. 48, 1941, p. 204. In the list of New Books Received, after *A Survey of Methods of Apportionment in Congress*, insert "By E. V. Huntington."

Vol. 48, 1941, p. 258 and p. 262. In the list of New Books Received and in the book review, the title of the book should read *Displacement, Velocity, and Acceleration Factors for Reciprocating Motion.*

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